MA 180 Lecture
Chapter 7
College Algebra and Calculus by Larson/Hodgkins
Limits and Derivatives

7.1) Limits

An important concept in the study of mathematics is that of a limit. It is often one of the harder concepts to understand. A limit is a bound, it is a value that we approach (but often do not achieve.) One example is carrying capacity. Consider a contained environment (think fruit flies in a corked test tube,), if we introduce a species and wait to see how they reproduce we will find that there is a limited number of resources and space and that their population will reach a value that becomes bounded. (We call this number the carrying capacity in this situation.) This is a limit. There is a limit to which the population grows to. A second example would be cooling temperatures. Imagine a room temperature object that you place in the fridge. If we monitor its temperature over time we will find that the temperature decreases and eventually reaches the same temperature as the fridge. It will not become any colder than the temperature in the fridge. This is a limit. The final example is one we have already discussed in class and that is the value of *e* in terms of continuous compounding. We said that if we evaluated an amount of money invested at a certain interest rate for a year with different compounding times per year, after a certain point we were making the same amount of money. Here is the math for that example using an investment of $1 and a 100% interest rate for a duration of one year. We say *n* is the number of times compounded per year. Note what happens to our total amount as *n* gets bigger and bigger.

n Values	$A = P\left(1+\frac{1}{n}\right)^{nt}$
n=1	$2
n=4	$2.44…
n=12	$2.61…
n=52	$2.692596…
n=365	$2.714567…
n=8760 (hourly)	$2.71812669…
n=525600 (once per minute)	$2.7182792425…
n=31536000 (once per second)	$2.71828178536…

As you can see it looks like there is a limit to the value this will take on. And this is what we define to be the number *e*. It is also the limit. We use the following notation for limits:

$\lim_{x\to c}\left(f(x)\right) = L$ and is read "the limit of *f(x)* as *x* approaches *c* is *L*."

Graphically, the limit is the *y* value we approach as we let the *x* value approach a number *c*. That *y* value is called the limit *L*.

The first method we will discuss for evaluating a limit is by making a table. This method is both tedious and lengthy so we will only use it as a way to understand a limit.

Find $\lim_{x\to 1}\left(x^2+1\right)$. To find this limit we will use a table of values. First we will plug in values that are approaching *c=1* and evaluate the function at those values. We will then observe the behavior of those *y* values.

x values	0.900	0.990	0.999		1.001	1.010	1.100
f(x)	1.810	1.980	1.998		2.002	2.020	2.210

Note that we plugged in values that approached *1* with *x* values smaller and larger than one. When first studying limits we might notice that we can simply plug in the number into the function and get the right limit. This is true *some* of the time and in the next section we will determine when that is true. For this section, we should assume we cannot simply plug it in and try to fully understand a limit and how to evaluate it.

The second method for finding a limit is graphically. Consider the graph of $y = x^2$. Describe the behavior as *x* approaches *2*. Then determine the following: $\lim\limits_{x \to 2}\left(x^2\right)$.

Sometimes a limit exists even if the function does not exist at that *x* value. Consider for example $\lim\limits_{x \to 1}\left(\dfrac{x^2 - 1}{x - 1}\right)$. Graph the function then discuss the limit.

It is possible that a limit does not exist. Consider a piecewise defined function. Find the following $\lim\limits_{x \to 1}\left(f(x)\right)$ if $f(x) = \begin{cases} x + 2 & \text{if } x < 1 \\ x - 2 & \text{if } x > 1 \end{cases}$

Does this problem change if the function is defined at one?

Find the following limit.

$$\lim_{x\to 1}\left(f(x)\right) \text{ if } f(x)=\begin{cases} 2 & if\ x=1 \\ -2 & if\ x\neq 1 \end{cases}$$

The take-away…

1.) Saying that the limit of *f(x)* approaches *L* as *x* approaches *c* means that the value of *f(x)* may be made arbitrarily close to the number *L* by choosing *x* closer and closer to *c*.
2.) For a limit to exist, you must allow *x* to approach *c* from either side of *c*. If *f*(x) approaches a different number as x approaches c from the left than it does as x approaches *c* from the right, then the limit does not exist.
3.) The value of *f(x)* when *x=c* has no bearing on the existence or nonexistence of the limit of *f(x)* as *x* approaches *c*.

Definition of the limit of a function

If *f(x)* becomes arbitrarily close to a single number *L* as *x* approaches *c* from either side, then $\lim_{x\to c}\left(f(x)\right)=L$ which is read as "the **limit** of *f(x)* as *x* approaches *c* is *L*."

Properties of Limits:

Whenever the limit of *f(x)* as *x* approaches *c* is simply *f(c)*, we say the limit can be evaluated by **direct substitution**. In the next section we will learn that this happens for *continuous* functions. We will need to figure out what functions have this property. We can use the following list of properties to get us started.

Properties of Limits
Let *b* and *c* be real numbers, and let *n* be a positive integer.
1. $\lim_{x\to c}(b)=b$
2. $\lim_{x\to c}(x)=c$
3. $\lim_{x\to c}(x^n)=c^n$
4. $\lim_{x\to c}\left(\sqrt[n]{x}\right)=\sqrt[n]{c}$ (If *n* is even, then *c* must be positive.)

