MA 131 Lecture Notes
Sections 1.1 and 1.2

Functions

Definition of a Function

A **function** *f* from a set A to a set B is a rule of correspondence that assigns to each element x in the set A exactly one element y in the set B. The set A is the **domain** (or set of inputs) of the function *f*, and the set B contains the **range** (or set of outputs.)

Things to note about functions:

- Each element of A (the domain) must be matched with an element of B (the range)
- Some elements of B may not be matched with any element of A
- Two or more elements of A may be matched with the same element of B
- An element of A cannot be matched with two different elements of B

We represent functions by equations or formulas involving two variables. For example, $y = 3x + 2$ represents the variable *y* as a function of the variable x. We say (in this example) x is the **independent variable** and *y* is the **dependent variable.**

Example: Which of these equations represents *y* as a function of x? We solve for *y* and determine if there is only one *y* for any given x.

$x^2 + 2y = 3$

$y^2 - x = 3$

We reference a function by giving it **function notation**. We use a letter to represent the function, often the letter *f*, and we say the input is x, so the function is $f(x) = y$. We read $f(x)$ as "*f* of *x*" or "the value of *f* at *x*."

We evaluate the function at a particular x by plugging that x value into the equation of the function.

For example: Given the function $f(x) = 3x + 5$ at $x = 4$ by replacing x with the number 4 every time it appears in the equation. So, $f(4) = 3(4) + 5 = 17$.

Examples:

Evaluate $f(3)$ and $f(-1)$ for $f(x) = \dfrac{x^2 - x + 3}{x + 5}$

Can you evaluate $f(-5)$?

Evaluate $f(x+h)$ for the above function.

A **piecewise defined function** is a function that is composed of two or more functions on a domain that is partitioned based on the functions. A real life example of this is for bulk purchasing. For example, suppose a company sells T-shirts and the larger the quantity you buy the cheaper each shirt costs. Suppose if you purchase less than ten shirts, the cost is twelve dollars each. But if you buy between ten and twenty five shirts the cost is only eight dollars, and more than twenty five shirts are only six dollars each. We can let x represent the number of shirts purchased and the following represents this function.

$$f(x) = \begin{cases} 12x \; if & 0 \le x < 10 \\ 8x \; if & 10 \le x \le 25 \\ 6x \; if & x > 25 \end{cases}$$

Using this function, determine the cost to buy twenty shirts.

Example: Evaluate $f(-3)$, $f(5)$, and $f(0)$ for the following function.

$$f(x)=\begin{cases} x+5 & if\ x<0 \\ 7 & if\ x=0 \\ 6x^2-2x & if\ x>0 \end{cases}$$

Domain of a Function

The domain of a function can be described explicitly or it can be implied by the expression used to define the function. The **implied domain** is the set of all real numbers for which the expression is defined. Basically this is the same as the domain as have already experienced. So far, we know that we cannot divide by zero and we cannot take a negative even root. The argument of a log must always be positive.

In the earlier example about t-shirts, it is explicitly shown that the function begins with x=0. This also makes sense from a practical view since we wouldn't sell a negative number of shirts.

Find the domain of the following functions.

$$f(x)=x^5-7x+3,\ x\geq 2$$

$$f(x)=\frac{4x-2}{x^2-25}$$

$$f(x)=\sqrt{x-5}$$

Notation:

$\Re`\{2,5\}$means all real numbers except (minus) the numbers two and five. This is the notation I will use in class to exclude a finite set of numbers from the domain. We do not use this notation to exclude intervals.

Graphs of Functions

The **graph of a function *f*** is the collection of ordered pairs (x,f(x)) such that *x* is in the domain of *f*. Remember *x* is the directed distance in the horizontal direction and *f(x)* is the directed distance in the vertical direction.

If the graph of a function has an x-intercept, *(a,0)*, then *a* is a **zero** of the function. The **range** of a function (the set of values assumed by the dependent variable) is often easier to determine graphically than algebraically.

Vertical Line Test

A set of points in a coordinate plane is the graph of *y* as a function of *x* if and only if no vertical line intersects the graph at more than one point.

Increasing, Decreasing, and Constant Functions

A function *f* is **increasing** on an interval if, for any x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) < f(x_2)$.

A function *f* is **decreasing** on an interval if, for any x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) > f(x_2)$.

A function *f* is **constant** on an interval if, for any x_1 and x_2 in the interval $x_1 < x_2$ implies $f(x_1) = f(x_2)$.

Definition of Relative Minimum and Relative Maximum

A function value *f(a)* is called a **relative minimum** of *f* if there exists and interval (x_1, x_2) that contains *a* such that $x_1 < x < x_2$ implies $f(a) \le f(x)$.

