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# Packing of complex spheres

Dov Aharonov

*Department of Mathematics*
*Technion - Israel Institute of Technology, Haifa 32000, Israel*
*e-mail: dova@techunix.technion.ac.il*
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**Abstract**

In what follows we deal with a "complex version" of packing of spheres. Among other things we present a "complex" generalization of the beautiful "Four Circle Theorem of Descarte" for arbitrary finite dimension. The "Boul Of Integer Property" (BOIP) of F. Soddy is extended in various directions. We also present a new approach to "Mauldon Inclination Theorem". This includes a consideration of the hyperbolic space as a projection of a "complex sphere" in an higher dimension. Our methods are mainly geometrical, but in some cases it is more natural to use the classical "Darboux-Frobenious" theorem, which is purely algebraic.

## 1 Introduction

This paper deals with packing of spheres. The starting point of this investigation goes back to the Ring Lemma of Rodin and Sullivan [1] and, in particular, to the finding of the sharp constants (for each $n$) that appear in this lemma [2–4].

The Four Circle Theorem of Descartes is a cornerstone in finding the precise value of these constants [3, 4].

Another important ingredient in what follows is the "Bowl of integer property" (BOIP from now on). This property goes back to F. Soddy [5]. Roughly, it states that if the first four bends (which mean reciprocal of radii) in the Apollonian packing are integers, then *all* bends in the Apollonian packing are integers. This property is the reason why all constants in the Ring Lemma are reciprocal of integers [3, 4].

Among other things, we discuss in this paper many ramifications of BOIP, in particular in connection to reflected Apollonian packing, dual Apollonian packing, etc. We note in passing that a very interersting connection between BOIP and number theory was established in a recent paper [6].

The research on Apollonian packing is quite broad. For a survey of various ramifications, the reader is referred to [4].

We recall that the beautiful Four Circle Theorem of Descartes states that the bends $\{b_j\}_{j=1}^4$ of four mutually disjoint tangent discs satisfy

$$\left(\Sigma_{j=1}^4 b_j\right)^2 = 2\Sigma_{j=1}^4 b_j^2. \tag{1.1}$$

This fundamental theorem was extended to $n$ dimension by Gossett [4] and later by Mauldon [7] to $n+2$ spheres in $R^n$, $n \geq 2$, having mutual inclination $\gamma$. The case $\gamma = -1$ is the Gosset theorem, i.e., the case of tangency. One of the ramifications arising out of what follows is to extend the Mauldon result to $n = 1$ as well. In other words, we deal with packings on the real line after we define what we mean by an angle between two segments. In fact, this opens a window to some invetigation of packing of the real line by segments.

One of the central topics in our paper is the complex approach. We define complex inclination between two generalized spheres in the space $C^n$ and show that it is invariant under a (generalized) Möbius map. We also take a different approach to the hyperbolic version of Mauldon's theorem [7].

In the sequel it will be useful to change the notation of the space $C^n$ to $G^n$. This is done in order to emphasize the fact that we replace the usual distance $C^n$ by a different concept of a "distance".

Let $\{a_j\}_{j=1}^n$, $\{b_j\}_{j=1}^n$ be complex numbers. We define

$$d^2 = \Sigma_{j=1}^n (a_j - b_j)^2 \tag{1.2}$$

as the square of the "distance" between $\bar{a} = (a_1, \ldots, a_n)$ and $\bar{b} = (b_1, \ldots, b_n)$. Of course, in general, $d^2$ is not real, and thus $d^2$ hardly may be considered as a square of a distance. Also, it may happen that $d^2 = 0$ but $\bar{a} \neq \bar{b}$. (For instance, take $\bar{a} = (0, 0)$ and $b = (1, i)$ in $G^2$.) In addition, if $d^2 \neq 0$, $d$ may take two different values. Hence we may say that $d(a, b)$ is not uniquely defined (but $d^2(a, b)$ is). Similarly, we define a "sphere" in $G^n$ as follows. Let $\bar{a} = (a_1, \ldots, a_n) \in G^n$, $R \in \mathbb{C}$ be a complex number. A sphere in $G^n$ is $S = S(a, R)$ where

$$\Sigma_{j=1}^n (z_j - a_j)^2 = R^2. \tag{1.3}$$

The center of the sphere is at $\bar{a}$ and its radius is $R$. As mentioned above, in case $R \neq 0$, we get two different values for the radius of the sphere. If $R = 0$ and $\bar{a} = 0$, we get

$$\Sigma_{j=1}^{n} z_j^2 = 0. \tag{1.4}$$

This is *not* a single point but an entire variety in $G^n$. This obvious observation is important when dealing with the concept of Möbius maps in $G^n$. Indeed, it follows that the inverse of "infinity" with respect to a sphere is the variety of the sort described in (1.4). Nevertheless, for every point in $G^n$, i.e., not infinity, the inverse is a unique single point. Later on we shall discuss a space $\hat{G}^n \supset G^n$ such that on $\hat{G}^n$ the Möbius map will be $1-1$ (see section 13.3).

It is worth noting that there are two approaches to the investigation of these matters. One is geometric, taken by most researchers. A very good representative of this approach is Coxeter [8]. Another approach is algebraic. This approach was taken a long time ago by Darboux-Frobenious [9]. In the present paper we mainly take the geometric approach but in some cases we shall use the algebraic approach as well.

# 2 The space $G^n$

## 2.1 Definition of $G^n$

Let $a_j \in \mathbb{C}$, $1 \leq j \leq n$, $\bar{a} = (a_1, \ldots, a_n)$. We then define the space $G^n$ as the space $C^n$ of all vectors $\bar{a}$, but with the "distance" $d$, where $\bar{d}^2 = (a-b)^2 = d^2(a,b) = \Sigma_{j=1}^{n}(a_j - b_j)^2$, $\bar{a} = (a_1, \ldots, a_n)$, $\bar{b} = (b_1, \ldots, b_n)$. (Note that $(a-b)^2 = (a-b, a-b)$ is the scalar product and $\bar{a}$ is replaced by $a$. When there is no danger of confusion, we occasionally shall use this notation. Obviously $d(a,b) = 0$ does not imply $a = b$. Also, there is no meaning to the triangle rule. Thus certainly $d$ may be hardly regarded as a distance in the usual sense. Moreover, in case $d(a,b) = 0$, $d(b,c) = 0$ it does not follow in general that $d(a,c) = 0$. (Take as an example the space $G^2$, $a = (0,0)$, $b = (i,1)$, $c = (0,2)$.) In other words, the property of having a distance zero is not transitive. (Of course, in $G^1$, $z^2 = 0$ implies that $z = 0$.) As already pointed out above, we shall introduce at a later stage an extension $\hat{G}^n = G^n \cup M$, where $M$ is a colleciton of points, all of them inverse to points satisfying $z^2 = 0$. This process of extension will be done in such a way as to assure the $1-1$ property of Möbius maps on $\hat{G}^n$ and to keep the property that inverse points on $\hat{G}^n$ will be mapped onto inverse points under a generalized Möbius map.

## 2.2 Spheres and planes in $G^n$

Let $\bar{a} = (a_1, \ldots, a_n) \in G^n, R \in \mathbb{C}$. A sphere in $G^n$ is

$$S = \{z, \Sigma(z_j - a_j)^2 = R^2\} = S(a, R). \tag{2.1}$$

Note that it may occur that $R = 0$ and we then get for $S = S(a, 0)$ : $S = \{z, \Sigma_{j=1}^n (z_j - a_j)^2 = 0\}$. Of course, $S$ is an entire variety and *not* the single point $\bar{a} = (a_1, \ldots, a_n)$. If $R = 0, \bar{a} = 0$, we get

$$\Sigma_{j=1}^n z_j^2 = 0. \tag{2.2}$$

These are the points on the sphere $S = S(\bar{0}, 0)$. As mentioned above, in what follows $\bar{a}$ is replaced by $a$ when there is no danger of confusion.

Similarly, we now define a plane in $G^n$.

*Let* $\{\alpha_j\}_{j=1}^n \in \mathbb{C}, \beta \in \mathbb{C}$, $\bar{\alpha} = (\alpha_1, \ldots, \alpha_n)$; then $P = P(\bar{\alpha}, \beta)$ is a plane in $G^n$ where

$$P = \left\{z, \Sigma_{j=1}^n z_j \alpha_j = (z, \alpha) = \beta\right\}. \tag{2.3}$$

We differentiate between two cases: $\alpha^2 = 0$ and $\alpha^2 \neq 0$.

If $\alpha^2 = 0$ but $\bar{\alpha} \neq 0$, the situation in some cases may cause confusion. Therefore, we usually only deal with the case $\alpha^2 \neq 0$. In the sequel, when we speak about a plane in $G^n$, we mean only the case $\alpha^2 \neq 0$, unless we specifically say that the case $\alpha^2 = 0$ is under consideration.

Given a plane, it may occur that $\beta = 0$. This means that the plane "passes" through the origin $\bar{0} = (0, \ldots, 0)$ in $G^n$. If this is not so, we may write the equation of the given plane in a more convenient form, namely, in a "normal form". Indeed, if $P = \{(z, \alpha) = \beta, \alpha^2 \neq 0, \beta \neq 0\}$, then $(z, \alpha\frac{\beta}{\alpha^2}) = \beta\frac{\beta}{\alpha^2} = \frac{\beta^2}{\alpha^2}$, and putting $\gamma = \frac{\beta}{\alpha^2}\alpha$, we get $(z, \gamma) = \gamma^2$. Note that both conditions $\alpha^2 \neq 0$, $\beta \neq 0$, have been used. Indeed, we multiply by the factor $\frac{\beta}{\alpha^2}$. Since it is not zero, it may be applied only for $\beta \neq 0$. Since we divide by $\alpha^2$, it is applicable only for $\alpha^2 \neq 0$.

The normal form is more convenient in proving some of our statements, but we have to keep in mind that the case of a plane passing through the origin is not covered and has to be handled separately. Geometrically $(z, \gamma) = \gamma^2$ or $(z - \gamma, \gamma) = 0$ means that the vector $\gamma$ is perpendicular to $z - \gamma$. As in the real case, we may say that $\gamma$ measures the distance of the origin from the given plane.

## 2.3 Orthogonal transformations in $G^n$

Let $T = (t_{ij})_{i,j=1}^n$ be an $n \times n$ orthogonal complex matrix. We define an *orthogonal transformation* in $G^n$ by $w = Tz$ for $\bar{z} = (z_1, \ldots, z_n), \bar{w} =$

$(w_1, \ldots, w_n)$.

**Lemma 2.1.** *Let $T$ be a given orthogonal transformation in $G^n$. Then*

$$(a,b) = (Ta, Tb) \tag{2.4}$$

*for any $a, b \in G^n$.*

*Proof.* First note that for $a = b$ we get from $a^2 = (a,b) = (Ta, Tb) = (Ta)^2$. We may say that we have an *isometry*. To prove (2.4) we write $T = (t_{ij})_{i,j=1}^n$ and we get

$$\begin{aligned} (Ta, Tb) = \Sigma_{i=1}^n (Ta)_i (Tb)_i &= \Sigma_{i=1}^n \big(\Sigma_{j=1}^n t_{ij} a_j\big)\big(\Sigma_{k=1}^n t_{ik} a_k\big) \\ &= \Sigma_{j,k=1}^n \big(\Sigma_{i=1}^n t_{ij} t_{ik}\big)(a_j, b_k) \\ &= \Sigma_{j,k=1}^n \delta_{jk} (a_j, b_k) = (a,b). \end{aligned}$$

**Lemma 2.2.** *An orthogonal transformation $T$ preserves spheres in $G^n$ and the radius is not changed.*

*Proof.* Let $S = \{z, (z-a)^2 = R^2\}$ be a sphere in $G^n$. Let $T = (t_{ij})_{i,j=1}^n$, $M = T^{-1} = (m_{ij})_{i,j=1}^n$. Since $w = Tz$ we have $z = Mw$.

Let $b = Ta$ or $a = Mb$. Then, by Lemma 2.1, $R^2 = (z-a)^2 = (z-a, z-a) = z^2 + a^2 - 2(z,a) = (Mw, Mw) + (Mb, Mb) - 2(Mb, Mw) = w^2 + b^2 - 2(b,w) = (w-b)^2$. This ends the proof of Lemma 2.2. Note that the center $b$ is $Ta$, i.e.,. the image of the center $a$ by $T$.

**Lemma 2.3.** *An orthogonal transformation $T$ preserves planes in $G^n$.*

*Proof.* Let $(\alpha, z) = \beta$ be the equation of $P$ in $G^n$. Let $T = (t_{ij})_{i,j=1}^n$, $M = T^{-1} = (m_{ij})_{ij=1}^n$. Then for $\alpha = M\gamma, w = Tz, z = Mw$, we get, by Lemma 2.1, $\beta = (\alpha, z) = (M\gamma, Mw) = (\gamma, w)$. Hence the image of $P$ is $(\gamma, w) = \beta$.

In adddition to orthogonal transformations it will be useful to consider translations, i.e., $w = z + b$. Obviously translations preserve spheres and planes as well.

Indeed, $(\alpha, z) = \beta$ and $z = w - b$ implies $(\alpha, w - b) = \beta$ or $(\alpha, w) = \beta + (\alpha, b)$. Similarly, $(z-\alpha)^2 = R^2$ is transformed to $(w - b - \alpha)^2 = R^2$.

## 2.4 Inverse points in $G^n$

Let $S = S(a, R) = \left\{z, \Sigma_{j=1}^n (z_j - a_j)^2 = R^2\right\}$ be a given sphere in $G^n$. Assume further that $R \neq 0$. Let $c = (c_1, \ldots, c_n) \in G^n$. Assume that

$(c-a)^2 \neq 0$. then we define $d$ to *be the inverse point of* $c$ *with respect to* $S$ if and only if

$$d - a = \lambda(c-a), \lambda = \frac{R^2}{(c-a)^2} \tag{2.5}$$

we have $(d-a)^2(c-a)^2 = \lambda^2(c-a)^4 = \frac{R^4}{(c-a)^4}(c-a)^4 = R^4$.

Note that if $d$ is the inverse point of $c$, then $c$ is the inverse point of $d$. Indeed,

$$c - a = \frac{1}{\lambda}(d-a) = \frac{(c-a)^2}{R^2}(d-a) = \frac{(d-a)}{R^2}\frac{R^4}{(d-a)^2} = \frac{R^2}{(d-a)^2}(d-a).$$

We say that $c$ and $d$ *are inverse points of each other with respect to* $S$.

In the particular case of $S = \left\{z, \Sigma_{j=1}^n z_j^2 = 1\right\}$, (i.e., $S$ is the unit sphere in $G^n$) we get from (2.5),

$$d = \frac{c}{c^2}. \tag{2.5'}$$

Next, we deal with a plane in $G^n$. Let $P = \{z, (\alpha, z) = \beta\}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$,
$\beta \in \mathbb{C}$. We assume further that $P$ is a "proper" plane, i.e., $\alpha^2 \neq 0$. Motivated by the real case, it is natural to consider $b = (b_1, \ldots, b_n)$, as the inverse point of $\bar{a} = (a_1, \ldots, a_n)$ with respect to $P$, provided $\frac{a+b}{2} \in P$, and also $b - a = \lambda\alpha$ for some parameter $\lambda$. Hence $(\alpha, \frac{a+b}{2}) = \beta$ or $(a, \alpha) + (b, \alpha) = 2\beta$. Also $b - a = \lambda\alpha$ implies $(b, \alpha) - (a, \alpha) = \lambda\alpha^2$. Thus $2(a, \alpha) = 2\beta - \lambda\alpha^2$ or $\lambda = \frac{2\beta - 2(a,\alpha)}{\alpha^2}$ and we are led to the definition:
*Given a plane* $P = \{z, (\alpha, z) = \beta\}$ *in* $G^n$ *and* $\bar{a} \in G^n$, *then the inverse point of* $\bar{a}$ *with respect to* $P$ *is defined as*

$$\bar{b} = \bar{a} + \left[\frac{2\beta - 2(\bar{a}, \alpha)}{\alpha^2}\right]\bar{\alpha}. \tag{2.6}$$

*If* $P$ *is given in its normal form* $P = \{z, (z, \gamma) = \gamma^2\}$, *then* $\gamma = \alpha, \gamma^2 = \beta$ *and*

$$\bar{b} = \bar{a} + 2\gamma\big(1 - \frac{(a, \gamma)}{\gamma^2}\big) \tag{2.6'}$$

## 2.5 Preservation of inverse points under translation and magnification

**Lemma 2.4.** *Translations and magnification preserve inverse points.*

*Proof.* Let $\bar{w} = \bar{z} + \bar{\beta}$, $w = (w_1, \ldots, w_n)$, $z = (z_1, \ldots, z_n)$, $\beta = (\beta_1, \ldots, \beta_n)$, $S = \{z, (z-a)^2 = R^2\}$ a given sphere in $G^n$. Let $c, d$ be inverse with respect to $S$. If $C, D$ are the images of $c, d$ respectively, then $C = \bar{c} + \bar{\beta}$, $\bar{D} = \bar{d} + \bar{\beta}$, $\bar{D} - \bar{C} = \bar{d} - \bar{c}$. Also $S$ is mapped onto onto $S_1 = \{w, (w-A)^2 = R^2\}$ where $A = a + \beta$ (as $(z-a)^2 = (w - \beta - a)^2 = R^2$). $D - A = (d + \beta) - (a + \beta) = d - a = \lambda(c - a) = \lambda(C - A)$ where $\lambda = \frac{R^2}{(c-a)^2} = \frac{R^2}{(C-A)^2}$. Thus $D$ is the inverse point of $C$ with respect to $S_1$.

