MA 181 Lecture
Chapter 7
College Algebra and Calculus by Larson/Hodgkins
Limits and Derivatives

7.5) Rates of Change: Velocity and Marginals

Previously we learned two primary applications of derivatives.

1. **Slope** The derivative of f is a function that gives the slope of the graph of f at a point $(x,f(x))$.
2. **Rate of Change** The derivative of f is a function that gives the rate of change of $f(x)$ with respect to x at the point $(x,f(x))$.

In this section we will look at several real life applications of the derivative. Several of these examples will use time as our independent variable. We will consider the rate of change with respect to time. When we discuss rate of change there are two different types, and it is important to be able to distinguish between the two types, average rate of change and instantaneous rate of change.

The key to remember is **average rate of change** is over an interval, or it is the slope of the secant line. **Instantaneous rate of change** is at a single point and is the slope of the tangent line.

Definition of Average Rate of Change
If $y=f(x)$, then the **average rate of change** of y with respect to x on the interval $[a,b]$ is
Average Rate of Change = $\dfrac{f(b)-f(a)}{b-a}=\dfrac{\Delta y}{\Delta x}$
Note that $f(a)$ is the value of the function at the *left* endpoint of the interval, $f(b)$ is the value of the function at the *right* endpoint of the interval, and $b-a$ is the width of the interval.

Example:

The concentration C (in milligrams per milliliter) of a drug in a patient's bloodstream is monitored over 10-minute intervals for 2 hours, where t is measured in minutes, as shown in the table. Find the average rate of change over each interval.

t	0	10	20	30	40	50	60	70	80	90	100	110	120
C	0	2	17	37	55	73	89	103	111	113	113	103	68

[0,10]

Note the average rate of change will have milligrams per milliliter per minute units.

[0,30]

Note a positive average rate of change implies that the concentration is increasing while a negative average rate of change occurs when the concentration decreases.

[100,110]

A common application of an average rate of change is to find the average **velocity** of an object that is moving in a straight line. That is:

$$Average\ Velocity = \frac{Change\ in\ Distance}{Change\ in\ Time}$$

Example:

If a free-falling object is dropped from a height of *100* feet, and *air resistance is neglected*, the height *h* (in feet) of the object at time *t* (in seconds) is given by

$$h = -16t^2 + 100$$

Find the average velocity of the object over each interval. Be sure to include units.

a.) *[1,2]* b.) *[1,1.5]* c.) *[1,1.1]*

Use the table as a reference for the function values:

t (in seconds)	0	1	1.1	1.5	2
h (in feet)	100	84	80.64	64	36

We are often interested in the rate of change at a particular time, for example when *t=1*. This rate is the **instantaneous rate of change**. We could approximate the instantaneous rate of change by calculating the average rate of change repeatedly with values of *t* that get closer and closer to *t=1*. The following table shows the results of such calculations, where we let Δt approaches zero (so the points are getting closer to each other.) What can you conclude about the average rate of change, $\frac{\Delta h}{\Delta t}$?

Δt	1	0.5	0.1	0.01	0.001	0.0001	0
$\frac{\Delta h}{\Delta t}$	-48	-40	-33.6	-32.16	-32.016	-32.0016	?

Definition of Instantaneous Rate of Change
The **instantaneous rate of change** (or simply **rate of change**) of *y=f(x)* at *x* is the limit of the average rate of change on the interval $[x, x+\Delta x]$, as Δx approaches zero. $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ If *y* is a distance and *x* is time, then the rate of change is **velocity**.

Verify the instantaneous velocity in the above example.

Consider the following:

If you drive from Potsdam to Massena, it takes 30 minutes to travel 20 miles. Find the average rate of change for this drive. What can you tell about the instantaneous rate of change at the fifteenth minute?

The general **position function** for a free-falling object, neglecting air resistance, is

$$h = -16t^2 + v_0 t + h_0$$

where *h* is the height (in feet), *t* is the time (in seconds), v_0 is the initial velocity (in feet per second), and h_0 is the initial height (in feet.) Positive velocities indicate movement upwards while negative velocities indicate downward motion.

The derivative $h' = -32t + v_0$ is the **velocity function**. The absolute value of the velocity is the **speed** of the object.

Example:

At time *t=0*, a diver jumps from a diving board that is 32 feet high. Given that the diver's initial velocity is 16 feet per second, find his position function.

When does the diver hit the water?

What is the diver's velocity at impact?

Rates of Change in Economics: Marginals

When we take the derivative of the *cost*, *revenue*, or *profit* functions in economics, we call the derivative functions *marginal cost*, *marginal revenue*, or *marginal profit* respectively. This describes the rate of change with respect to the number *x* of units produced or sold.

We use the following equation to relate the three quantities:
$P = R - C$
where *P* is the total profit, *R* is the total revenue, and *C* is the total cost.

We say the following:

$\frac{dP}{dx} = \text{marginal profit}$
$\frac{dR}{dx} = \text{marginal revenue}$
$\frac{dC}{dx} = \text{marginal cost}$

Often in business or economics applications we consider the units we are selling or producing to be restricted to positive integers (in other words we don't sell 5.7 t-shirts.) We say that this variable denotes such units is called a **discrete variable** To analyze a function of a discrete variable , you can temporarily assume that x is a **continuous variable** and is able to take on any real value in a given interval. Then you can use the methods of calculus to find the x-value that corresponds to the marginal revenue, marginal cost, or marginal profit and round the solution to the nearest sensible x-value, if we are talking about money, please round to the nearest cent.

Example:

The profit P (in dollars) from selling x units of calculus textbooks is given by

$$P = -0.05x^2 + 20x - 1000$$

Find the additional profit when the sales increase from *150* units to *151* units.

Find the marginal profit when $x=150$.

Compare the two answers above.

The number of units x that consumers are willing to purchase at a given price per unit p is given by the **demand function**
$p = f(x)$.

The total revenue R is then related to the price per unit and the quantity demanded (or sold) by the equation
$R = xp$

Example:

A business sells 2000 items per month at a price of $10 each. It is estimated that monthly sales will increase 250 units for each $0.25 reduction in price. Use this information to find the demand function and total revenue function.

$$p = 10 - .25\left(\frac{x - 2000}{250}\right) = 10 - .001(x - 2000) = 10 - .001x + 2 = 12 - \frac{x}{1000}$$

Example:

The demand function for a product is given by $p = \frac{50}{\sqrt{x}}$ for $1 \le x \le 8000$, and the cost function is given by $C = 0.5x + 500$ for $0 \le x \le 8000$.

Find the marginal profits for

$x = 900$	$x = 1600$	$x = 2500$	$x = 3600$

If you were in charge of setting the price for this product, what price would you set? Explain your reasoning.

Example:

The cost C of producing x units is modeled by $C = v(x) + k$, where v represents the variable cost and k represents the fixed cost. Show that the marginal cost is independent of the fixed cost.

Example:

The annual inventory cost for a manufacturer is given by $C = \frac{1{,}008{,}000}{Q} + 6.3Q$ where Q is the order size when the inventory is replenished. Find the change in annual cost when Q is increased from 350 to 351, and compare this with the instantaneous rate of change when $Q = 350$.