A function value *f(a)* is called a **relative maximum** of *f* if there exists and interval (x_1, x_2) that contains *a* such that $x_1 < x < x_2$ implies $f(a) \ge f(x)$.

Symmetries of a Graph

We say that a graph is **symmetric** if it is a mirror image once reflected across an axis (or both.) A graph can be symmetric with respect to (wrt) the x-axis, the y-axis, or the origin (symmetric wrt both axes.)

A function (because of the vertical line test) cannot be symmetric wrt y-axis. But if it is symmetric wrt x-axis we call that function an **even function**. If it is symmetric wrt the origin, we say the function is **odd.**

Tests for Even and Odd Functions

A function given by $y = f(x)$ is even if, for each *x* in the domain of *f*, $f(-x) = f(x)$.

A function given by $y = f(x)$ is odd if, for each *x* in the domain of *f*, $f(-x) = -f(x)$.

We determine whether a function is even or odd (or neither) by using the following process.

1. The function you are given is *f(x)*.
2. Evaluate *f(-x)* by plugging a negative *x* into the equation. If it is the same as the original, the function is even.
3. If it's not then negate the original (find *–f(x)*). If *f(-x)* is the same, then the function is odd.
4. Otherwise, the function is neither even nor odd.

Example: Determine if the following functions are even, odd, or neither.

$$f(x) = 3x^2 + 5$$

f(x)	
f(-x)	
-f(x)	

$$f(x) = \frac{x^4 - 3x^2 + 2}{x}$$

f(x)	
f(-x)	
-f(x)	

$$f(x) = (x+5)^2$$

f(x)	
f(-x)	
-f(x)	

A **mathematical model** is a mathematical description (often by means of a function or an equation) of a real world phenomenon. We often begin with a real life problem, formulate a mathematical model, solve it and come up with a mathematical conclusion that we interpret to make real world predictions. Often after this final step we begin the process all over again either to make a better model or to further study the problem.

An **empirical model** is one based entirely on collected data. It can help us formulate a model. We often seek to find a mathematical model that will fit the empirical data.

There are several different types of mathematical models. We will briefly discuss each of them here.

Linear Models

A linear model is represented with a **linear function**. We say *y* is a linear function of *x*, then the graph of the functions is a line. We use the slope-intercept form of the equation of the line to write the formula for the function as $y = f(x) = mx + b$ where *m* is the slope of the line and *b* is the *y*-intercept.

Recall that the slope of a line is $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_2}{x_1 - x_2}$ and that from this we arrive at the point slope equation of the line: $y - y_1 = m(x - x_1)$ where $(x_1, y_1)$is a point on the line.

Polynomials

Definition of a Polynomial in *x*

Let $a_n, \ldots, a_2, a_1, a_0$ be real numbers and let *n* be a *nonnegative integer*. A **polynomial in *x*** is an expression of the form $a_n x^n + \ldots + a_2 x^2 + a_1 x + a_0$ where $a_n \neq 0$. The polynomial is of **degree** *n*, and the number a_n is the **leading coefficient**. The number a_0 is the **constant term**. The constant term is considered to have a degree of zero.

Note: We define a **monomial** as an algebraic expression that is of the form $a_n x^n$ where a_n is a real number (coefficient) and x^n is the integer exponent part. A polynomial then is simply the sum of multiple monomials.

We define *n* to be the **degree** of the polynomial. We write a polynomial in **standard form** by expressing it such that the powers decrease when read left to right. A polynomial with all zero coefficients is called the **zero polynomial** (and has no degree.)

Exercise:
Determine if the following are polynomials. If they are, state the degree.

$6x^4 - 3x^2 + 1$	$\sqrt{2}x^3 - 5$	$\sqrt{2x^3} - 5$
7	$\frac{3x^5 - 6x^2 + 5}{x}$	$\frac{3x^5 - 6x^2 + 5}{4}$

Quadratic and Cubic Functions

Of particular interest are two specific polynomial functions, the quadratic and cubic functions. The **quadratic function** is $y = f(x) = ax^2 + bx + c$ is a polynomial of degree two and the graph is called a parabola. The **cubic function** is a polynomial of degree three.

Power Functions

A function of the form $f(x) = x^a$, where *a* is a constant, is called a **power function**. There are three possible scenarios.

If $a = n$ where *n* is a positive integer, then the function is a polynomial function as noted above.

If $a = \frac{1}{n}$ where *n* is a positive integer, then this is called a **root function**. We will see examples of this later.

If $a = -n$ where *n* is a positive integer, then this is the **reciprocal function**. We will see examples of this later.

Rational Functions

A **rational function** is a ratio of two polynomials: $f(x) = \frac{p(x)}{q(x)}$ where *p(x)* and *q(x)* are polynomials.