If $w = \mu z$, $\mu \neq 0$ is a given magnification, then the proof is very similar and we omit the simple details.

If we consider a plane $P = \{z, (z, \alpha) = \beta\}$ and a translation or magnification, the discussion is also very simple and details are again omitted.

Next we prove

**Lemma 2.5.** *Orthogonal transformations preserve inverse points.*

*Proof.* Let $w = Tz$, $z = Mw$ for $T = (t_{ij})_{i,j=1}^n$, $T$ an orthogonal matrix. We first consider a sphere $S = S(a, R)$, $S = \{z, (z-a)^2 = R^2\}$. Let $c, d$ be inverse points with respect to $S$. Then $d - a = \frac{R^2}{(c-a)^2}(c - a)$. Applying $T$ to this relation, we get $Td - Ta = \frac{R^2}{(c-a)^2}(Tc - Ta)$ and this means $D - A = \frac{R^2}{(C-A)^2}(C - A)$ where $A, C, D$ are the images of $a, c, d$ respectively. To end the proof we use the isometry $(c-a)^2 = (Tc - Ta)^2 = (C - A)^2$.

The proof for a plane $P = \{z, (z, \alpha) = \beta\}$ is very similar and details are omitted.

## 2.6 Transformation of spheres and planes under inversion

Let $S = \{z, (z-a)^2 = R^2\}$ be a given sphere. Our aim is to find its inversion (we note that sometimes it is called "reflection") with respect to another sphere

$S_1$, $S_1 = \{z, (z-c)^2 = \rho^2\}$ we have $(z-a)^2 = ((z-c) + (c-a))^2 = (z-c)^2 + (c-a)^2 + 2(z-c)(c-a) = R^2$. Hence for $\tau^2 = R^2 - (c-a)^2$

$$(z-c)^2 + 2(z-c)(c-a) = R^2 - (c-a)^2 = \tau^2 \tag{2.7}$$

we separate between two cases, $\tau \neq 0$, $\tau = 0$.

First consider the case $\tau \neq 0$.

Let $w$ be the inverse point of $z$ with respect to $S_1$. Then $z - c = \frac{\rho^2}{(w-c)^2}(w - c)$, putting this in (2.7) $\frac{\rho^4}{(w-c)^2} + \frac{2(w-c)}{(w-c)^2}\rho^2(c-a) = \tau^2$. Thus $(w - c)^2 = \frac{\rho^4 + 2(c-a)(w-c)\rho^2}{\tau^2}$ or $\left(w - c - \frac{(c-a)\rho^2}{\tau^2}\right)^2 = \frac{\rho^4}{\tau^2} + \frac{(c-a)^2\rho^4}{\tau^4} = \rho^4\frac{[\tau^2 + (c-a)^2]}{\tau^4} =$

$\frac{\rho^4}{\tau^4}R^2 = \left(\frac{\rho^2 R}{\tau^2}\right)^2$. Hence the image of $S$ with this transformation (i.e., inversion with respect to $S_1$)

$$(w-A)^2 = \mu^2, \quad A = c + \frac{(c-a)\rho^2}{\tau^2}, \quad \mu = \pm\frac{\rho^2 R}{\tau^2} = \pm\frac{\rho^2 R}{R^2-(c-a)^2}. \quad (2.8)$$

We next consider the other case $\tau^2 = R^2 - (c-a)^2 = 0$. From (2.7) we see that $z = c$ is a point of $S$ in this case. In other words, we may say that $S$ "passes" through the center of $S_1$ if $\tau = 0$. Thus it is expected that the image of $S$ will be a plane in this case. Indeed, from (2.7) $(z-c)^2+2(z-c)(c-a) = 0$ and putting $z - c = \frac{\rho^2}{(w-c)^2}(w-c)$, we get $\frac{\rho^4}{(w-c)^2} + \frac{2\rho^2}{(w-c)^2}(w-c)(c-a) = \tau^2 = 0$. Thus, $(w-c)(c-a) = \frac{-\rho^2}{2}$ or

$$(w, c-a) = c(c-a) - \frac{\rho^2}{2}. \quad (2.9)$$

For the particular case $c = 0$, $\rho = 1$, i.e., $S_1$ is the unit sphere, we get

$$(w-A)^2 = \mu^2, \quad A = \frac{a}{a^2-R^2}, \quad \mu = \pm\frac{R}{a^2-R^2} \quad (2.8')$$

$$(w, a) = \frac{1}{2}. \quad (2.9')$$

(Note that since $a^2 \neq 0$, we can rewrite $(2.9')$ in the normal form, $\left(w, \frac{a}{2a^2}\right) = \frac{1}{(2a)^2}$.)

## 2.7 Invariance property of inverse points

As in the real case, we expect that the property of two points being inverse to each other with respect to a given sphere or a plane is invariant under an inversion with respect to a given sphere or a plane.

(In the future we will use the word "reflection" instead of "inversion".)

As in the real case, it is convenient to talk about a *generalized sphere.* By this we mean that we have either a sphere or a plane. Generalized spheres will be denoted by $\Sigma, \Sigma'$, etc. Spheres will be denoted, as before, by $S$, and planes by $P$.

**Theorem 2.1.** *Let $\Sigma$ be a generalized sphere in $G^n$. Let $\Sigma_0$ be another generalized sphere in $G^n$. Denote by $\Sigma'$ the inverse of $\Sigma$ with respect to $\Sigma_0$.*

*If $c, d$ denote the inverse points with respect to $\Sigma$, and $C, D$ denote the images under this transformation, then $C, D$ are inverse points with respect to the image $\Sigma'$ of $\Sigma$.*

*Proof.* The complete proof of the above theorem will be given at a later stage, after we define properly the space $\hat{G}^n$. For the time being we confine ourselves to "non zero" points. More specifically, we do not divide by zero. This means that in the following, when we consider $\frac{c}{c^2}$, we assume that $c^2 \neq 0$. Similarly, when we consider $\frac{c-a}{(c-a)^2}$, we assume that $(c-a)^2 \neq 0$, and so on. After defining the space $\hat{G}^n$, these missing cases will be proved as well.

It is possible to prove the assertion by an analytic continuation from the real case, but we prefer to give a direct independent proof.

There are various cases to consider.

$\Sigma$ may be a sphere or a plane. The same is true for $\Sigma_0$. In fact, there are six cases to check;

(a) $S-S-S$

(b) $S-S-P$

(c) $P-S-S$

(d) $P-S-P$

(e) $S-P-S$

(f) $P-P-P$

(The above notation means the following: say for the case (d) we reflect a plane P with respect to a sphere S to get another plane P.)

Note that if $\Sigma_0$ is $P$, we get only the two cases (e) and (f). This is due to the fact that inversion with respect to a plane preserves planes and spheres and cannot, like in the case when $\Sigma_0$ is $S$, transform a plane into a sphere.

Also note that (b) and (c) are in fact the same case because of the duality property of inverse points.

In addition, the case (d) is trivial.

Indeed, it is enough to consider the unit sphere $S = \{z, \Sigma_{j=1}^n z_j^2 = 1\}$. If $P = \{z, (z, \gamma) = \beta\}$, then if we are in the case (d), necessarily we must have $\beta = 0$. Thus $z = \frac{w}{w^2}$ leads to $(w, \gamma) = 0$, and the image of $P$ is $P$ itself. Thus the tranformation simply reverses $c$ and $d$.

Hence, it remains to check (a), (b), (e) and (f). We will check only (a) and (f), since the cases (b) and (e) are very similar. For the proof of (a) consider $S = \{z, (z-a)^2 = R^2\}$, a given sphere in $G^n$. We may assume that we reflect $S$ in $S_1 = \{z, z^2 = 1\}$, the unit sphere in $G^n$. There is no loss of generality in doing so, as we have shown that translation and magnification preserve inverse points. Since we are interested in case (a), we assume $a^2 \neq R^2$. By (2.8$'$), we know that the image of $S$ after inversion in $S_1$ is the sphere $\hat{S} = \{w, (w-A)^2 = \rho^2\}$ for $A = \frac{a}{a^2-R^2}$ and $\rho = \frac{R}{a^2-R^2}$. If $c, d$ are the inverse points with respect to $S$, we have to show that $C = \frac{c}{c^2}$, $D = \frac{d}{d^2}$ are inverse points with respect to $\hat{S}$, or

$$\begin{aligned} D - A &= \frac{\rho^2}{(C-A)^2}(C-A), \quad D = \frac{d}{d^2}, C = \frac{c}{c^2}, \\ A &= \frac{a}{a^2-R^2}, \quad \rho = \frac{R}{a^2-R^2}. \end{aligned} \tag{2.10}$$

We put the notation

$$\lambda = \frac{R^2}{(c-a)^2}, \quad \mu = \frac{\rho^2}{(C-A)^2}. \tag{2.11}$$

Thus we have to check

$$C - A = \frac{1}{\mu}(D-A) = \frac{(C-A)^2}{\rho^2}(D-A). \tag{2.10'}$$

From (2.11)

$$\frac{1}{\lambda} = \frac{c^2 + a^2 - 2(a,c)}{R^2}, \quad 2(a,c) = c^2 + a^2 - \frac{R^2}{\lambda}. \tag{2.12}$$

Similarly, $\frac{1}{\mu} = \frac{C^2+A^2-2(A,C)}{\rho^2}$. Using $C^2 = \left(\frac{c}{c^2}\right)^2 = \frac{1}{c^2}$, $A^2 = \left(\frac{a}{a^2-R^2}\right)^2$, $2(A,C) = 2\left(\frac{a}{a^2-R^2}, \frac{c}{c^2}\right) = \frac{2(a,c)}{c^2(a^2-R^2)}$ we get

$$\begin{aligned} \frac{1}{\mu} &= \frac{1}{\rho^2}\left[\frac{1}{c^2} + \frac{a^2}{(a^2-R^2)^2} - \frac{2(a,c)}{c^2(a^2-R^2)}\right] \\ &= \frac{1}{\rho^2 c^2}\left[\frac{(a^2-R^2)^2 + a^2c^2 - 2(a,c)(a^2-R^2)}{(a^2-R^2)^2}\right] \\ &= \frac{1}{\rho^2 c^2 (a^2-R^2)^2}\left[a^4 + R^4 - 2a^2R^2 + a^2c^2 - (c^2 + a^2 - \frac{R^2}{\lambda})(a^2-R^2)\right] \end{aligned}$$

where we have used (2.12).

Using $\rho^2(a^2-R^2)^2 = R^2$ we get after simple manipulations

$$\frac{1}{\mu} = 1 + \tau, \quad \tau = \frac{(a^2-R^2)^2}{c^2}\left(\frac{1}{\lambda} - 1\right). \tag{2.13}$$

From $(2.10')$ and (2.13) it remains to show

$$C - A = (1+\tau)(D-A) \quad \text{or} \quad C + \tau A = (1+\tau)D,$$

where $\tau A = \frac{(a^2-R^2)}{c^2}\big(\frac{1}{\lambda}-1\big)\frac{a}{a^2-R^2} = \frac{1}{c^2}\big(\frac{1}{\lambda}-1\big)a$. Hence $C+\tau A = \frac{c}{c^2}+\frac{1}{c^2}\big(\frac{1}{\lambda}-1\big)a$
$= \frac{1}{c^2}\big(c+(\frac{1}{\lambda}-1)a\big) = \frac{1}{c^2}\big(\frac{\lambda c+(1-\lambda)a}{\lambda}\big) = \frac{1}{\lambda c^2}d$. (The last step follows from $d-a = \lambda(c-a)$ as $c, d$ are inverse points with respect to $S$.) Thus, putting $C+\tau A = \frac{\bar{d}}{\lambda c^2}$ in the above, it remains to check (using $D = \frac{\bar{d}}{d^2}$ and cancelling $\bar{d}$) that $\frac{1}{\lambda c^2} = \frac{1+\tau}{d^2}$, or

$$d^2 = (1+\tau)\lambda c^2. \tag{2.14}$$

But we have seen that $d = \lambda c + (1-\lambda)a$. Hence, from (2.12)

$$d^2 = \lambda^2 c^2 + (1-\lambda)^2 a^2 + 2\lambda(1-\lambda)(a,c) = \lambda^2c^2+(1-\lambda)^2a^2+\lambda(1-\lambda)(c^2+a^2-\frac{R^2}{\lambda}).$$

Also from (2.13) $(1+\tau)\lambda c^2 = \lambda c^2 + \lambda\tau c^2 = \lambda c^2 + (a^2-R^2)(1-\lambda)$. Hence (2.14) is reduced to checking whether $\lambda^2c^2+(1-\lambda)^2a^2+\lambda(1-\lambda)\big(c^2+a^2-\frac{R^2}{\lambda}\big)$ $= \lambda c^2 + (a^2-R^2)(1-\lambda)$, which is an identity as can easily be varified. Thus (2.14) is confirmed and this ends the proof of case (a).

In order to check (f), we consider the planes $P = \{z, (z,\gamma) = \beta\}$ and $P_0 = \{z, (z,\eta) = \delta\}$. We reflect $P$ in $P_0$. Also, let $c, d$ be inverse with respect to $P$, i.e.,

$$\bar{d} = \bar{c} + 2\bar{\gamma}\frac{\beta - (\bar{\gamma},\bar{c})}{\gamma^2} = c + \lambda\bar{\gamma} \quad \text{for} \quad \lambda = 2\frac{\beta-(\gamma,c)}{\gamma^2}. \tag{2.15}$$

Reflecting $P$ in $P_0$, we get $z = w + \left[\frac{2\delta-2(w,\eta)}{\eta^2}\right]\bar{\eta}$. Hence $\beta = (z,\gamma) = (w,\gamma) + \frac{2\delta-2(w,\eta)}{\eta^2}(\eta,\gamma)$. Thus the equation after reflecting

$$(\bar{w},\bar{\gamma}_1) = \beta_1, \quad \text{where} \quad \bar{\gamma}_1 = \bar{\gamma} - 2(\gamma,\eta)\frac{\bar{\eta}}{\eta^2}, \quad \beta_1 = \beta - \frac{2\delta(\eta,\gamma)}{\eta^2}. \tag{2.16}$$

Denote by $C, D$ the image of $c, d$ respectively. Then

$$\bar{C} = \bar{c} + \left[\frac{2\delta - 2(c,\eta)}{\eta^2}\right]\bar{\eta}, \quad \bar{D} = \bar{d} + \left[\frac{2\delta-2(d,\eta)}{\eta^2}\right]\bar{\eta}. \tag{2.17}$$

We have from (2.17)

$$\bar{D} - \bar{C} = \bar{d} - \bar{c} + \frac{2(c,\eta)}{\eta^2}\bar{\eta} - \frac{2(d,\eta)}{\eta^2}\bar{\eta}.$$

>From (2.15)

$$\bar{D} - \bar{C} = \lambda\bar{\gamma} - \frac{2\bar{\eta}}{\eta^2}(\bar{d} - \bar{c}, \bar{\eta}) = \lambda\bar{\gamma} - \frac{2\bar{\eta}}{\eta^2}(\lambda\bar{\gamma}, \bar{\eta}) = \lambda\big(\bar{\gamma} - \frac{2\bar{\eta}}{\eta^2}(\bar{\gamma}, \bar{\eta})\big).$$

>From (2.16) we conclude

$$\bar{D} - \bar{C} = \lambda\bar{\gamma}_1. \tag{2.18}$$

Our aim is to show that $\bar{C}, \bar{D}$ are inverse points with respect to the image $(\bar{w}, \bar{\gamma}_1) = \beta_1$. We thus have to show that

$$\bar{D} - \bar{C} = \frac{2[\beta_1 - (\bar{\gamma}_1, \bar{C})]}{2\gamma_1^2}\bar{\gamma}_1. \tag{2.19}$$

Comparing (2.18) and (2.19), we have to confirm

$$\lambda = \frac{2[\beta_1 - (\bar{C}, \bar{\gamma})]}{\gamma_1^2}. \tag{2.20}$$

But $\gamma_1^2 = \gamma^2$, as follows at once from (2.16). Also $\lambda = 2[\beta - (\gamma, c)]\frac{1}{\gamma^2}$ by (2.15). Hence (2.20) is reduced to $\beta - \beta_1 = (c, \gamma) - (C, \gamma_1)$. By (2.16) and (2.17) this means

$$\begin{aligned}
\frac{2\delta(\eta,\gamma)}{\eta^2} &= (\bar{c}, \bar{\gamma}) - \left(\bar{c} + \frac{2\delta - 2(c,\eta)}{\eta^2}\bar{\eta}, \bar{\gamma} - \frac{2(\gamma,\eta)\bar{\eta}}{\eta^2}\right) \\
&= (c,\gamma) - \Bigg[(c,\gamma) - \frac{2(\gamma,\eta)(c,\eta)}{\eta^2} + \frac{2\delta(\eta,\gamma)}{\eta^2} \\
&\qquad - \frac{4\delta(\gamma,\eta)\eta^2}{\eta^4} - \frac{2(c,\eta)(\eta,\gamma)}{\eta^2} + \frac{4(c,\eta)(\eta,\gamma)\eta^2}{\eta^4}\Bigg],
\end{aligned}$$

which is easily confirmed.