We can also combine the above to evaluate more limits.

Operations with Limits
Let *b* and *c* be real numbers, let *n* be a positive integer, and let *f* and *g* be functions with the following limits. $\lim_{x\to c}\left(f(x)\right)=L$ and $\lim_{x\to c}\left(g(x)\right)=K$
1. Scalar multiple: $\lim_{x\to c}\left(b\cdot f(x)\right)=bL$
2. Sum or difference: $\lim_{x\to c}\left(f(x)\pm g(x)\right)=L\pm K$
3. Product: $\lim_{x\to c}\left(f(x)\cdot g(x)\right)=L\cdot K$
4. Quotient: $\lim_{x\to c}\left(\frac{f(x)}{g(x)}\right)=\frac{L}{K}$, provided $K\neq 0$
5. Power: $\lim_{x\to c}\left(f(x)\right)^n=L^n$
6. Radical: $\lim_{x\to c}\left(\sqrt[n]{f(x)}\right)=\sqrt[n]{L}$ If *n* is even, then *L* must be positive.

Find the limit of the polynomial:

$$\lim_{x\to 2}\left(3x^2+5x-7\right)$$

The limit of a polynomial function

If *p* is a polynomial function and *c* is any real number, then $\lim_{x\to c}\left(p(x)\right)=p(c)$.

Techniques for Evaluating Limits

The following theorem helps us evaluate limits, it basically states that if two functions agree at all but a single point *c*, then they have identical limit behavior at *x=c*.

The Replacement Theorem

Let *c* be a real number and let *f(x)=g(x)* for all $x\neq c$. If the limit of *g(x)* exists as $x\to c$, then the limit of *f(x)* also exists and $\lim_{x\to c}\left(f(x)\right)=\lim_{x\to c}\left(g(x)\right)$.

Examples:

$$\lim_{x\to 2}\left(\frac{x^2-4}{x-2}\right)$$

$$\lim_{x\to 5}\left(\frac{x-5}{x^2-6x+5}\right)$$

We sometimes refer to this method as **dividing out**. Find the limit.

$$\lim_{x\to 1}\left(\frac{x^3-1}{x-1}\right)$$

There is a process for evaluating limits. We will begin that process now.

1. Try plugging in the *c* value. If it is a polynomial, rational, root function that is continuous at *c* then we get the limit by plugging it in directly.
2. If we get zero on the top and zero on the bottom of a rational expression then we should try dividing out.

The next option involves roots. If we see a radical expression we should think to multiply by the conjugate. Remember the conjugate of $\sqrt{a}+b$ is $\sqrt{a}-b$ and when we multiply them and use FOIL, we get $\left(\sqrt{a}+b\right)\left(\sqrt{a}-b\right)=a-b^2$ thus eliminating the radical.

Use that concept to find the following limit.

$$\lim_{x\to 0}\left(\frac{\sqrt{x+1}-1}{x}\right)$$

The final scenario is that involving fractions. If you come across a limit that you can't use direct substitution on, you may need to combine fractions and simplify.

Find the following limit.

$$\lim_{x\to 0}\left(\frac{\frac{1}{x+5}-\frac{1}{5}}{x}\right)$$

Here is the final process:

Strategies for finding limits
1. Direct substitution.
2. Divide it out. (Factor and cancel.)
3. Multiply radicals by their conjugate.
4. Combine and simplify all fractions.

One sided-limits

When evaluating a limit as *x* approaches *c* we need to consider *x* values that occur before *c*, so values that are less than *c*, or to the *left* of *c*, as well as values that are larger and to the *right* of *c*. This is the general concept behind one-sided limits. We consider what is happening to the **right** and **left** and find the limits independently. We say that if both the right and left limits exist and are equal then the (two – sided) limit exits. We use the following notation.

$\lim_{x\to c^-}\left(f(x)\right)=L$ is the **left hand limit** and considers what occurs to the left of *c*

$\lim_{x\to c^+}\left(f(x)\right)=L$ is the **right hand limit** and considers what occurs to the right of *c*

Find the following:

$$\lim_{x\to 0^+}\left(\frac{|x|}{x}\right)$$

$$\lim_{x\to 0^-}\left(\frac{|x|}{x}\right)$$

Existence of a limit

If f is a function and *c* and *L* are real numbers, then $\lim_{x \to c}\left(f(x)\right) = L$ if and only if the left and right limits both exist and are equal to *L*.

Find the following limit. $\lim_{x \to 8}\left(f(x)\right)$ with *f(x)* as defined below.

$$f(x) = \begin{cases} 2x - 3 & \text{if } x < 8 \\ x + 5 & \text{if } x > 8 \end{cases}$$

Unbounded Behavior

Look at the graph of the following function to determine its behavior.

$$\lim_{x \to 0^+}\left(\frac{1}{x^2}\right)$$

We say this is unbounded and thus the limit does not exist. We may use infinity or negative infinity to describe the behavior.