Some Matrix Applications

This page is a brief introduction to two applications of matrices - the solution of multiple equations, and eigenvalue/eigenvector problems (don't worry if you haven't heard of the latter). Before reading this you should feel comfortable with basic matrix operations. If you are confident in your ability with this material, feel free to skip it.

Note that there is no description of how the operations are done -- it is assumed that you are using a calculator that can handle matrices, or a program like MatLab. Also, this page typically only deals with the most general cases, there are likely to be special cases that aren't covered at all.

Solution of Linear Systems of Algebraic Equations

Consider a linear system of algebraic equations with n equations and n unknowns:

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$$

In these equations, a_{ij} and b_i are constants, and the unknowns are x_i. We can rewrite the equations as:

$$\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$$

where

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \ddots & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

We can find all the unknowns in the vector **x** by doing a few simple matrix manipulations. If we premultiply both sides of the matrix equation by the inverse of the **A** matrix we get:

$$A \cdot x = b$$
$$A^{-1} \cdot A \cdot x = A^{-1} \cdot b$$
$$I \cdot x = A^{-1} \cdot b$$
$$x = A^{-1} \cdot b$$

which gives us our solution.

Note: This discussion has assumed that all the equations are independent, i.e. that you can't form one of the equations from combinations of other equations.

An Example

Consider the system of 3 equations and 3 unknowns:

$$2x_1 + 3x_2 - x_3 = -1$$
$$-x_1 + 2x_2 + 3x_3 = 9$$
$$x_2 + 2x_3 = 5$$

Clearly,

$$A = \begin{bmatrix} 2 & 3 & -1 \\ -1 & 2 & 3 \\ 0 & 1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} -1 \\ 9 \\ 5 \end{bmatrix}, \quad \text{so } x = A^{-1} \cdot b = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

Using MatLab

```
» A=[2 3 1;-1 2 3;0 1 2]
A =
     2    3    1
    -1    2    3
     0    1    2

» b=[-1 9 5]'
b =
    -1
     9
     5

» x=inv(A)*b
x =
```

```
-1.5714
-0.1429
 2.5714
```

Eigenvalues and Eigenvectors

Many problems present themselves in terms of an eigenvalue problem:

$$\mathbf{A} \cdot \mathbf{v} = \lambda \cdot \mathbf{v}$$

In this equation $\mathbf{A}$ is an n-by-n matrix, $\mathbf{v}$ is a non-zero n-by-1 vector and λ is a scalar (which may be either real or complex). Any value of λ for which this equation has a solution is known as an eigenvalue of the matrix $\mathbf{A}$. It is sometimes also called the characteristic value. The vector, $\mathbf{v}$, which corresponds to this value is called an eigenvector. The eigenvalue problem can be rewritten as

$$\mathbf{A} \cdot \mathbf{v} - \lambda \cdot \mathbf{v} = 0$$
$$\mathbf{A} \cdot \mathbf{v} - \lambda \cdot \mathbf{I} \cdot \mathbf{v} = 0$$
$$(\mathbf{A} - \lambda \cdot \mathbf{I}) \cdot \mathbf{v} = 0$$

If v is non-zero, this equation will only have a solution if

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = 0$$

This equation is called the characteristic equation of $\mathbf{A}$, and is an nth order polynomial in λ with n roots. These roots are called the eigenvalues of $\mathbf{A}$. We will only deal with the case of n distinct roots, though they may in fact be repeated. For each eigenvalue there will be an eigenvector for which the eigenvalue equation is true.

Example

If

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

then the characteristic equation is

$$|\mathbf{A} - \lambda \cdot \mathbf{I}| = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = 0$$

$$\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = \lambda^2 + 3\lambda + 2 = 0$$

and the two eigenvalues are

$$\lambda_1 = -1, \quad \lambda_2 = -2$$

All that's left is to find the two eigenvectors. Let's find the eigenvector, $\mathbf{v}_1$, associated with the eigenvector, λ_1, first.

$$\mathbf{A} \cdot \mathbf{v}_1 = \lambda_1 \cdot \mathbf{v}_1$$

$$(\mathbf{A} - \lambda_1 \cdot \mathbf{I}) \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} -\lambda_1 & 1 \\ -2 & -3-\lambda_1 \end{bmatrix} \cdot \mathbf{v}_1 = 0$$

$$\begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \cdot \begin{bmatrix} v_{1,1} \\ v_{1,2} \end{bmatrix} = 0$$

so clearly

$$v_{1,1} + v_{1,2} = 0, \quad \text{and}$$

$$-2 \cdot v_{1,1} + -2 \cdot v_{1,2} = 0, \quad \text{so}$$

$$v_{1,1} = -v_{1,2}$$

and the first eigenvector is any 2 element column vector in which the two elements have equal magnitude and opposite sign.

$$\mathbf{v}_1 = k_1 \cdot \begin{bmatrix} +1 \\ -1 \end{bmatrix}$$

where k_1 is an arbitrary constant. Note that we didn't have to use +1 and -1, we could have used any two quantities of equal magnitude and opposite sign.

Going through the same procedure for the second eigenvalue:

$$\mathbf{A} \cdot \mathbf{v}_2 = \lambda_2 \cdot \mathbf{v}_2$$

$$\begin{bmatrix} 2 & 1 \\ -2 & -1 \end{bmatrix} \cdot \begin{bmatrix} v_{2,1} \\ v_{2,2} \end{bmatrix} = 0$$

$$2 \cdot v_{2,1} + v_{2,2} = 0, \quad \text{and}$$

$$-2 \cdot v_{2,1} + -v_{2,2} = 0, \quad \text{so}$$

$$2 \cdot v_{2,1} = -v_{2,2}$$

and

$$\mathbf{v}_2 = k_2 \cdot \begin{bmatrix} +1 \\ -2 \end{bmatrix}$$

Using MatLab

```
» A=[0 1;-2 -3]
A =
    0    1
   -2   -3

» [v,d]=eig(A)
v =
    0.7071   -0.4472
   -0.7071    0.8944

d =
   -1    0
    0   -2
```

The eigenvalues are the diagonal of the "d" matrix; $\lambda_1 = -1$, $\lambda_2 = -2$. The eigenvectors are the columns of the "v" matrix. Note that MatLab chose different values for $v_{1,1}$, etc..., but that the ratio of $v_{1,1}$ to $v_{1,2}$ and the ratio of $v_{2,1}$ to $v_{2,2}$ are the same as our solution. (MatLab chooses the values such that the sum of the squares of the elements of the eigenvector equals unity).