## 2.8 Points with mutually equal distance in $G^n$

In this section we develop some material needed for a later stage. While the information we deduce is fairly obvious in $R^n$, it seems that in $G^n$ it demands some care.

**Lemma 2.6.** *Let $\{\bar{a}_j\}_{j=1}^n$ be $n$ points in $G^n$, such that $a_j \neq 0, 1 \leq j \leq n$. Also let*

$$(a_j - a_k)^2 = \lambda, \quad j \neq k, \quad 1 \leq j, \quad k \leq n, \quad \lambda \neq 0. \tag{2.21}$$

*Then*

$$\{\bar{b}_j\}_{j=1}^n, \quad \bar{b}_j = \bar{a}_j - \bar{a}_1, \quad 2 \leq j \leq n \tag{2.22}$$

*are $n-1$ linearly independent vectors.*

*Proof.* We have for $j \neq k$, $2 \leq j, k \leq n$, $(b_j - b_k)^2 = (a_j - a_k)^2 = \lambda = b_j^2 = b_k^2$. Hence, $\lambda = b_j^2 + b_k^2 - 2(b_j, b_k) = b_j^2$. Thus

$$b_k^2 = 2(b_j, b_k) \quad j \neq k, \quad 2 \leq j, k \leq n. \tag{2.23}$$

Denote

$$(b_j, b_k) = X, \quad b_k^2 = 2X = \lambda, \quad j \neq k, \quad 2 \leq j, k \leq n. \tag{2.24}$$

Assume now that, in contrast to the assertion of the lemma, we have $\Sigma_{j=2}^{n} \lambda_j \bar{b}_j = 0$ (where not all $\lambda_j$ are zero). Then multiplying by $b_k$ we get for $2 \leq k \leq n$ the system

$$\begin{array}{llll}
\lambda_2(b_2, b_2) & + \lambda_3(b_3, b_2) + \cdots + & \lambda_n(b_n, b_2) = 0 \\
\lambda_2(b_2, b_3) & + \lambda_3(b_3, b_3) + \cdots + & \lambda_n(b_n, b_3) = 0 \\
\quad \vdots & & \quad \vdots \\
\lambda_2(b_2, b_n) & + \lambda_3(b_3, b_n) + \cdots + & \lambda_n(b_n, b_n) = 0.
\end{array}$$

Hence from (2.24)

$$\begin{array}{lll}
\lambda_2(2X) + \lambda_3 X + & \cdots & + \lambda_n X = 0 \\
\lambda_2 X + \lambda_3(2X) + & \cdots & + \lambda_n X = 0 \\
\quad \vdots & & \quad \vdots \\
\lambda_2 X \qquad\qquad + & \cdots & + \lambda_n(2X) = 0.
\end{array}$$

Thus the determinant is

$$\begin{vmatrix}
2 & 1 & 1 & \cdots & 1 \\
1 & 2 & 1 & \cdots & 1 \\
1 & 1 & 2 & \cdots & 1 \\
\vdots & & \vdots & & \vdots \\
1 & 1 & & \cdots & 2
\end{vmatrix}$$

But this determinant is easily shown to be different from zero for each $n$. Hence we derive a contradiction to the assumption of dependence.

**Corollary 2.1.** *The set $\{\bar{b}_n, \ldots, \bar{b}_n\}$ defined above may be orthonormalized.*

*Indeed, let*

$$\begin{aligned}
c_2 &= b_2 \\
c_3 &= b_2 \quad -2b_3 \\
&\vdots \qquad \vdots \\
c_k &= b_2 \quad + \cdots + b_{k-1} - (k-1)b_k \\
&\vdots \qquad \vdots \qquad \vdots \\
c_n &= b_2 \quad + \cdots + b_{n-1} - (n-1)b_n.
\end{aligned}$$

We first show that this system is orthogonal. Indeed, $(c_3, c_2) = (b_2, b_2) - 2(b_3, b_2) = 2X - 2X = 0$, as follows from (2.21). More generally, for $j < k$,

$$\begin{aligned}
(c_k, b_j) &= \left(\Sigma_2^{k-1} b_\ell - (k-1)b_k, b_j\right) \\
&= \Sigma_{\ell=2}^{k-1}(b_\ell, b_j) - (k-1)(b_k, b_j) \\
&= (k-3)X + 2X - (k-1)X \\
&= 0.
\end{aligned}$$

We leave it to the reader to show that $(c_j, c_j) \neq 0$. Hence, multiplying by suitable constants, we can produce an orthonormal set.

We now have

**Lemma 2.7.** *Given the set $A = \{\bar{0}, \bar{b}_2, \dots, \bar{b}_n\}$ in $G^n$ for the $\{b_j\}_2^n$ as above, there exists an orthogonal matrix $T = (t_{ij})_{i,j=1}^n$ such that applying $T$ to the set $A$ we get a new set $B = \{\bar{0}, \bar{d}_1, \dots, \bar{d}_{n-1}\}$ where the $n^{th}$ coordinate of each $\{d_j\}_{j=1}^{n-1}$ is zero.*

*Proof.* Our aim is to find an orthogonal $T$ such that

$$\begin{pmatrix} t_{11} & t_{12}\cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} 0 & b_{12} & b_{13} & \cdots & b_{1n-1} \\ 0 & b_{22} & b_{23} & \cdots & \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & b_{n2} & b_{n3} & \cdots & b_{nn-1} \end{pmatrix} = \begin{pmatrix} 0\cdots & \\ 0\cdots & \\ \vdots & \vdots \\ 0\cdots & 0. \end{pmatrix}$$

For $(t_{11}, \dots, t_{1n})$ we take $\bar{c_2}$, for $(t_{21} \cdots t_{2n})$ we take $c_3$ and so on, where the $\bar{c}_k$ are defined as in Corollary 2.1.

Hence, the first $n-1$ rows of $T$ are $\bar{c}_2, \bar{c}_3, \dots, \bar{c}_n$. Our aim is now to show that it is possible to "complete" the last row and form an orthogonal matrix

$T$. Indeed, for the orthonormal system $\bar{c}_2 \dots, \bar{c}_n$ we create the following linear system of equations:

$$\begin{cases} x_1c_{12} & + & x_2c_{22} & + & x_3c_{32} & + & \cdots & + & x_nc_{n2} & = 0 \\ x_1c_{13} & + & x_2c_{23} & + & \cdots & + & & & x_nc_{n3} & = 0 \\ \vdots & & & & & & & & \vdots & \\ x_1c_{1n} & + & x_2c_{2n} & + & \cdots & + & & & x_nc_{nn} & = 0. \end{cases} \tag{2.25}$$

This system, as a linear homogeneous system of $n-1$ equations with $n$ unknowns, has a non trivial solution $\bar{x} = (x_1, \dots, x_n)$. Hence the vector $\bar{x}$ is orthogonal to $\bar{c}_2, \dots, \bar{c}_n$. We may assume that $(\bar{x}, \bar{x}) \neq 0$. Indeed, in the case $(\bar{x}, \bar{x}) = 0$ it follows that $\bar{x}$ is orthogonal to $\{\bar{c}_2, \dots, \bar{c}_n, \bar{x}\}$, i.e., to a set of $n$ vectors. Since $\bar{x}$ by construction does *not* depend on $\{\bar{c}_k\}_2^n$, it follows that $\bar{x}$ must be zero in contrast to the above. Thus, by an additional normalization, the last missing row of $T$ is easily constructed. Since the last row, $T_n$, is now orthogonal to the set $\{\bar{c}_2, \dots, \bar{c}_n\}$, we have the same property of orthogonality to $\{\bar{b}_n, \dots, \bar{b}_n\}$. Hence the last row of the resulting matrix after multiplication is indeed composed of zeros as wanted. This ends the proof of the lemma.

Based on the above, we can now formulate the main result in this section that will be needed later on.

**Theorem 2.2.** *Let $\{\bar{a}_1, \bar{a}_n, \dots, \bar{a}_n\}$ be a set of $n$ nonzero points in $G^n$. Assume further that $d^2(a_j, a_k) = (a_j - a_k)^2 = \lambda$ for $j \neq k$, $1 \leq j, k \leq n$, $\lambda \neq 0$ (i.e., all mutual distances are the same). Then, by a combination of orthogonal transformations and translations plus magnifications, we can transform the given set $\{a_j\}_{j=1}^n$ to the basis $\{(0, \dots, \underset{j-th}{1}, 0, \dots)\}_{j=1}^n$*

*Proof.* We first translate the set by $-\bar{a}_1$ to create
$(\bar{a}_1, \bar{a}_n, \dots, \bar{a}_n) \to (0, \bar{a}_2 - \bar{a}_1, \dots, \bar{a}_n - \bar{a}_1)$. Using Lemma 2.6 we get that the set $\{\bar{b}_2 = \bar{a}_2 - \bar{a}_1, \bar{b}_3 = \bar{a}_3 - \bar{a}_1, \dots, \bar{b}_n = \bar{a}_n - \bar{a}_1\}$ is independent. Using Lemma 2.7 we deduce that the $n^{\text{th}}$ coordinate of all vectors $T\bar{b}_i, 2 \leq j \leq n$ will be zero for suitable orthogonal $T$. Also, using $(Tb_k - Tb_j)^2 = (b_k - b_j)^2$ we conclude that the given property of equal distances remains valid after applying $T$. Hence, without loss of generality, we may assume that the given vectors $b_j = a_j - a_1$, $2 \leq j \leq n$ have all $n^{\text{th}}$ coordinate zero. As in Lemma 2.6 we have:

$$b_k^2 = 2(b_k, b_j), \quad j \neq k, \quad 2 \leq j, \quad k \leq n. \tag{2.26}$$

Denoting $\bar{b} = \frac{1}{n}\Sigma_{j=2}^n \bar{b}_j = \frac{1}{n}\Sigma_{j=1}^n \bar{b}_j$ (for $\bar{b}_1 = \bar{a}_1 - \bar{a}_1 = 0$) we have

$$(b - b_j)^2 = \mu, \quad \mu \neq 0, \quad 1 \leq j \leq n \ \text{ for some } \ \mu. \tag{2.27}$$

Indeed, for $j = 1$ we get

$$\begin{aligned}(b-b_1)^2 &= b^2 = \frac{1}{n^2}\big(\Sigma_{j=2}^n b_j\big)^2 \\ &= \frac{2(n-1)X + [(n-1)^2 - (n-1)]X}{n^2} = X\big(\frac{n-1}{n}\big)\end{aligned}$$

(where we recall that $X = (b_j, b_k), 2X = (b_j, b_j)$.)

For $j \geq 2$ we claim that:

$$b_j^2 = \frac{2}{n}\Sigma_{\ell=2}^n (b_j, b_\ell) = 2(\bar{b}_\ell, \bar{b}). \tag{2.28}$$

Indeed, $2X = \frac{2}{n}[(n-2)X + 2X]$ where, again, we use $(b_j, b_\ell) = X$ for $j \neq \ell$, $(b_j, b_j) = 2X$. Hence, from (2.28) we have $(b - b_j)^2 = b^2 + b_j^2 - 2(b, b_j) = b^2 = \big(\frac{n-1}{n}\big)X$. Thus (2.27) is confirmed and, moreover, $\mu = \big(\frac{n-1}{n}\big)X$.

We next use another translation, namely, we move $\bar{b}$ to zero. So let $z_j = b_j - b, \ \ 1 \leq j \leq n$. We get from (2.27),

$$z_j^2 = \mu, \quad 1 \leq j \leq n. \tag{2.29}$$

Now we put the notation

$$z_j = (z_{j1} z_{j2} \cdots z_{jn-1}, 0), \ \ 1 \leq j \leq n.$$

Also, let

$$\bar{\alpha} = (0, \ldots, 0, \alpha_n).$$

Our aim now is to find $\alpha_n$ such that

$$(z_j - \alpha, z_k - \alpha) = 0, \ \ j \neq k, \ \ 1 \leq k, \ j \leq n. \tag{2.30}$$

In other words we claim that for some value of $\alpha_n$ all vectors $\{z_j - \alpha\}_{j=1}^n$ are mutually orthogonal. In order to show the existence of such an $\alpha$, we have first to show some preliminary facts.

First note that

$$(z_j, \alpha) = 0, \ \ 1 \leq j \leq n. \tag{2.31}$$

Indeed, $(z_j, \alpha) = \Sigma_{k=1}^{n-1} z_{jk} \cdot 0 + 0 \cdot \alpha_n = 0$. Next, observe that

$$(z_j, z_k) = \text{const for each } j \neq k, \ 1 \leq j, \ k \leq n. \tag{2.32}$$

Indeed, $(z_j - z_k)^2 = z_j^2 + z_k^2 - 2(z_j, z_k)$. Since by construction $(z_j - z_k)^2$ are all equal for $j \neq k$, (2.32) follows from (2.29). We are now ready to show the existence of $\alpha_n$.

For $j \neq k$,

$$(z_j - \alpha, z_k - \alpha) = (z_j, z_k) - (z_k, \alpha) - (z_j, \alpha) + \alpha^2.$$

But $(z_k, \alpha) = (z_j, \alpha) = 0$ by construction. Hence, denoting the constant appearing in (2.32) by $\tau$, we have $(z_j - \alpha)^2 = \tau + \alpha^2$. Now taking $\alpha_n^2 = \alpha^2$ to satisfy $\alpha^2 + \tau = 0$, we get that (2.30) is valid.

In contrast to the real case, we have to be careful with the possibility that the length of $z_j - \alpha$ might be zero or, in other words, that $(z_j - \alpha)^2 = 0$. We now show that this is not the case. Indeed, by (2.29) and (2.31)

$$(z_j - \alpha)^2 = z_j^2 + \alpha^2 - 2(z_j, \alpha) = z_j^2 + \alpha^2 = \mu + \alpha^2.$$

If $(z_j - \alpha)^2 = 0$, it follows that $\mu = z_j^2 = -\alpha^2$ for each $j$. But we have seen above that the constant value $\tau$ of $(z_j, z_k)$ satisfies $\tau = (z_j, z_k) = -\alpha^2,\ j \neq k$. Hence

$$(z_j, z_k) = -\alpha^2 = z_j^2 \quad j \neq k, \quad 1 \leq j, k \leq n. \tag{2.33}$$

Now, by our construction, $(z_j - z_k)^2 \neq 0$. Thus

$$(z_j - z_k)^2 = z_j^2 + z_k^2 - 2(z_j, z_k) = [z_j^2 - (z_j, z_k)] + [z_k^2 - (z_j, z_k)] \neq 0,$$

in contradiction to (2.33).

We have, by the above discussion,

$$\mu + \alpha^2 = (z_j - \alpha)^2 \neq 0, \quad 1 \leq j \leq n. \tag{2.34}$$

Using (2.34), we now may assume that $(z_j - \alpha)^2 = 1$, since otherwise we can multiply by a suitable factor to get it. Now making the additional translation $w = z - \alpha,\ w_j = z_j - \alpha,\ 1 \leq j \leq n$, we derive the set $\{w_1, w_2, \ldots, w_n\}$ which is an orthonormal set by the above and by (2.30).

Summing up what we have up to now, we see that we can get by a sequence of admissible transformations (i.e., orthogonal, translation, magnification) from our original set, the set $\{w_1, w_2, \ldots, w_n\}$ which is orthonormal. Denoting $w_j = (w_{1j}, w_{2j}, \ldots, w_{nj}),\ 1 \leq j \leq n$, we want to show that by additional orthogonal transformation we can arrive at the set $\{e_j\}_1^n = \{(0, \ldots, \underset{j-th}{1}, 0, \ldots\}_1^n$, i.e., we want to find $T = (t_{ij})_{i,j=1}^n$ such that

$$I = \begin{pmatrix} 1 & & & 0 \\ & 1 & & \\ & & \ddots & \\ 0 & & & n \end{pmatrix} = \begin{pmatrix} t_{11} & t_{12} \cdots & t_{1n} \\ \vdots & & \vdots \\ t_{n1} & \cdots & t_{nn} \end{pmatrix} \begin{pmatrix} w_{11} & w_{21} & \cdots & w_{n1} \\ w_{21} & w_{22} & & \\ \vdots & \vdots & & \vdots \\ w_{n1} & w_{n2} & \cdots & w_{nn} \end{pmatrix} = TW.$$

Since the matrix $W$ is orthogonal, as the set $\{\bar{w}_j\}_{j=1}^n$ is orthonormal, it follows at once that $T = W^{-1} \circ I = W^{-1}$ is orthogonal.

