MA 131 Lecture Notes
Derivative of Trig Functions and the Chain Rule

Derivatives of Trigonometric Functions

We are also interested in finding the derivatives of trig functions. Let us study the graph of $f(x)=\sin x$. Draw the graph and estimate the slope of the tangent line to the graph at all x that are multiples of $\frac{\pi}{2}$. What do you see?

As it turns out $\frac{d}{dx}(\sin x) = \cos x$. What about $f(x)=\cos x$?

We can see that $\frac{d}{dx}(\cos x) = -\sin x$.

We can use trig identities to find $\frac{d}{dx}(\tan x)$.

In fact for each of the last three trig functions we can simply rewrite them and differentiate using the quotient rule. We find the following to be true.

$\frac{d}{dx}(\sin x) = \cos x$	$\frac{d}{dx}(\csc x) = -\csc x \cot x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\frac{d}{dx}(\sec x) = \sec x \tan x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\frac{d}{dx}(\cot x) = -\csc^2 x$

(One way to remember them is to notice that if you learn the derivatives of *tanx* and *secx*, you get the co-function derivatives by adding "co" to the derivatives and a negative. Also note that each co-function's derivative is negative.)

Examples

$$\frac{d}{dx}(5\sec x)$$

$$\frac{d}{dx}(x\cot x)$$

$$\frac{d}{dx}(\sin x \tan x)$$

The Chain Rule

So far we have learned some basic differentiation rules that help us take the derivatives of constant multipliers, sums and differences, exponential and trig functions, and products and quotients. We have one more major type of function that requires a special rule and that is compositions of functions. We will call this rule the **chain rule** and it will help us find derivatives of composite functions.

> **The Chain Rule**
>
> If $y=f(u)$ is a differentiable function of u, and $u=g(x)$ is a differentiable function of x, then $y=f(g(x))$ is a differentiable function of x, and
>
> $$\frac{dy}{dx}=\frac{dy}{du}\cdot\frac{du}{dx}$$
>
> or, equivalently,
>
> $$\frac{d}{dx}\left(f(g(x))\right)=f'(g(x))\cdot g'(x)$$

This rule says that if we can identify the function as a composition of functions, then identify the inner function and the outer function. We get the derivative by doing the following, take the derivative of the outer function, copy the inner function, and multiply by the derivative of the inner function.

First we'll practice identifying the inner and outer functions.

Example:

Given $y=\left(3x^2+5\right)^7$ then we can say that $u=g(x)=3x^2+5$ and this is the inner function. Then $y=u^7$ is our outside function.

Find the inner and outer functions of the following:

$$y=\frac{1}{x^5-4}$$

$$y=\sqrt{x-9}$$

$$y=\frac{1}{(7x-2)^2}$$

$y = \ln\left(3x + 5x^2\right)$

$y = \sin(5x^2 + 7)$

$y = e^{8x^3}$

Currently we do not have the tools to know how to differentiate the logarithmic function, but it is still important to identify that is an example of a composite function.

Let us revisit the first example to see how the chain rule works.

$y = \left(3x^2 + 5\right)^7$ with $u = g(x) = 3x^2 + 5$ and $y = u^7$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$\frac{dy}{dx} = 7u^6 \cdot 6x = 7\left(3x^2 + 5\right)^6 6x = 42x\left(3x^2 + 5\right)$$

Often in practice we skip the first step of writing it in terms of *u*.
Consider the following example, find the derivate using two different methods and compare your answers.

$y = \left(3x + 2\right)^2$

The General Power Rule

If $y = [u(x)]^n$, where *u* is a differentiable function of *x* and *n* is a real number, then

$$\frac{dy}{dx} = n[u(x)]^{n-1} \cdot \frac{du}{dx}$$

or, equivalently,

$$\frac{d}{dx}\left(u^n\right) = nu^{n-1}u'$$

Find

$$\frac{d}{dx}\left(\left(4x^3 - 5x + 3\right)^9\right)$$

$$\frac{d}{dx}\left(\left(5x^2 + 2x + 6\right)^3\right)$$

$$\frac{d}{dx}\left(\sqrt{3x^5 - 4x + 2}\right)$$

Use two different methods to find the following derivative

$$\frac{d}{dx}\left(\frac{5}{(7x^2-3x+1)^8}\right)$$

Consider exponential functions and how they can be part of a composition of functions. Can you find a general rule?
Find

$$\frac{d}{dx}\left(e^{3x}\right)$$

$$\frac{d}{dx}\left(e^{x^5}\right)$$

The General Exponential Rule

If $y = e^u$, where u is a differentiable function of x then

$$\frac{dy}{dx} = e^u \cdot \frac{du}{dx}$$

Or equivalently

$$\frac{d}{dx}\left(e^u\right) = e^u \cdot u'$$

(Copy the function and multiply by the derivative of the power.)

Find the following derivatives

$\frac{d}{dx}\left(\frac{5}{7x^3-4}\right)$
$\frac{d}{dx}\left(e^{3x^2+5x}\right)$
$\frac{d}{dx}\left(\sin(7x^3+4)\right)$
$\frac{d}{dx}\left(\tan\left(\frac{\pi}{x}\right)\right)$
$\frac{d}{dx}\left(e^{5x}\cos(x^2)\right)$

$$\frac{d}{dx}\left(\sec(5e^{x^3})\right)$$

$$\frac{d}{dx}\left(\frac{\left(x^2-5\right)^3}{6e^x}\right)$$

$$\frac{d}{dx}\left(\left(\frac{2x^4-3x}{1-x^2}\right)^3\right)$$

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$\frac{d}{dx}\left(\frac{2x^4-3x}{(1-x^2)^3}\right)$
$\frac{d}{dx}\left((4x^2-5x+3)(6x+8)^3\right)$