The domain is all real numbers except any *x* that make the denominator zero.

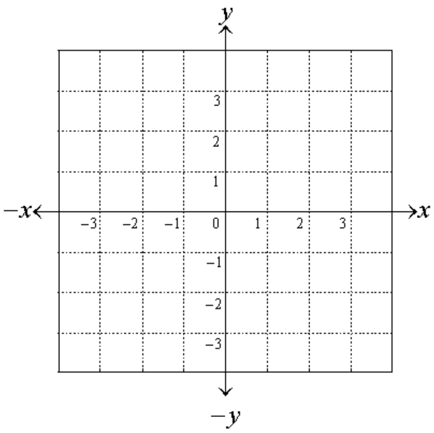
Algebraic Functions

A function *f* is called an **algebraic function** if it can be constructed using algebraic operations (such as addition, subtraction, multiplication, division, and taking roots.)

Trigonometric Functions and Exponential Functions

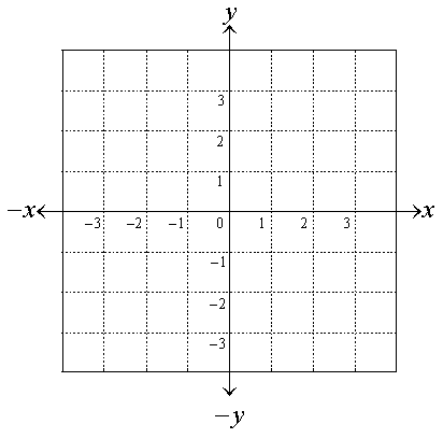
Trigonometric, exponential, and logarithmic functions are examples of **transcendental functions**. Transcendental functions are those that cannot be defined as algebraic functions. We will discuss these further.

Common Graphs



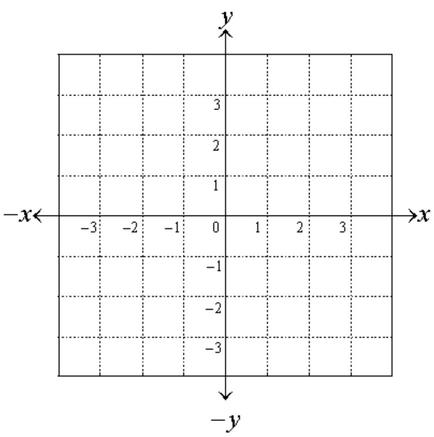
$f(x) = c$

Constant Function



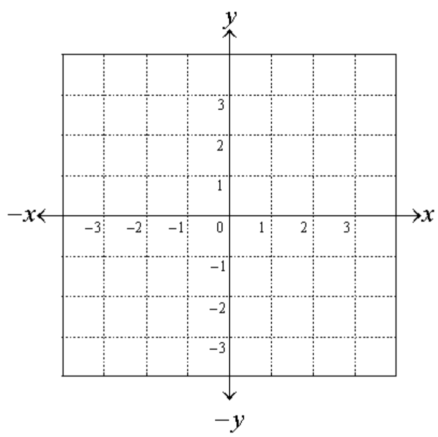
$f(x) = x$

Identity Function



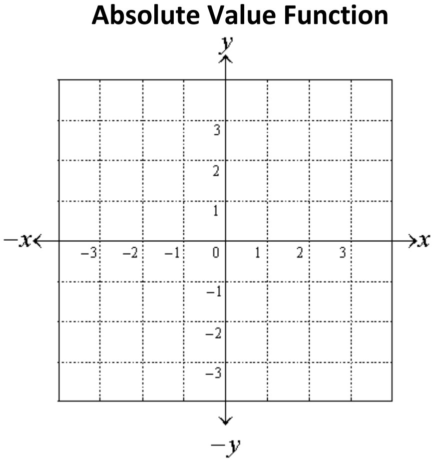
$f(x) = |x|$

Absolute Value Function



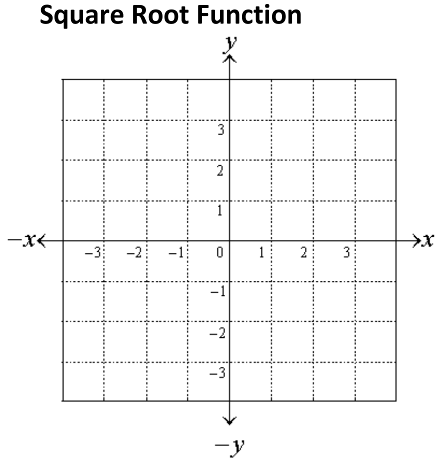
$f(x) = \sqrt{x}$

Square Root Function



$f(x) = x^2$

Squaring Function



$f(x) = x^3$

Cubing Function