This ends the proof of Theorem 2.2.

# 3 Möbius transformation in $G^n$ and inclination

By a *Möbius transformation* in $G^n$ we mean a chain of inversions (reflections) with respect to (generalized) spheres, translations and magnifications.

**Remark.** In [9] it is shown that there is a clear connection between Möbius maps and orthogonal transformations. Also a Möbius transformation can be written as a chain of reflections alone. In addition, there is a classical treatment of further connection between isometry and orthogonal transformation. All this may be easily generalized to $G^n$. We omit the details, mainly because we will not need any of these later on.

## 3.1 Inclination in $G^n$

Our next aim is to define the concept of *inclination.*

Given two spheres in $G^n$, $S = \{z, (z-a)^2 = R^2\}, S_1 = \{z, (z-b)^2 = \rho^2\}$, the *inclination* $\lambda$ *between* $S$ *and* $S_1$ is defined (for $R \neq 0, \rho \neq 0$) as

$$\lambda = \frac{R^2 + \rho^2 - (a-b)^2}{R\rho} = \lambda(S, S_1). \tag{3.1}$$

Note that in order to make the inclination $\lambda$ uniquely defined, we must choose a definite value of $R$ and $\rho$.

Given two planes in $G^n$, $P = \{z, (z, \gamma) = \beta\}$, $P_1 = \{z, (z, \eta) = \delta\}$, we define the *inclination* $\lambda$ *between* $P$ *and* $P_1$ as

$$\lambda = \lambda(P, P_1) = \frac{(\bar{\gamma}, \bar{\eta})}{\gamma\eta}. \tag{3.2}$$

Note that as usual we assume $\eta^2 \neq 0$, $\gamma^2 \neq 0$, i.e., we have "proper" planes. Also, again as in the above case, in order to make $\lambda$ uniquely determined, we have to choose definite values of $\gamma, \eta$ (otherwise, since $\gamma = \sqrt{\Sigma_{j=1}^u \gamma_j^2}$, $\eta = \sqrt{\Sigma_{j=1}^u \eta_j^2}$, the inclination is determined only up to a sign, and can get two possible values).

Given a plane $P = \{z, (z, \gamma) = \beta\}$ and a sphere $S = \{z, (z-a)^2 = R\}$, we define the *inclination* $\lambda$ *between* $P$ *and* $S$ *as*

$$\lambda = \lambda(P, S) = \frac{(a, \gamma) - \beta}{\gamma R}. \tag{3.3}$$

(Again we assume that $\gamma, R$ are not zero.) Later on we will show that inclination is invariant under Möbius transsformation. In section 2.5 we have shown that a generalized sphere is transformed to a generalized sphere under a Möbius map. In order to show the invariance, we need to fix the sign of $\gamma$ (or $R$) after inversion. We now clarify what we mean by that.

We take as an example the sphere $S = \{z, (z-a)^2 = R^2\}$ in $G^n$. We reflect $S$ in the unit sphere $S_0 = \{z, z^2 = 1\}$. Let $a^2 \neq R^2$. We have seen in section 2.5 that the result in the image plane is $(w - A)^2 = \mu^2$, where $\mu = \pm\frac{R}{a^2 - R^2}$.

We state as follows: if for the original sphere $S$ the choice of the radius was one of the two values that we call $R$, then the radius of the image sphere is $\frac{R}{a^2-R^2}$. The motivation for this choice will be clear when we later motivate our discussion by looking at the situation in the classical real case. Similarly, given a plane $P = \{z, (z, \gamma) = \gamma^2\}$, $\gamma^2 \neq 0$, with a specific choice of one of the two values of $\gamma = \sqrt{\Sigma_{j=1}^{u} \gamma_j^2}$, we get after reflection with respect to the unit sphere, the image sphere $\gamma^2 = (z, \gamma) = \left(\frac{w}{w^2}, \gamma\right)$ or $\left(w, \frac{\gamma}{2\gamma^2}\right) = \left(\frac{1}{2\gamma}\right)^2$.

Again we state that the image sphere has the radius $\frac{1}{2\gamma}$ and not $-\frac{1}{2\gamma}$.

## 3.2 The real case

Our aim in this section is to recall the concept of inclination in the real case, and to analyze it geometrically.

We start our discussion with two intersecting spheres in $R^n$. Hence, let
$S = \{\bar{x}, (\bar{x} - \bar{a})^2 = R^2\}$
$S_1 = \{\bar{x}, (\bar{x} - \bar{b})^2 = \rho^2\}$
$a = (x_1, \ldots, x_n), \quad b = (y_1 \ldots, y_n)$
$d^2(a, b) = \Sigma_{j=1}^{n}(x_j - y_j)^2 = (a - b)^2$.

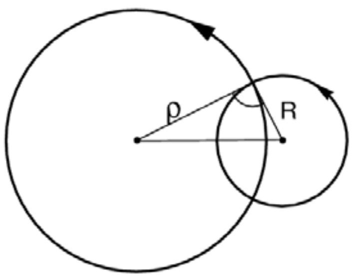


Figure 1: Inclination: the real case.

We see that $\lambda = \lambda(S, S_n) = \frac{\rho^2+R^2-(a-b)^2}{2\rho R}$ is $\cos\phi$, where $\phi$ is the angle of intersection between the two spheres. It is well known that $\lambda = \lambda(S, S_1) = \cos\phi$ is invariant under conformal tranformation, and in particular under a Möbius map. In fact, this invariant property is valid even in the non-intersection case.

We now point out the fact that even in the real case $R \neq 0$ may get two possible values, namely, $R$ may be positive or negative. $R > 0$ means that the sphere bounds a finite ball in $R^n$, $R < 0$ means that the sphere (in $\hat{R}^n$) bounds the complement of a ball in $\hat{R}^n$, i.e., containing the point of infinity. If $R > 0$, $\rho < 0$, $\lambda(S, S_1)$ changes its sign (see Figure 2).

We next want to give the geometrical interpretation of (3.2) and (3.3) in the real case. We start with (3.2). It is easy to see that $\frac{(\bar{\gamma},\bar{\eta})}{\gamma\eta}$ is $\cos\varphi$ for $\varphi$ the angle of intersection between the two planes, or what is the same, the angle between the two perpendicular vectors to $P, P_1$, i.e., $\gamma, \eta$ respectively.

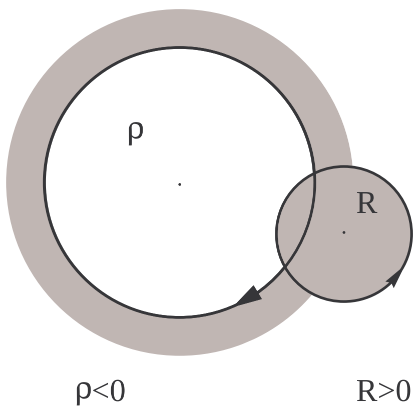


Figure 2: Two possible values for radii.

To explain (3.3) we start with two spheres and let one of these spheres change continuously to a plane. Let $S$ be the sphere $S = \{z, (z-a)^2 = R^2\}$ and $P = \{z, (z, \gamma) = \beta\}$ be a plane in $R^n$. We denote the distance between $a$ and $P$ by $h$ (see Figure 3).

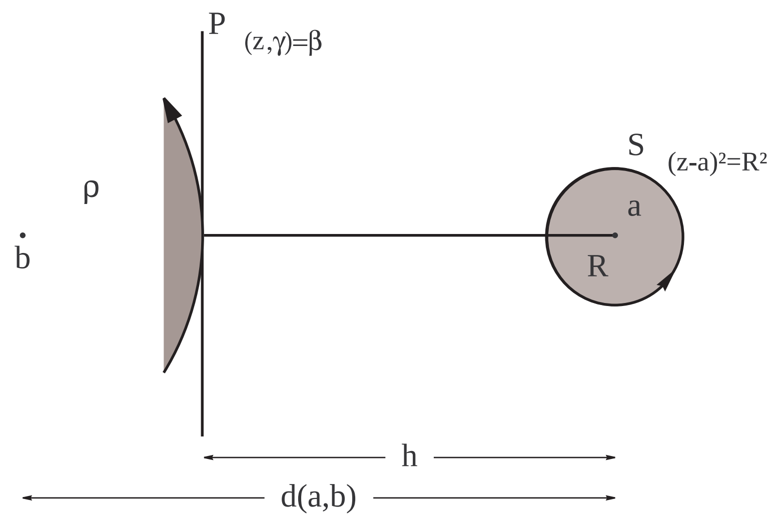


Figure 3: Convergence to a limit case.

We have for the tangent sphere $S_1$ to the plane $P$: $S_1 = \{z, (z-b)^2 = \rho^2\}$

$$\lambda(S, S_1) = \frac{R^2 + \rho^2 - (b-a)^2}{2R\rho}. \tag{3.4}$$

In the particular case of the situation described in Figure 3 we have $\rho > 0$, $R > 0$. Also, $d^2 = d^2(a, b) = (\rho + h)^2 = (b - a)^2$. Hence from (3.4):

$$\lambda(S, S_1) = \frac{R^2 + \rho^2 - (\rho + h)^2}{2\rho R} = \frac{R^2 - h^2 - 2\rho h}{2\rho R}.$$

We now let $\rho \to \infty$ continuously. Obviously $S_1$ converges to $P$. In the limit we get $\lambda(S, S_1) \to -\frac{h}{R}$. Our aim is now to show that this last expression, i.e., $-\frac{h}{R}$ is exactly the expression of $\lambda = \lambda(P, S)$ appearing in (3.3).

To show that again we take a specific situation in order to illustrate the geometry (see Figure 4) we first define $\mu$ such that $\bar{a} - \mu\bar{\gamma}$ is orthogonal to $\bar{\gamma}$, i.e., $(\bar{a} - \mu\bar{\gamma}, \bar{\gamma}) = 0$ or $(\bar{a}, \bar{\gamma}) = \mu\gamma^2$. Hence

$$\mu = \frac{(\bar{a}, \bar{\gamma})}{\gamma^2}. \tag{3.5}$$

$S = \{z, (z-a)^2 = R^2\}$
$P = \{z, (z, \gamma) = \gamma^2\}$
$P$ is given in its normal form

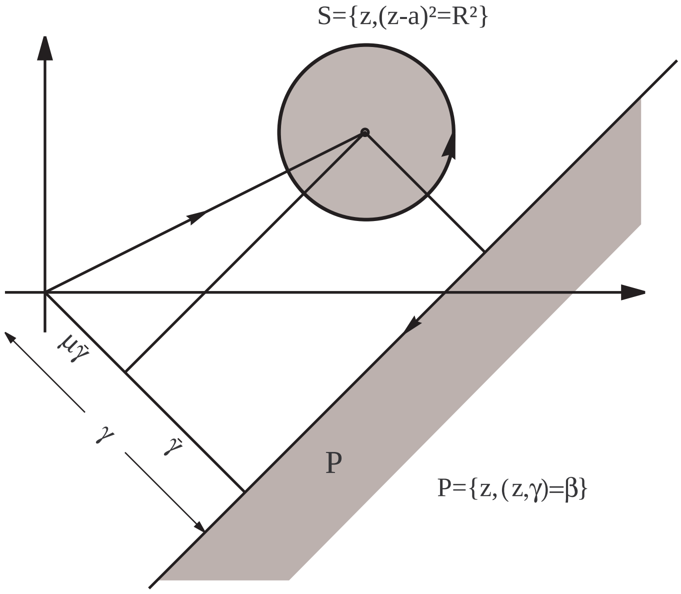


Figure 4: Normal form.

Clearly, the distance from the center $\bar{a}$ to $P$ is $h = \gamma - \mu\gamma$ or, by (3.5), $-h = \mu\gamma - \gamma = (\mu - 1)\gamma = \left[\frac{(\bar{a}, \bar{\gamma})}{\gamma^2} - 1\right]\gamma$, i.e., $-\frac{h}{R} = \frac{(\bar{a}, \bar{\gamma}) - \gamma^2}{\gamma R} = \frac{(\bar{a}, \bar{\gamma}) - \beta}{\gamma R}$ (where we have used the connection $\gamma^2 = \beta$ as we took the normal form of $P$). Thus we can conclude that $\lambda$ appearing in (3.3) is exactly $-\frac{h}{R}$ in accordance with the discussion above.

After dealing with the geometrical meaning of (3.1), (3.2) and (3.3), we are ready to prove the invariance of $\lambda(\Sigma, \Sigma_1)$ for $\Sigma, \Sigma_1$ two generalized spheres. This will be done in the next section. But before turning to that, we want to point out a few additional facts:

**Remarks**

(1) Inclination depends linearly on the cross ratio, but we will not make any use of that [10].

(2) Inclination is closely related to the concepts of *separation* and *inversive distance*, [9]. *Separation* is the same as inclination, but with a negative sign. *Inversive distance* is also a very close concept. It is just the absolute value of inclination. Thus obvioiusly both separation and inversive distance are conformal invariant as well.

(3) The case of tangency is of particular interest. If $S = \{z, (z-a)^2 = R^2\}$, $S_1 = \{z, (z-b)^2 = \rho^2\}$ are two tangent spheres with $R,\ \rho > 0$ (meaning that their interiors are disjoint), we get from (3.1):

$$\lambda(S, S_1) = \lambda = \frac{R^2 + \rho^2 - (R+\rho)^2}{2R\rho} = -1. \tag{3.6}$$

In the case $R > 0$, $\rho < 0$ we get $\lambda = 1$ (see Figure 5)

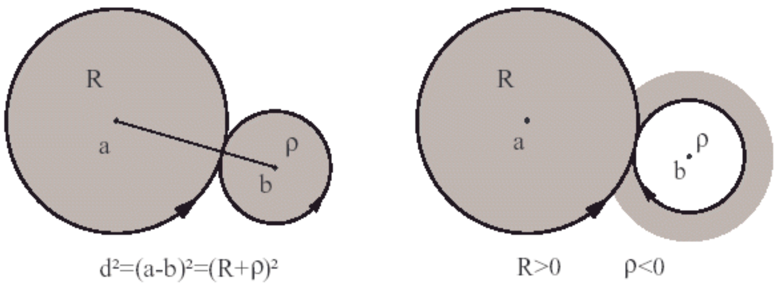


Figure 5:

(4) In case of tangency between a plane and a sphere, the situation is very similar (see Figure 6).

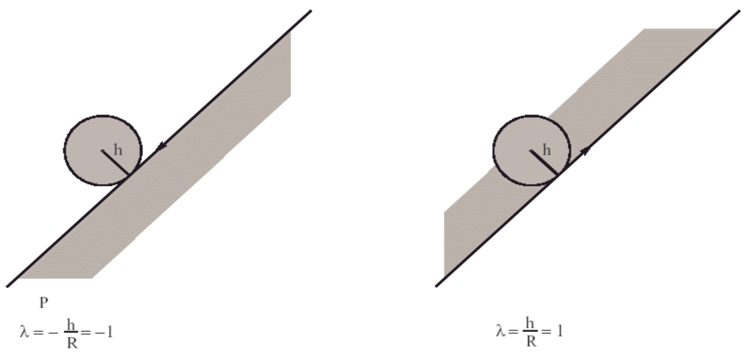


Figure 6:

(5) As mentioned in section 3.1, if we reflect the sphere $S = \{z, (z-a)^2 = R^2\}$ in the unit sphere, we choose one of the two possible values of $R$, and then the new center after reflection is $\frac{a}{a^2-R^3}$ (provided, of course, that $a^2 \neq R^2$) and the new radius is chosen to be $\frac{R}{a^2-R^2}$ where $R$ is the specific chosen value for $S$. We now motiviate this in $\hat{R}^n$. Indeed, we separate between two caps: $a^2 > R^2$, $a^2 < R^2$. Suppose $R > 0$. Then in the first case the image is bounded and thus $\frac{R}{a^2-R^2}$ is the right choice. In the other case, i.e., $a^2 < R^2$, the image contains infinity and thus the radius of the image is negative, and thus, again, $\frac{R}{a^2-R^2} < 0$ is the right choice (see Figure 7)

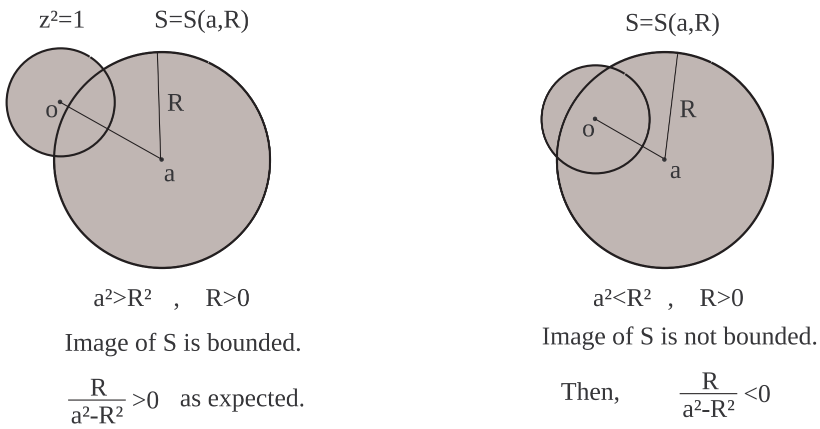


Figure 7:

### 3.3 Invariance of inclination

We are now back in $G^n$. Our aim is to show invariance of inclination between two (generalized) spheres $\Sigma, \Sigma'$ under reflection in another (generalized) sphere $\Sigma_0$. In fact, the invariance property is more general. Any orthogonal transformation, translation or magnification, keeps this invariance as well.

**Lemma 3.1.** *Given two generalized spheres in $G^n$, and applying an orthogonal $n \times n$ matrix $T = (t_{ij})_{i,j=1}^n$ acting on $G^n$, the inclination remains invariant. Similarly, translation or magnification keep the inclination invariant.*

*Proof.* Let $S = S(a, R)$, $S_1 = S_1(b, \rho)$ be two given spheres and $T$ an orthogonal transformation. By (3.1), $\lambda = \lambda(S, S_1) = \frac{R^2+\rho^2-(a-b)^2}{2R\rho}$. We have shown that applying $T$ keeps $R, \rho$ the same. Also $T(a-b) = Ta - Tb$ and $(Ta - Tb)^2 = (a-b)^2$. This ends the proof of this case.

Now let $S = S(a, R)$, $P = \{z, (z, \gamma) = \beta\}$. We have by (3.3) $\lambda(P, S) = \frac{(a,\gamma)-\beta}{\gamma R}$. Applying $T$ we get for the image of $P$: $(Tz, T\gamma) = (z, \gamma) = \beta$. Also, $Ta$, the image of $a$, is the center of the image of $S = S(a, R)$. In addition, $R$ remains the same. Also, $\gamma = \sqrt{\Sigma\gamma_j^2} = \sqrt{\Sigma_{j=1}^{u}(T\gamma)_j^2} = \sqrt{(T\gamma, T\gamma)}$. Putting all this together, we have for the inclination of the two images of $P, S$:

$$\lambda = \frac{(Ta, T\gamma) - \beta}{\sqrt{(T\gamma)^2}R} = \frac{(a, \gamma) - \beta}{\gamma R} = \lambda(P, S),$$

which ends the proof of this case as well.

The case of two planes (i.e., (3.2)) is even simpler and details are omitted.

The proof for translations and magnifications is really not difficult. We give one example and leave the rest to the reader. Let $S = S(a, R)$, $S_1 = S_1(b, \rho)$ be two spheres. Suppose we impose magnification by $M$ on $G^n$. Thus $(z-a)^2 = R^2$ is replaced by $(Mz - Ma)^2 = (MR)^2$.

Similarly, $(Mz - Mb)^2 = (M\rho)^2$. Hence the new inclination satisfies

$$\lambda = \frac{(MR)^2 + (M\rho)^2 - (Ma - Mb)^2}{(MR)(M\rho)} = \lambda(S, S_1).$$

**Theorem 3.1.** *Given two (generalized) spheres in $G^n$, $\Sigma, \Sigma'$ and another (generalized) sphere $\Sigma_0$, it follows that $\lambda(\Sigma, \Sigma')$ remains invariant after reflection of $\Sigma, \Sigma'$ with respect to $\Sigma_0$.*

*Proof.* We first note that by combination of Theorem 3.1 and Lemma 3.1, it follows that any Möbius transformation (i.e., this means a combination of translations, magnifications and reflections) or orthogonal transformation keep the inclination invariant.

In order to prove Theorem 3.1, we have again to check various different cases. In fact, the proofs are simple and we show only one case, namely, $S = \Sigma = \{z, (z-a)^2 = R^2\}$, $S_1 = \Sigma' = \{z, (z-b)^2 = \rho^2\}$, $\Sigma_0 = S_0 = \{z, z^2 = 1\}$ and, in addition, $a^2 \neq R^2$, $\rho^2 \neq b^2$. We have for the new $\lambda$:

$$\lambda = \frac{\frac{R^2}{(a^2-R^2)^2} + \frac{\rho^2}{(b^2-\rho^2)^2} - \left(\frac{a}{a^2-R^2} - \frac{b}{b^2-\rho^2}\right)^2}{2\frac{R}{a^2-R^2}\cdot\frac{\rho}{b^2-\rho^2}}$$

or

$$\lambda = \frac{\frac{R^2-a^2}{(a^2-R^2)^2} + \frac{\rho^2-b^2}{(\rho^2-b^2)^2} + \frac{2(a,b)}{(a^2-R^2)(b^2-\rho^2)}}{2\frac{R\rho}{(a^2-R^2)(b^2-\rho^2)}}$$

$$\lambda = \frac{\frac{1}{(R^2-a^2)} + \frac{1}{\rho^2-b^2} + \frac{2(a,b)}{(a^2-R^2)(b^2-\rho^2)}}{2\frac{R\rho}{(a^2-R^2)(b^2-\rho^2)}}$$

$$\lambda = \frac{(\rho^2-b^2)+(\rho^2-a^2)-2(a,b)}{2R\rho} = \frac{\rho^2+R^2-(a-b)^2}{2R\rho}.$$

### 3.4 Mutual inclination in $G^n$

Given $n$ different (generalized) spheres $\{\Sigma_j\}_{j=1}^n$, we say that they have *mutual inclination* $\gamma$, if and only if $\lambda(\Sigma_j, \Sigma_\ell) = \gamma$ for $j \neq \ell$, $1 \leq j,\ \ell \leq k$. For later purposes we will need

**Theorem 3.2.** *Let $n+1$ (generalized) spheres be given in $G^n$ such that they have mutual inclination $\lambda \neq 1$. Then, by a chain of Möbius maps and orthogonal transformations, we may assume that all radii of the spheres are the same.*

For the proof of Theorem 3.2 we will need to use Theorem 3.1. In addition we will use two lemmas that are stated and proved in what follows.

**Lemma 3.2.** *If $\Sigma, \Sigma'$ are two (generalized) spheres in $G^1$ such that $\lambda(\Sigma, \Sigma') \neq 1$, then by a suitable reflection it is possible to assume that the two radii of $\Sigma, \Sigma'$ are the same.*

*Proof.* First note that $\lambda \neq 1$ is a necessary conditon. Indeed, given two (different) spheres $\Sigma, \Sigma'$ such that $\Sigma = \{z, (z-a)^2 = R^2\}, \Sigma = \{z, (z-b)^2 = R^2\}$, then $\lambda(\Sigma, \Sigma') = \frac{R^2+R^2-(a-b)^2}{2R\cdot R} = 1 - \frac{(a-b)^2}{2R^2}$. Since $\Sigma, \Sigma'$ are assumed to be different spheres with the same radius $R$, it necessarily follows that $a \neq b$. Thus $\lambda(\Sigma, \Sigma') \neq 1$. Assume now that $\lambda \neq 1$. Our aim is to show that in the space $G^1$ we can make an inversion such that the two new radii will be equal.

First, it is obviously allowed to assume that the two given spheres (actually segments) in $G^1$ are bounded, i.e., $S = \{z, (z-a)^2 = R^2\}, S_1 = \{z, (z-b)^2 = \rho^2\}$ we put the notation $d = b - a$. We want to invert with respect to $S_0 = \{z, (z-c)^2 = 1\}$ and find $c$. Changing the variable by translation it is equivalent to take $S_0 = \{z, z^2 = 1\}$, i.e., the unit sphere and find a suitable $a$ provided $d = b - a$ is fixed. After inversion with respect to $S_0$ we get the new radii $\frac{R}{a^2-R^2}$, $\frac{\rho}{b^2-\rho^2}$. In order to make them equal, we need to solve for $a$ the equation $\frac{R}{a^2-R^2} = \frac{\rho}{(a+d)^2-\rho^2}$. Since $a$ is considered as unknown, it is convenient to denote $a = x$. Of course, $a = x = R$ or $a + d = x + d = \rho$ are not admissible solutions. By a trivial calculation,

$$x^2(R-\rho) + 2xRd + (Rd^2 - R\rho^2 + \rho R^2) = 0, \quad x = a. \tag{3.7}$$

Since $\lambda \neq 1$, we have $R^2 + \rho^2 - d^2 \neq 2\rho R$ or

$$(R-\rho)^2 \neq d^2. \tag{3.8}$$

We first show that in view of (3.8) the equation (3.7) must have two different solutions. Indeed, the discriminant $\Delta$ is equal to $4[(Rd)^2 - (R-\rho)(Rd^2 - R\rho^2 + \rho R^2)]$. Thus $\frac{1}{4}\Delta = R^2d^2 - (R-\rho)Rd^2 - (R-\rho)^2R\rho = \rho R[d^2 - (R-\rho)^2] \neq 0$.

Now, if the two solutions to (3.7), (say, $x_1, x_2$) are both different from $R$, then at least one of the solutions is different also from $\rho - d$ and we are done. Hence, it remains to check the possibility that one of the solutions (say $x_1$) equals $R$. If the other one is diffeent from $\rho - d$, again we are done. Hence let us assume that $x_1 = R$, $x_2 = \rho - d$. In that case we get $(x-x_1)(x-x_2) = (x-R)(x-\rho+d) = 0$ or

$$x^2 - x(R+\rho-d) - R(d-\rho) = 0. \tag{3.9}$$

Since (3.9) is the same as (3.7), we easily get equating coefficients

$$\begin{cases} 2Rd = (-R-\rho+d)(R-\rho) \\ Rd^2 - R\rho^2 + \rho R^2 = -(d-\rho)(R-\rho)R. \end{cases}$$

The second equation leads to a contradiction. Indeed,
$d^2 + \rho(-\rho+R) = -(d-\rho)(R-\rho)$ leads to $d^2 = (R-\rho)(-d+\rho-\rho)$ or $d = \rho - R$ which contradicts (3.8).

**Lemma 3.3.** *Let $n+1$ spheres in $G^n$ have mutual inclination $\lambda$, $\lambda \neq 1$. Let $n$ of them have the same radius. Then the $n+1^{th}$ sphere has its center on the "perpendicular line" to the "$n$-plane" of the centers of the $n$ spheres, emanating from the "center of gravity" of the centers.*

*Proof.* First we note that all mutual distances among the given $n$-spheres are the same. Indeed, by assumption, all have mutual inclination $\lambda$. Also, radii are all the same. Now, $\lambda = \frac{\rho^2+R^2-(a-b)^2}{2PR}$. If $\rho = R = \tau$ where $\tau$ is the common value of the radii, we get $\lambda = \frac{2\tau^2-(a-b)^2}{2\tau^2} = 1 - \frac{(a-b)^2}{2\tau^2}$, which means that $(a-b)^2$ is equal for all distances, or putting this differently, all mutual distances are the same. We now make use of Theorem 2.2, and the invariance property of the inclination. We thus may assume that the centers of the $n$ spheres are located at $\{(0,0,\ldots,\underset{j^{\text{th}}\text{ place}}{1},0,\ldots,0)\}_{j=1}^{n}$ (see Figure 8).

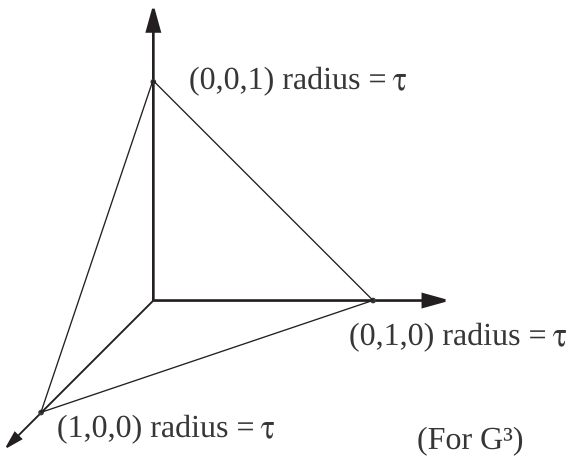


Figure 8:

Denote, as before, the common value of all equal radii of the $n$ spheres by $\tau$. Aso denote the coordinates of the center of the $n+1^{\text{th}}$ sphere by $(z_1, z_2, \ldots, z_n)$. Since we are given that the inclination between each of the $n$ spheres and this $n+1^{\text{th}}$ sphere is again equal to the same value $\lambda$, we have

$$\lambda = \frac{\rho^2+\tau^2-[\Sigma_{j\neq k} z_j^2+(z_k-1)^2]}{2\rho\tau} = \frac{\rho^2+\tau^2-\Sigma_{j=1}^{n} z_j^2+1-2z_k}{2\rho\tau}.$$

Hence we get at once that $z_1 = z_2 \cdots = z_n$. Denoting this common value by $\alpha$ we get that the center of the $n+1^{\text{th}}$ sphere is located at $(\alpha, \alpha, \ldots, \alpha)$. Now, the center of gravity of the plane determined by the centers is $\left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)$. Since $(\alpha - \frac{1}{n}, \alpha - \frac{a}{n}, \ldots, \alpha - \frac{a}{n})$ is a vector perpendicular to each of the vectors $(0, \ldots, \underset{j-th}{1}, 0, \ldots, ) - \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right) = \left(-\frac{1}{n}, \ldots, 1-\frac{1}{n}, -\frac{1}{n}, \ldots, -\frac{1}{n}\right)$ (as can easily be checked), we are done.

**Proof of Theorem 3.2**. The proof of Theorem 3.2 is by an induction process. We have shown that given the mutual inclination $\lambda, \lambda \neq 1$, then we can prove the claim of Theorem 3.2 for $n = 1$, i.e., for two spheres in $G^1$ (Lemma 3.2). We now proceed, in the induction process, from $n$ to $n+1$. Hence assume that we are given $n+1$ spheres in $G^n$, such that they have mutual inclination $\lambda$, $\lambda \neq 1$. By the induction assumption we may assume that $n$ spheres have the same radius, denoted by $R$. Without loss of generality we may assume that all centers of $n$ spheres are located in the plane $z_n = 0$. Indeed, if this is not so, we can arrive at this situation by applying translations and orthogonal transformations (see Lemma 2.7). Since the property of equal radii remains invariant by translations and orthogonal transformations, there is, indeed, no loss of generality to assume that all centers of the $n$ spheres are located at the plane $z_n = 0$. We have seen already that if all $n$ spheres have the same radius $R$, and mutual inclination $\lambda$, then all mutual distances are the same.

Our aim now is to use Lemma 3.3. It will be convenient to assume that the center of gravity is at zero and that the center of the first sphere is located at $(1, 0, \ldots, 0)$. Also, we assume that mutual distances among the $n$ centers satisfy $d^2 = 2$. (In fact, what we do is, rotating the plane appearing in Lemma 3.3 and, after translation, arrive at the situation described above). Using Lemma 3.3, we know that, necessarily, the center of the $n+1^{\text{th}}$ sphere is located at $(0, 0, \ldots, 0, \alpha)$ for some $\alpha$. We have $\lambda = \frac{R^2+R^2-d^2}{2R\cdot R} = 1 - \frac{d^2}{2R^2} = 1 - \frac{1}{R^2}$, i.e.,

$$\lambda = 1 - \frac{1}{R^2}. \tag{3.10}$$

Also, denoting by $\rho$ the radius of the remaining $n+1^{\text{th}}$ sphere and using the fact that the inclination of this sphere with all other spheres is equal to $\lambda$, we get

$$\lambda = \frac{R^2 + \rho^2 - (1 + \alpha^2)}{2R\rho}. \tag{3.11}$$

It follows at once from (3.10) and (3.11) that

$$\alpha^2 = (\rho - R)^2 - 1 + \frac{2\rho}{R}. \tag{3.12}$$

Our next aim is to make an inversion with respect to a sphere with radius 1 and a center $\bar{a} = (0, 0, \ldots, x)$, $x$ to be fixed later. Obviously, $\bar{a}$ has the same distance from all centers of the first $n$ spheres. Hence all radii of the $n$ spheres will be equal, after the inversion, to $R_1 = \frac{R}{1+x^2-R^2}$. In addition, inversion ofs the $n+1^{\text{th}}$ sphere changes the radius $\rho$ of this sphere to the radius $\rho_1 = \frac{\rho}{(\alpha-x)^2-\rho^2}$. It now remains to show that it is possible to choose $x$ such that $R_1 = \rho_1$. Doing so, the proof will be finished, because this will mean that *all* radii of the $n+1$ spheres are equal.

We need to solve $\rho_1 = R_1$ or

$$\frac{R}{1+x^2-R^2} = \frac{\rho}{(\alpha-x)^2-\rho^2}. \tag{3.13}$$

This is equivalent to $R(\alpha^2+x^2-2\alpha x-\rho^2)-\rho(1+x^2-R^2)$ or $x^2(R-\rho)-2\alpha Rx + (R\alpha^2-R\rho^2-\rho+\rho R^2) = 0$. Hence, $x = \frac{\alpha R \pm \sqrt{\Delta}}{R-\rho}$, where $\Delta = \alpha^2R^2-(R-\rho)(R\alpha^2-R\rho^2-\rho+\rho R^2) = \rho R\alpha^2-(R-\rho)(-R\rho^2-\rho+\rho R^2)$. >From (3.12) we get, after an easy calculation,

$$\Delta = \rho^2. \tag{3.14}$$

Hence, $x = \frac{\alpha R\pm\sqrt{\Delta}}{R-\rho} = \frac{\alpha R\pm\rho}{R-\rho}$. Thus

$$x_1 = \frac{\alpha R+\rho}{R-\rho}, \quad x_2 = \frac{\alpha R-\rho}{R-\rho} \tag{3.15}$$

are two possible solutions for $x$. This ends the proof of Theorem 3.2.

Our next aim is to analyze the (more restricted) situation in $R^n$. Hence consider the real version of Theorem 3.2.

**Theorem 3.3.** *Let $n+1$ (generalized) spheres be given in $R^n$ such that they have mutual inclination $\lambda \neq 1$. Then $\lambda < 1$ is a necessary and sufficient condition that these spheres may be transformed by a chain of Möbius maps and orthogonal transformation to another set of $n+1$ spheres having equal radii $R$, $0 < R < \infty$.*

*Proof.* The process of proving Theorem 3.2 is essentially the same in our case. If $\lambda < 1$, one needs to show that we get a real solution. To check the first step of induction (i.e., Lemma 3.2) we recall that $\frac{1}{4}\Delta = \rho R[d^2-(R-\rho)^2]$. We need to check that for $\lambda < 1$ we get two different real solutions. But $\lambda = \frac{R^2+\rho^2-d^2}{2\rho R}$ and thus $\lambda < 1$ is equivalent to $(R-\rho) < d^2$ which implies for $\Delta$ appearing in Lemma 3.2 that $\frac{1}{4}\Delta = \rho R[d^2-(R-\rho)^2] > 0$. Hence $\sqrt{\Delta}$ is real, which is what we need. The next step of the induction causes no problem, as the passage from $n$ to $n+1$ in the induction process implies by (3.14) that $\Delta = \rho^2$, i.e., $\Delta > 0$ for real $\rho \neq 0$. This ends one direction of the proof, i.e., showing that if $\lambda < 1$, then indeed we can find $n+1$ spheres with equal radii. Proving the other part is much shorter. Indeed, by (3.10) we see that $\lambda = 1 - \frac{1}{R^2}$. Hence, if $\lambda > 1$, $R$ is not real.

We note that in [8] a special case of the above theorem is proved, namely, the case $\lambda = -1$, i.e., the case of tangency. The proof is entirely different.

**Remark**. There is one point in the proof of Theorem 3.2 and Theorem 3.3 that needs further explanation. One may argue that the $n+1^{\text{th}}$ (generalized) sphere appearing in the proof of Lemma 3.2 is not necessarily a sphere, but may be a plane. To complete the argument for this case one needs only to use an additional reflection with respect to a sphere with a center at any point of the form $(\alpha, \alpha, \ldots, \alpha, t)$ (for some complex $\alpha$ and $t$) which is not on the plane. Then each of the $n$ spheres is reflected to a new sphere and all $n$ spheres have the same radius after reflection. The plane is reflected to a sphere and we are back in the previous situation. Indeed, by additional magnifications, translations and orthogonal transformations, spheres cannot be transformed to planes.

Thus, only one case remains to be checked, namely, if there is not any point of the form $(\alpha, \alpha, \ldots, \alpha, t)$ that is not on the plane. But we now show that such a case contradicts the condition of mutual inclination. Indeed, $\lambda = \frac{(a,\gamma)-\beta}{\gamma R} = \frac{(a,\gamma)}{\gamma R}$ as $\beta = 0$ for the case under consideration. $a$ may be any of the $n$ unit vectors $(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, \ldots, 1)$. Thus $\gamma$, the vector perpendicular to the plane under consideration, must be of the form $(\tau, \tau, \ldots, \tau, \eta)$. Since $\gamma$ is perpendicular to any vector of the form $(\alpha, \alpha \ldots, \alpha, t)$, we must have $\eta = 0$. Hence the orthogonality condition turns out to be $\Sigma_{k=1}^{n-1} \alpha\tau = (n-1)\alpha\tau = 0$. This leads to one of the two possibilities: $\alpha = 0$ or $\tau = 0$. Hence we get the desired contradiction.

# 4 Inclination theorems

Let $n+1$ (generalized spheres $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma$, $\gamma \neq 1$, and $\Sigma_{n+2}$ be another sphere in $G^n$ having an inclination $\mu$ with each of $\{\Sigma_j\}_{j=1}^{n+1}$, such that $\mu$ satisfies a certain condition, to be specified later. Now let $\Sigma$ be an arbitrary sphere in $G^n$ which we call "*a reference sphere*". Denote by $\{\lambda_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+1}$ with $\Sigma$ respectively. In what follows, we shall prove some theorems concerning relations among $\{\Sigma\lambda_j\}_{j=1}^{n+2}, \gamma, \mu$. Theorem 4.2 is a generalization of Mauldon's theorem [7] and was announced (for the real case) in [11] without proof.

**Theorem 4.1.** *Let $n+1$ spheres $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma, \gamma \neq 1$, and $\gamma \neq -\frac{1}{n}$. Let $\Sigma_{n+2}$ be another sphere in $G^n$ with inclination $\mu$ with each of $\{\Sigma_j\}_{j=1}^{n+1}$ where*

$$\mu^2 = \frac{1+\gamma n}{1+n}. \tag{4.1}$$

*Let $\Sigma$ be any sphere of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}$ the inclinations of $\Sigma$ with $\{\Sigma_j\}_{j=1}^{n+2}$ respectively. Then*

$$\Sigma_{k=1}^{n+1}\lambda_k = (n+1)\mu\lambda_{n+2}. \tag{4.2}$$

*Proof.* Following a model of Coxeter [8], we now take the plane $\Sigma_{j=1}^{n+1}z_j = 1$ in $G^{n+1}$ and we may assume that all centers of the spheres, $\Sigma$, $\{\Sigma_j\}_{j=1}^{n+2}$ are in this plane. By Theorem 3.2 we may assume that all radii of $\{\Sigma_j\}_{j=1}^{n+1}$ are the same, say $R$, and further, we may assume that the centers of $\{\Sigma_j\}_{j=1}^{n+1}$ are located at $(1,0,\dots,),(0,1,0\dots,),\dots,(0,0,\dots,1)$. Hence

$$\gamma = \frac{R^2+R^2-2}{2R^2} = 1-\frac{1}{R^2},\quad 1-\gamma = \frac{1}{R^2}. \tag{4.3}$$

Since $\Sigma_{n+2}$ has the same inclination $\mu$ with each of the set of spheres $\{\Sigma_j\}_{j=1}^{n+1}$, we have, denoting by $\rho$ the radius of $\Sigma_{n+2}$ its center's coordinates by $(\alpha_1,\alpha_2,\dots,\alpha_{n+1})$,

$$\mu = \frac{R^2+\rho^2-\left(\Sigma_{j\neq k}\alpha_j^2+(\alpha_k-1)^2\right)}{2\rho R} = \frac{R^2+\rho^2-\Sigma\alpha_j^2-1+2\alpha_k}{2\rho R}.$$

Hence all $\{\alpha_k\}$ are equal. Also $(\alpha_1,\alpha_2,\dots,\alpha_{n+1})$ satisfy $\Sigma_{j=1}^{n+1}\alpha_j = 1$ by our construction. Thus $\alpha_1=\alpha_2=\cdots=\alpha_{n+1}=\frac{1}{n+1}$. The distance $d_k$ of this center (i.e., $(\frac{1}{n+1},\dots,\frac{1}{n+1})$) from the points $(0,\dots,\underset{k^{\text{th}}}{1},0,\dots,)$ satisfies

$$\begin{aligned} d_k^2 = \Sigma_{j\neq k}\frac{1}{(n+1)^2}+\left(1-\frac{1}{n+1}\right)^2 &= \Sigma_{j=1}^{n+1}\frac{1}{(n+1)^2}+1-\frac{2}{n+1} \\ &= \frac{n+1}{(n+1)^2}+1-\frac{2}{n+1}. \end{aligned}$$

Hence

$$d_k^2 = \frac{n}{n+1} \text{ for each } k,\quad 1\le k\le n+1. \tag{4.4}$$

Thus $\mu = \frac{R^2+\rho^2-d_k^2}{2\rho R} = \frac{R^2+\rho^2-\frac{n}{n+1}}{2\rho R}$. Using the given condition (4.1) for $\mu$, we get

$$R^2+\rho^2-\frac{n}{n+1} = 2\rho R\mu,\quad \mu^2 = \frac{1+\gamma n}{1+n}.$$

Solving the equation $\rho^2-(2R\mu)\rho+R^2-\frac{n}{n+1}=0$ for $\rho$, we get $\rho = R\mu\pm\sqrt{\Delta}$, $\Delta = (R\mu)^2-\left(R^2-\frac{n}{n+1}\right)$. Putting $\mu^2=\frac{1+\gamma n}{1+n}$ and $\gamma = 1-\frac{1}{R^2}$, by (4.3) we very easily get $\Delta = 0$. Hence finally we have

$$\rho = R\mu. \tag{4.5}$$

Note that $\rho = 0$, $\mu = 0$ and $\gamma = -\frac{1}{n}$ are equivalent. This explains the condition $\gamma \neq -\frac{1}{n}$. Our aim is now to show (4.2).

Denote the coordinates of the center of the sphere of reference $\Sigma$ by $(y_1, \ldots, y_{n+1})$. Obviously $\Sigma_{j=1}^{n+1} y_j = 1$, since this point is located also in the plane
$\Sigma_{j=1}^{n+1} z_j = 1$. Denote by $r$ the radius of $\Sigma$. By definition of $\{\lambda_k\}_{k=1}^{n+1}$ we get

$$\lambda_k = \frac{R^2 + r^2 - \left(\Sigma_{j \neq k} y_j^2 + (y_k - 1)^2\right)}{2rR}$$

or

$$\lambda_k = \frac{R^2 + r^2 - \Sigma_{j=1}^{n+1} y_j^2 - 1 + 2y_k}{2rR}, \quad 1 \leq k \leq n+1. \tag{4.6}$$

For $\lambda_{n+2}$ we similarly get $\lambda_{n+2} = \frac{r^2 + \rho^2 - \Sigma_1^{n+1}\left(y_j - \frac{1}{n+1}\right)^2}{2r\rho}$ or

$$\lambda_{n+2} = \frac{\rho^2 + r^2 - \Sigma_{j=1}^{n+1} y_j^2 + \frac{1}{n+1}}{2r\rho} \tag{4.7}$$

(where we have used $\Sigma_{j=1}^{n+1} y_j = 1$).

>From (4.6), and again applying $\Sigma y_j = 1$, we have

$$\Sigma_{k=1}^{n+1} \lambda_k = \frac{(n+1)\left(R^2 + r^2 - \Sigma_{j=1}^{n+1} y_j^2\right) - (n-1)}{2rR}. \tag{4.8}$$

Using (4.5), (4.7) and (4.8) we get that (4.2) is reduced to

$$\frac{(n+1)\left(R^2 + r^2 - \Sigma_j y_j^2\right) - (n-1)}{2rR} = \frac{(n+1)\left(\rho^2 + r^2 - \Sigma_{j=1}^{n+1} y_j^2 + \frac{1}{n+1}\right)}{2r\rho} \cdot \frac{\rho}{R},$$

or $(n+1)R^2 - (n-1) = (n+1)\rho^2 + 1$. This is equivalent to $(n+1)(R^2 - \rho^2) = n$. This is the same (from (4.5)) as $(n+1)(R^2 - R^2\mu^2) = n$ or $(n+1)R^2(1-\mu^2) = n$. Putting $\mu^2 = \frac{1+\gamma n}{1+n}$ by (4.1) and $\gamma = 1 - \frac{1}{R^2}$, this follows at once. (Alternatively we can use $\Delta = 0$ which is equivalent and was shown above.) This ends the proof of Theorem 4.1 in the case where $\Sigma$ is a sphere.

If $\Sigma$ is a plane, say, $P = \{z, (z, \eta) = \beta,\ \Sigma_{j=1}^{n+1} z_j = 1\}$ we have (see (3.3))

$$\lambda_j = \frac{\eta_j - \beta}{R\eta}, \quad j = 1, 2, \ldots, n+1. \tag{4.9}$$

(Indeed, $\frac{(a,\eta) - \beta}{R\eta} = \frac{0 \cdot \eta_1 + \cdots + 1 \cdot \eta_j + 0 \cdot \eta_{j+1} + \cdots - \beta}{R\eta} = \frac{\eta_j - \beta}{\eta R}$). Similarly,

$$\lambda_{n+2} = \frac{\left(\Sigma_{j=1}^{n+1} \eta_j\right) \frac{1}{n+1} - \beta}{\eta\rho}. \tag{4.10}$$

We need to check (4.2), or (using (4.9, and 4.10),

$$\begin{aligned}\frac{1}{R\eta}\left(\Sigma_1^{n+1}\eta_j-(n+1)\beta\right) &= (n+1\mu\left[\frac{1}{n+1}(\Sigma_{j=1}^{n+1}\eta_j)-\beta\right]\frac{1}{\eta\rho}\\ &= \frac{\mu}{\eta\rho}\left(\Sigma_{j=1}^{n+1}\eta_j-(n+1)\beta\right).\end{aligned}$$

But this follows at once from (4.5), i.e., $\rho = R\mu$.

Note that $\gamma \neq 1$ is a natural condition, as for $\gamma = 1$ there is no chance to prove any relation (cf. [7]). This remark applies also for other theorems as well. Thus the proof of Theorem 4.1 is now complete.

**Remark**. The discussion of the "plane" case could be omitted using the remark at the end of Theorem 3.3.

**Theorem 4.2.** *Let $n+1$ spheres $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma \neq 0$, $\gamma \neq 1$, $\gamma \neq \frac{-1}{n}$. Let $\Sigma_{n+2}$ be another sphere which is orthogonal to each of $\{\Sigma_j\}_{j=1}^{n+1}$. Let $\Sigma$ be any sphere of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}$ the inclination of $\{\Sigma_j\}_{j=1}^{n+2}$ with $\Sigma$ respectively. Then*

$$\left(\Sigma_{k=1}^{n+1}\lambda_k\right)^2-\left(n+\frac{1}{\gamma}\right)\Sigma_{k=1}^{n+1}\lambda_k^2=\left(n+\frac{1}{\gamma}\right)(1-\gamma)(\lambda_{n+2}^2-1). \qquad (4.11)$$

*Proof.* Comparing Theorem 4.2 with Theorem 4.1 we see that the condition (4.1) on $\mu$ — appearing in Theorem 4.1 — is now replaced by the orthogonality condition, i.e., $\mu = 0$. Then we get instead of a linear relation (Theorem 3.3) the relation (4.11). Like in the proof of Theorem 4.1 we separate the two cases, namely, $\Sigma$ is a sphere or a plane. Of course we may use various formulas developed for the proof of Theorem 4.1.

Hence we now turn to the first case, i.e., $\Sigma$ is a sphere. We use the same notation as in Theorem 4.1, i.e., the center of $\Sigma$ is $(y_1,\dots,y_{n+1})$, $\Sigma_{j=1}^{n+1}y_j = 1$. Like in the proof of Theorem 4.1, the fact that $\Sigma_{n+1}$ has the same inclination $\mu$ (in this case $\mu = 0$) implies that the coordinates of $\Sigma_{n+2}$ are $\left(\frac{1}{n+1},\dots,\frac{1}{n+1}\right)$. Thus $d_k^2$, the distances of the center of $\Sigma$ from each of the $\{\Sigma_j\}_{j=1}^{n+1}$, satisfy (4.4), i.e., $d_k^2 = \frac{\eta}{n+1}$. Since in our case $\Sigma_{n+2}$ is orthogonal to $\{\Sigma_j\}_{j=1}^{n+1}$, we have $\mu = 0$, or

$$\rho^2+R^2-\frac{n}{n+1}=0. \qquad (4.12)$$

Note that if $\rho = 0$, then $R^2 = \frac{1}{1-\gamma} = \frac{n}{n+1}$ or $\gamma = -\frac{1}{n}$, which explains the condition $\gamma \neq -\frac{1}{n}$. The radius of $\Sigma$ is, again, denoted by $r$. To make the calculation easier, it will be convenient to put the following notation,

$$a = R^2 - \rho^2 - \frac{1}{n+1}, \quad b = \Sigma_{j=1}^{n+1} y_j^2 - r^2 - R^2. \tag{4.13}$$

>From (4.6) and (4.13)

$$\lambda_k = (2y_k - 1 - b)\frac{1}{2rR}, \quad 1 \leq k \leq n+1 \tag{4.14}$$

and (4.7) and (4.14) yield

$$\lambda_{n+2} = \left(\rho^2 - b - R^2 + \frac{1}{n+1}\right)\frac{1}{2r\rho} = \frac{-b-a}{2r\rho}. \tag{4.15}$$

>From (4.14) we easily get (using $\Sigma_{k=1}^{n+1} y_k = 1$)

$$\left(\Sigma_{k=1}^{n+1}\lambda_k\right) = \frac{1}{4r^2R^2}[(n+1)^2b^2 + (n-1)^2 + 2(n-1)(n+1)b]. \tag{4.16}$$

>From (4.13) and (4.14) we have

$$\left(\Sigma_{k=1}^{n+1}\lambda_k^2\right) = \frac{1}{4r^2R^2}[(n+1)b^2 + (n+1) + 2(n+1)b + 4(r^2 + R^2 - 1)]. \tag{4.17}$$

Using the orthogonality condition (4.12) and $\gamma = 1 - \frac{1}{R^2}$ (from (4.3)) we get

$$\rho^2 = \frac{-1-n\gamma}{(n+1)(1-\gamma)}, \quad \frac{\rho^2}{R^2} = \frac{-1-n\gamma}{n+1}. \tag{4.18}$$

>From (4.12) and (4.13), using (4.3) we get

$$a = 2R^2 - 1 = \frac{1+\gamma}{1-\gamma}. \tag{4.19}$$

In order to prove our theorem for the sphere case we have to confirm (4.11).

In view of (4.15), (4.16), (4.17), (4.18) and (4.19) this means checking if

$$\begin{aligned} \frac{1}{4r^2R^2}\big[(n+1)^2b^2 + (n-1)^2 + 2(n-1)(n+1)b\big] - \left(n + \frac{1}{\gamma}\right)\frac{1}{4r^2R^2} \\ \times \big[(n+1)b^2 + (n+1) + 2b(n+1) + 4(r^2 + R^2 - 1)\big] \\ = \left(n + \frac{1}{\gamma}\right)(1-\gamma)\left[\left(\frac{1+\gamma}{1-\gamma} + b\right)^2 \frac{1}{4r^2\rho^2} - 1\right]. \end{aligned}$$

This will be done by equating coefficients.

In order to check the coefficient of $b^2$, we need to confirm $\frac{(n+1)^2}{4r^2R^2} - (n+\frac{1}{\gamma})\frac{1}{4r^2R^2}(n+1) = (n+\frac{1}{\gamma})(1-\gamma)\frac{1}{4r^2\rho^2}$. This is valid in view of (4.18).

Equating the coefficient of $b$, we have

$$\begin{aligned}\frac{2(n-1)(n+1)}{4r^2R^2} \quad &- \quad \left(n+\frac{1}{\gamma}\right)\frac{1}{4r^2R^2}2(n+1) \\ &= \quad \frac{\left(n+\frac{1}{\gamma}\right)(1-\gamma)2\left(\frac{1+\gamma}{1-\gamma}\right)}{4r^2\rho^2}\end{aligned}$$

which is, again, easily confirmed using (4.18).

It remains to check the free coefficient of $b$, i.e.,

$$\begin{aligned}\frac{1}{4r^2R^2}\Big[(n-1)^2 \quad &- \quad (n+\frac{1}{\gamma})\big((n+1)+4(r^2+R^2-1)\big)\Big] \\ &= \quad (n+\frac{1}{\gamma})(1-\gamma)\Big[(\frac{1+\gamma}{1-\gamma})^2\frac{1}{4r^2\rho^2}-1\Big].\end{aligned}$$

We first consider the coefficient of $r^2$ on the left hand side. Then $\frac{-(n+\frac{1}{\gamma})4r^2}{4r^2R^2} = (n+\frac{1}{\gamma})(1-\gamma)(-1)$, since $\frac{1}{R^2} = 1-\gamma$ by (4.3). This cancels $\left(n+\frac{1}{\gamma}\right)(1-\gamma)(-1)$ on the right hand side.

Thus it remains to check

$$(n-1)^2 - \left(n+\frac{1}{\gamma}\right)\Big[(n+1)+4(R^2-1)\Big] = \frac{R^2}{\rho^2}\left(n+\frac{1}{\gamma}\right)(1-\gamma)(\frac{1+\gamma}{1-\gamma})^2.$$

Using (4.3) and (4.18), this is the same as

$$(n-1)^2 - \left(n+\frac{1}{\gamma}\right)\Big[(n+1)+4\left(\frac{1}{1-\gamma}-1\right)\Big] = \left(n+\frac{1}{\gamma}\right)\frac{(1+\gamma)^2}{(1-\gamma)}\cdot\frac{(n+1)}{-(n\gamma+1)}$$

or

$$(n-1)^2\gamma(1-\gamma) - (n\gamma+1)\big[(n+1)-\gamma(n-3)\big] = -(n+1)(1+\gamma)^2,$$

which is easily seen to be an identity. This ends the case of $\Sigma$ equal to a sphere.

We now move on to the case where $\Sigma$ is a plane $P$. The direct proof is omitted, as we can again use the argument given in the remark at the end of Theorem 4.2, showing that there is no loss of generality to assume that $\Sigma$ (i.e., the reference sphere) is really a sphere and not a plane.

**Remark**. It is worthwhile to consider the situation in $R^n$ instead of $G^n$. As Mauldon pointed out (see [7, section 5]), if three (generalized) spheres in $R^n$ have mutual inclination $\lambda$, then necessarily $\lambda > 1$ is impossible. Thus one can use Theorem 3.3 to give a "real" proof of Theorem 4.2, provided $n \geq 2$, since for such $n$ there are at least three spheres having mutual inclination $\lambda$, which cannot be bigger than 1, as explained above, which enables one to use Theorem 3.3. On the other hand, if $n = 1$, $\lambda > 1$ is not excluded as we now show. This means that for $n = 1$ we need to use the "complex" proof to get the real result.

We now show that $\lambda > 1$ is, indeed, possible for $n = 1$. Consider the spheres $S_1 = \{z, (z+a)^2 = R_1^2\}$, $S_2 = \{z, (z-a)^2 = R_2^2\}$, $S_3 = \{z, (z-b)^2 = \rho^2\}$. We have for $\lambda = \lambda(S_1, S_2)$, $\quad \lambda = \frac{R_1^2+R_2^2-4a^2}{2R_1R_2}$.

Comparing with the notation of Theorem 4.2 (for $n = 1$), we have $S_1 = \Sigma_1$, $S_2 = \Sigma_2$, and $S_3 = \Sigma_3$. This means that $S_3$ is orthogonal to $\{S_j\}_{j=1,2}$. In other words, $R_1^2 + \rho^2 = (b+a)^2$, $R_2^2 + \rho^2 = (b-a)^2$. Hence $R_1^2 + R_2^2 - 4a^2 = 2x$ for $x = -a^2 + b^2 - \rho^2$. For $\lambda$ we get $\lambda = \frac{x}{R_1R_2} = \frac{x}{\sqrt{(b+a)^2-\rho^2}\sqrt{(b-a)^2-\rho^2}}$. By a trivial calculation $\frac{1}{\lambda^2} = 1 - \frac{4a^2}{x^2}\rho^2$. >From the definition of $x$, taking $a \neq 0$, $a^2 < b^2$, and $\rho$ small enough, we have $0 < \frac{1}{\lambda^2} < 1, \lambda > 0$, which means $\lambda > 1$. Thus, indeed, $\lambda > 1$ is possible for $n = 1$, in contrast to the other cases, namely, $n \geq 2$, and thus, indeed, in view of the limitations of Theorem 3.3 (i.e., $\lambda \leq 1$), our proof of Theorem 4.2 cannot be "translated" to a real proof for this particular case.

Our next aim is to prove a lemma which will be useful for "translating" inclination theorems to results about radii. In many similar cases we will omit the details of this translation, which is, indeed, very simple.

**Lemma 4.1.** *Let $S_j = \{z, (z-a_j)^2 = R_j^2\}_{j=1,2}$ be two spheres in $G^n$. Let $S_u = \{z, (z-cu)^2 = R^2\}$ be a sphere of reference in $G^n$ such that $c^2 \neq 0$. Denote by $\lambda_j$, $j = 1, 2$, the inclinations of $S_j$ with $S_u$ respectively. Then, if $u \to \infty$, we have*

$$\frac{\lambda_1}{\lambda_2} \to \frac{R_2}{R_1}, \quad n \to \infty. \tag{4.20}$$

*Proof.* Clearly, $\lambda_j = \frac{R^2+R_j^2-(a_j-cu)^2}{2RR_j}$, $j = 1, 2$. Hence $\frac{\lambda_1}{\lambda_2} = \frac{R_2}{R_1} \cdot \frac{R^2+R_1^2-(a_1-cu)^2}{R^2+R_2^2-(a_2-cu)^2}$. But $\frac{R^2+R_1^2-(a_1^2+c^2u^2-2(a_1,cu))}{R^2+R_2^2-(a_2^2+c^2u^2-2(a_2,cu))} \to 1$, as $u \to \infty$ (since $c^2 \neq 0$). This ends the proof.

Our next theorem is a corollary of Theorem 4.2. We use the concept of Poincaré extension [12], or more precisely, the complex version of it, which is of the same nature as the real case. We note that instead of using our approach, we could give a direct proof which is very similar to the proof of Theorem 4.2. The present proof is considerably shorter.

Lemma 4.1 will be used to prove, from our next theorem, the complex version of Mauldon's theorem. (Using Lemma 4.1, one can give a "radii" version of Theorem 4.2 as well, but we omit the simple details.)

**Theorem 4.3.** *Let $\{\Sigma_j\}_{j=1}^{n+2}$ be $n+2$ spheres in $G^n$ having mutual inclination $\gamma \neq 0$, $\gamma \neq 1$. Let $\Sigma$ be another sphere in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$ with $\Sigma$ respectively. Then*

$$\left(\Sigma_{k=1}^{n+2}\lambda_k\right)^2 - \left(n+1+\frac{1}{\gamma}\right)\Sigma_{k=1}^{n+2}\lambda_k^2 = \left(n+1+\frac{1}{\gamma}\right)(\gamma-1). \qquad (4.21)$$

*Proof.* It will be more convenient to prove the theorem for $G^{n-1}$ and use Theorem 4.2 for $G^n$. Hence, let $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^{n-1}$ having mutual inclination $\gamma \neq 1$. $\Sigma$ is another sphere in $G^{n-1}$ and $\{\lambda_j\}_{j=1}^{n+1}$ the inclinations with respect to $\Sigma$. We now use the Poincaré extension from $G^{n-1}$ to $G^n$, and consider the extended spheres $\{\Sigma_j\}_{j=1}^{n+1}$ and $\Sigma$ in $G^n$. (See Figure 9.)

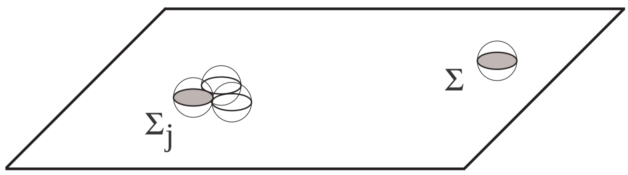


Figure 9: Poincare' extension.

Clearly, the inclinations of the extended $\{\Sigma_j\}_{j=1}^{n+1}$ with the extended $\Sigma$ remain the same, i.e., $\{\lambda_j\}_{j=1}^{n+1}$. Also, it is clear that $z_n = 0$ is an orthogonal plane to each of the extended $\{\Sigma_j\}_{j=1}^{n+1}$. It turns out that $\sigma$, the inclination of $z_n = 0$ with respect to the extended $\Sigma$, is also zero because of our construction. We are now in a position to apply Theorem 4.2 for the $n+1$ extended spheres $\{\Sigma_j\}_{j=1}^{n+1}$, the orthogonal sphere $z_n = 0$ to each of the (extended) $\{\Sigma_j\}_{j=1}^{n+1}$, and the sphere of reference (extended) $\Sigma$. Hence, from Theorem 4.2 (noting that the notation $\sigma$ replaces $\lambda_{n+2}$ in Theorem 4.2 and also that $\sigma = 0$), we get (from (4.11) using $\sigma = 0$)

$$\left(\Sigma_{k=1}^{n+1}\lambda_k\right)^2 - \left(n+\frac{1}{\gamma}\right)\Sigma_{k=1}^{n+1}\lambda_k^2 = -\left(n+\frac{1}{\gamma}\right)(1-\gamma) = (\gamma-1)\left(n+\frac{1}{\gamma}\right).$$

But this is exactly (4.21), replacing $n-1$ by $n$. This ends the proof of Theorem 4.3.

**Remark**. Since we have used Theorem 4.2, one might think that we have the limitation $n+1+\frac{1}{\gamma}\neq 0$. But since $\rho\neq 0$ for the case $n+1+\frac{1}{\gamma}=0$, we can use a simple continuity argument to complete the proof for this case as well.

We next apply Theorem 4.3 and Lemma 4.1 to get the complex version of Mauldon's theorem.

**Theorem 4.4.** *(Complex version of Mauldon's theorem). Let $\{S_j\}_{j=1}^{n+2}$ be $n+2$ spheres in $G^n$ having mutual inclination $\gamma\neq 0$, $\gamma\neq 1$. Denote their radii by $\{R_j\}_{j=1}^{n+2}$ respectively. Then*

$$\left(\Sigma_{j=1}^{n+2}\frac{1}{R_j}\right)^2-\left(n+1+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+2}\frac{1}{R_j^2}=0. \tag{4.22}$$

*Proof.* Using (4.21) we have

$$\begin{aligned}\left(\Sigma_{k=1}^{n+2}\frac{\lambda_k}{\lambda_{n+2}}\right)^2 &- \left(n+1+\frac{1}{\gamma}\right)\Sigma_{k=1}^{n+2}\left(\frac{\lambda_k}{\lambda_{n+2}}\right)^2\\ &= \left(n+1+\frac{1}{\gamma}\right)(\gamma-1)\frac{1}{\lambda_{n+2}^2}.\end{aligned}$$

Now taking the sphere of reference as in Lemma 4.1, and letting $u\to\infty$, we have $\frac{1}{\lambda_{n+2}}\to 0$, $\frac{\lambda_k}{\lambda_{n+2}}\to\frac{R_{n+2}}{R_k}$ for $k=1,2,\ldots,n+1$, and we get (4.22). For $\lambda=-1$, i.e., the tangency case, we get from (4.22)

$$\left(\Sigma_{j=1}^{n+2}\frac{1}{R_j}\right)^2-n\Sigma_{j=1}^{n+2}\frac{1}{R_j^2}=0. \tag{4.23}$$

This is the complex version of Gosset's theorem. For $n=2$ (i.e.., $G^2$) we get

$$\left(\Sigma_{j=1}^{4}\frac{1}{R_j}\right)^2-2\Sigma_{j=1}^{4}\frac{1}{R_j^2}=0, \tag{4.24}$$

which is the complex version of Descartes theorem.

Later on, we will give some more inclination theorems that will be, in fact, generalizations of the results in section 4. These generalizations will be concerned with two spheres of references instead of one, like in the above. But before doing so, we want to use some of our results to deduce new facts concerning hyperbolic space and its connection to inclinations.

## 5 Inclination and hyperbolic space

Let $(z_1, \ldots, z_n, z_{n+1}) \in G^{n+1}$. We introduce the notation

$$z_j = x_j + it_j, \quad 1 \le j \le n, \quad z_{n+1} = ix_0 + t_0 \tag{5.1}$$

for $\{x_j\}_{j=0}^n$, $\{t_j\}_{j=0}^n$ all real numbers. Consider the sphere $S = \big\{z, \Sigma_{j=1}^{n+1} z_j^2 = R^2$,
$R = i\big\}$. We consider the subspace of $G^{n+1}$ by putting the restrictions $t_j = 0$, $0 \le j \le n$. Then, in this subspace, $S$ is reduced to

$$\Sigma_{j=1}^n x_j^2 + (ix_0)^2 = i^2 = -1. \tag{5.2}$$

This is the same as

$$1 + \Sigma_{j=1}^n x_j^2 = x_0^2, \quad \Sigma_{j=1}^n x_j^2 = (x_0 - 1)(x_0 + 1). \tag{5.3}$$

This means that $S$ is "projected" onto a hyperboloid of the form (5.3). We recall the transformation [12]

$$x_j = y_j(1 + x_0), \ 1 \le j \le n. \tag{5.4}$$

>From (5.3) and (5.4) we get

$$\Sigma_{j=1}^n y_j^2 = \frac{x_0 - 1}{x_0 + 1}. \tag{5.5}$$

Consider now another sphere centered at $(\mu_1, \ldots, \mu_n, i\mu_0)$ and with radius $R$. Then

$$\Sigma_{j=1}^n (x_j - \mu_j)^2 - (x_0 - \mu_0)^2 = R^2, \tag{5.6}$$

which is another hyperboloid.

We will be particularly interested in the case of orthogonality of the two spheres introduced above, namely,

$$i^2 + R^2 = d^2 = \Sigma_{j=1}^n \mu_j^2 + (i\mu_0)^2, \quad R^2 - 1 = \Sigma_{j=1}^n \mu_j^2 - \mu_0^2. \tag{5.7}$$

Following the standard notation, (cf. [12]), we denote the (real) hyperbolic $n$ dimensional plane by $\Delta^n$. One of our aims in this section is to give a new proof for a theorem of Mauldon [7], i.e., the hyperbolic version of inclination theorem in $\Delta^n$. Our different proof will motivate some new insight later in the present paper.

**Theorem 5.1.** *(Mauldon hyperbolic inclination theorem). Let $n+2$ spheres of $\{S_1, S_2, \dots, S_{n+2}\}$ be given in the hyperbolic plane $\Delta^n$. Denote by $\{\beta_j\}_{j=1}^{n+2}$ the hyperbolic radii of $\{S_j\}_{j=1}^{n+2}$ respectively. Assume further that $\{S_j\}_{j=1}^{n+2}$ have mutual inclination $\gamma \neq 0$, $\gamma \neq 1$. Then*

$$\left(\Sigma_{j=1}^{n+2} \frac{1}{\tan h\beta_j}\right)^2 - \left(n + \frac{1}{\gamma} + 1\right)\Sigma_{j=1}^{n+2} \frac{1}{\tan h^2\beta_j} = \left(n + 1 + \frac{1}{\gamma}\right)(\gamma - 1). \quad (5.8)$$

For the proof of Theorem 5.1 we will need some preliminary results. First, we need the "translation" of Theorem 4.2 to the radii version (see Lemma 4.1). Following the notation of Theorem 4.2, and using it for $n+1$ instead of $n$, we further denote the radii of $\{\Sigma_j\}_{j=1}^{n+2}$ by $\{R_j\}_{j=1}^{n+2}$ respectively. Also, for our purposes we shall need to take the radius of $\Sigma$ to be $i$. Then the "radii" version of (4.11) (replacing $n$ by $n+1$, and noting that $R_{n+2} = i$),

$$\left(\Sigma_{j=1}^{n+2} \frac{1}{R_j}\right)^2 - \left(n + 1 + \frac{1}{\gamma}\right)\Sigma_{j=1}^{n+2} \frac{1}{R_j^2} = \left(n + 1 + \frac{1}{\gamma}\right)(\gamma - 1). \quad (5.9)$$

The idea of the proof of Theorem 5.1 will be by transformation of (5.9) to $\Delta^n$, using (5.4). Our aim is to derive (5.8) from (5.9). In order to do that we need to show first that the inclination in the original space $G^{n+1}$ remains invariant, i.e., that it is the same after the transformation into $\Delta^n$. As a matter of fact, this is generally not true, but if one uses the orthogonality conditions (5.7), the result is valid. Thus we have

**Lemma 5.1.** *Let $S_1, S_2$ be two "projected" spheres*

$$\begin{aligned} \Sigma_{j=1}^{n}(x_j - \mu_j)^2 + (ix_0 - i\mu_0)^2 &= R^2, \\ \Sigma_{j=1}^{n}(x_j - \eta_j)^2 + (ix_0 - i\eta_0)^2 &= \rho^2, \end{aligned} \quad (5.10)$$

*where both spheres are orthogonal to the "projected" sphere $\Sigma_{j=1}^{n} x_j^2 + (ix_0)^2 = i^2 = -1$. Assume further that these two spheres are transformed, as explained above, to two new spheres in $\Delta^n$. Then the inclination $\lambda$ between $S_1$ and $S_2$ is the same as the inclination $\gamma$ between their images in $\Delta^n$.*

*Proof.* It will be convenient to introduce the notation

$$\mu^2 = \Sigma_{j=1}^{n}\mu_j^2, \quad \eta^2 = \Sigma_{j=1}^{n}\eta_j^2, \quad (\mu, \eta) = \Sigma_{j=1}^{n}\mu_j\eta_j. \quad (5.11)$$

By the definition of $\lambda$ we have

$$\lambda = \lambda(S_1, S_2) = \frac{\rho^2 + R^2 - d^2}{2\rho R} = \frac{\rho^2 + R^2 - \left[\Sigma_{j=1}^{n}(\mu_j - \eta_j)^2 + (i\mu_0 - i\eta_0)^2\right]}{2\rho R}.$$

Hence
$$\lambda = \frac{\rho^2 + R^2 - \mu^2 - \eta^2 + 2(\mu,\eta) + \mu_0^2 + \eta_0^2 - 2\mu_0\eta_0}{2\rho R}.$$
Using the orthogonality condition (5.7) applied to $S_j, j = 1, 2$, we have
$$R^2 - 1 = \mu^2 - \mu_0^2, \quad \rho^2 - 1 = \eta^2 - \eta_0^2. \tag{5.12}$$
Putting this in the above expression for $\lambda$, we get
$$\lambda = \frac{2\mu,\eta) - 2\mu_0\eta_0 + 2}{2\rho R} = \frac{1 + (\mu,\eta) - \mu_0\eta_0}{\rho R}.$$
Again using (5.12) for $\rho$ and $R$, we have
$$\lambda = \frac{1 + (\mu,\eta) - \mu_0\eta_0}{\sqrt{1 + \mu^2 - \mu_0^2}\sqrt{1 + \eta^2 - \eta_0^2}}. \tag{5.13}$$
In order to confirm that $\lambda$ is equal to $\gamma$, i.e., the inclination between the images, we first need to find expression for the images of $S_j, j = 1, 2$. We have from (5.10) $\Sigma_{j=1}^n x_j^2 + \mu^2 - 2(x,\mu) - x_0^2 - \mu_0^2 + 2x_0\mu_0 = R^2$. We now use the transformation (5.3) to get $\mu^2 - 1 - 2(x,\mu) - \mu_0^2 + 2x_0\mu_0 = R^2$. By (5.12) $R^2 + \mu_0^2 - \mu^2 = 1$. Hence $-2 - 2(x,\mu) + 2x_0\mu_0 = 0$. Using (5.4) we have $0 = -1 - (x,\mu) + x_0\mu_0 = -1 - (1 + x_0)(y,\mu) + x_0\mu_0$. Denote
$$\tau = (y,\mu). \tag{5.14}$$
We have from (5.14) and the above
$$x_0 = \frac{1+\tau}{\mu_0 - \tau}. \tag{5.15}$$
>From (5.5) and (5.15) we easily get
$$\Sigma_{j=1}^n y_j^2 = \frac{2\tau}{1+\mu_0} + \frac{1-\mu_0}{1+\mu_0}. \tag{5.16}$$
>From (5.14) and (5.16) we get at once
$$\Sigma_{j=1}^n \left( y_j - \frac{\mu_j}{1+\mu_0} \right)^2 = \frac{1-\mu_0}{1+\mu_0} + \frac{\mu^2}{(1+\mu_0)^2} = \frac{1 - \mu_0^2 + \mu^2}{(1+\mu_0)^2}.$$
Using (5.12) we finally get
$$\Sigma_{j=1}^n \left( y_j - \frac{\mu_j}{1+\mu_0} \right)^2 = \left( \frac{R}{1+\mu_0} \right)^2, \tag{5.17}$$

which is the equation of the image sphere of $S_1$ in $\Delta^n$. In other words, the new radius is $\frac{R}{1+\mu_0}$ and the new center is $\left(\frac{\mu_1}{1+\mu_0}, \ldots, \frac{\mu_n}{1+\mu_0}\right)$. Similarly,

$$\Sigma_{j=1}^{n}\left(y_j - \frac{\eta_j}{1+\eta_0}\right)^2 = \left(\frac{\rho}{1+\eta_0}\right)^2 \tag{5.17'}$$

is the image of $S_2$ in $\Delta^n$.

Our aim is now to find $\gamma$, the inclination between these two images and confirm that $\gamma = \lambda$. For $\gamma$ we have

$$\gamma = \frac{\left(\frac{R}{1+\mu_0}\right)^2 + \left(\frac{\rho}{1+\eta_0}\right)^2 - \Sigma_{j=1}^{n}\left(\frac{\eta_j}{1+\eta_0} - \frac{\mu_j}{1+\mu_0}\right)^2}{2\left(\frac{R}{1+\mu_0}\right)\left(\frac{\rho}{1+\mu_0}\right)}$$

or

$$\gamma = \frac{\frac{R^2}{1+\mu_0} + \frac{\rho^2}{(1+\eta_0)^2} + \frac{2(\eta,\mu)}{(1+\eta_0)(1+\mu_0)} - \frac{\eta^2}{(1+\eta_0)^2} - \frac{\mu^2}{(1+\mu_0)^2}}{2\frac{R}{(1+\mu_0)} \cdot \frac{\rho}{(1+\eta_0)}}.$$

From (5.12) we have

$$\frac{R^2}{(1+\mu_0)^2} - \frac{\mu^2}{(1+\mu_0)^2} = \frac{1-\mu_0^2}{(1+\mu_0)^2} = \frac{1-\mu_0}{1+\mu_0}.$$

Similarly, $\frac{\rho^2}{(1+\eta_0)^2} - \frac{\eta^2}{(1+\eta_0)^2} = \frac{1-\eta_0}{1+\eta_0}$. Hence we get for $\gamma$

$$\gamma = \frac{\frac{1-\eta_0}{1+\eta_0} + \frac{1-\mu_0}{1+\mu_0} + \frac{2(\eta,\mu)}{(1+\eta_0)(1+\mu_0)}}{2\frac{R}{1+\mu_0} \cdot \frac{\rho}{1+\eta_0}},$$

or

$$\gamma = \frac{1-\eta_0\mu_0 + (\eta,\mu)}{R\rho} = \frac{1+(\mu,\eta)-\mu_0\eta_0}{\sqrt{1+\mu^2-\mu_0^2}\sqrt{1+\eta^2-\eta_0^2}}.$$

Hence $\gamma = \lambda$ by (5.13) and the proof of Lemma 5.1 is now complete.

Next, we shall find the connection between $R$ and the hyperbolic radius of its image in $\Delta^n$.

**Lemma 5.2.** *With the above notations the hyperbolic radius $\beta$ of the image sphere of $S_1$, given by (5.17), satisfies*

$$R = \tan h\beta. \tag{5.18}$$

*Proof.* Recalling that the new radius is $\frac{R}{1+\mu_0}$ and the new center is $\frac{\mu}{1+\mu_0}$ (see (5.17)), we have for $D$, the hyperbolic diameter of the image sphere,

$$D = \ell n \left[ \frac{1 + \left(\frac{\mu+R}{1+\mu_0}\right)}{1 - \left(\frac{\mu+R}{1+\mu_0}\right)} \cdot \frac{1 - \left(\frac{\mu-R}{1+\mu_0}\right)}{1 + \left(\frac{\mu-R}{1+\mu_0}\right)} \right]$$

(see Figure 10).

(By distance we mean, Euclidean)
Distance of $A$, the center, from the origin is $\frac{\mu}{1+\mu_0}$
Distance of $B$ from the origin is $\frac{\mu-R}{1+\mu_0}$
Distance of $C$ from the origin is $\frac{\mu+R}{1+\mu_0}$

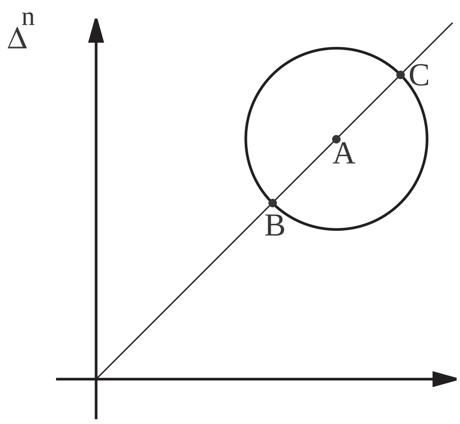


Figure 10:

Hence, $D = \ell n \left[ \frac{(1+\mu_0+\mu+R)}{(1+\mu_0-\mu-R)} \cdot \frac{(1+\mu_0-\mu+R)}{(1+\mu_0+\mu-R)} \right]$. But

$$\begin{aligned} (1+\mu_0+R+\mu)(1+\mu_0+R-\mu) &= (1+\mu_0+R)^2-\mu^2 \\ &= 1+\mu_0^2+R^2-\mu^2+2\mu_0+2R+2\mu_0R \\ &= 2+2\mu_0+2R+2\mu_0R \end{aligned}$$

where we have used (5.12). Similarly,

$$(1+\mu_0-\mu-R)(1+\mu_0+\mu-R) = 2+2\mu_0-2R-2\mu_0R.$$

Hence, for the diameter $D$,

$$D = \ell n \frac{1+\mu_0+R+\mu_0 R}{1+\mu_0-R-\mu_0 R} = \ell n \frac{(1+\mu_0)(1+R)}{(1+\mu_0)(1-R)} = \ell n \frac{1+R}{1-R}.$$

Thus we finally get for the hyperbolic radius $\beta = \frac{1}{2}D$,

$$\beta = \frac{1}{2}\ell n \frac{1+R}{1-R}. \tag{5.19}$$

>From (5.19) we get (5.18) by a simple calculation. (Indeed, $\frac{1+R}{1-R} = e^{2\beta}$ or $R = \frac{e^{2\beta}-1}{e^{2\beta}+1} = \frac{e^{\beta}-e^{-\beta}}{e^{\beta}+e^{-\beta}}$.) This ends the proof of Lemma 5.2. Having Lemma 5.1 and Lemma 5.2 at hand, it is now easy to prove our theorem.

**Proof of Theorem 5.1**. By the conditions of Theorem 5.1, we are given $n+2$ spheres $\{S_1, S_2, \ldots, S_{n+2}\}$ in $\Delta^n$ with hyperbolic radii $\{\beta_1, \ldots, \beta_{n+2}\}$ respectively. Also, it is given that the mutual inclination $\gamma$ satisfies $\gamma \neq 1$. Assume for a moment that this set of $n+2$ can be realized as images of $n+2$ spheres having radii $\{R_j\}_{j=1}^{n+2}$ respectively. Using Lemma 5.1 we have that these $n+2$ spheres have mutual inclination $\lambda$, $\lambda = \gamma$. Also, by Lemma 5.2, applied for each sphere of the set of $n+2$ spheres, we have by (5.18) that

$$R_j = \tan h\beta_j, \quad 1 \leq j \leq n+2. \tag{5.18'}$$

Hence (5.8) follows at once. It remains to show that realization is, indeed, possible. So let $S$ be a given sphere in $\Delta^n$, with the notation

$$\Sigma_{j=1}^{n}(y_j - \xi_j)^2 = r^2. \tag{5.20}$$

It has to be shown that the sphere can be viewed as an image of a suitable sphere in $G^n$. This means that we have to check the conditions

$$\xi_j = \frac{\mu_j}{1+\mu_0}, \quad 1 \leq j \leq n, \quad r^2 = \frac{R^2}{(1+\mu_0)^2}, \tag{5.21}$$

such that, in addition, the orthogonality condition (5.7) is satisfied, namely, $R^2 = 1 + \mu^2 - \mu_0^2$ for $\mu^2 = \Sigma_{i=1}^{n}\mu_j^2$. Thus, in view of (5.21)

$$R^2 = (1+\mu_0)^2 r^2 = 1 + \mu^2 - \mu_0^2 = 1 + \xi^2(1+\mu_0)^2 - \mu_0^2$$

(where $\xi^2 = \Sigma_{j=1}^{n}\xi_j^2$). Solving for $\mu_0$ we get at once

$$\mu_0 = \frac{-(r^2-\xi^2) \pm \sqrt{(r^2-\xi^2)^2 - [(r^2-\xi^2)^2 - 1]}}{\gamma^2 - \xi^2 + 1},$$

or

$$\mu_0 = \frac{-r^2 + \xi^2 \pm 1}{r^2 - \xi^2 + 1}.$$

Hence the two solutions, $\mu_0 = -1$ and $\mu_0 = \frac{-r^2+\xi^2+1}{r^2-\xi^2+1}$. $\mu_0 = -1$ is excluded, as $\mu_0 + 1 \neq 0$ is a necessary condition by (5.21). The other solution is possible, as obviously $1 + \left(\frac{1+\xi^2-r^2}{r^2-\xi^2+1}\right) = \mu_0 + 1 \neq 0$.

To complete the discussion, note also that the denominator $r^2 - \xi^2 + 1$ cannot be equal to zero, as $r^2 = \xi^2 - 1$ leads to a contradiction. Indeed, by construction, $\xi^2 < 1$ and thus $r^2 < 0$, which is not possible. This ends the proof of the theorem.

**Remark**. It is worthwhile to note that all $R_j$ satisfy $R_j < 1$. Indeed, this follows from (5.18′). It is possible to give a direct proof of this fact. We differentiate between two cases: $\mu_0 \leq |\mu|$, $\mu_0 > |\mu|$. If $\mu_0 \leq |\mu|$, then $|\xi| + \frac{R}{(1+\mu_0)} = \frac{|\mu|+R}{1+\mu_0} < 1$ yields $R < 1 + \mu_0 - |\mu| \leq 1$. If $\mu_0 > |\mu|$, then the orthogonality condition $R^2 = 1 + \mu^2 - \mu_0^2$ implies $R < 1$. This remark raises the question about equivalence of (5.8) and (5.9). In fact, in (5.9) no restriction is needed on $R_j$. One is led to the question: *Is it possible to give a hyperbolic translation of (5.9) in the general case as well (i.e., without the limitation $R_j < 1$, $1 \leq j \leq n+2$)?* Later on we will again discuss the issue and other matters concerning hyperbolic space and inclinations. In the meantime we go back to the previous line of reasoning.

# 6 Further results on inclinations

In this section we aim to deal with various topics. First, we plan to generalize some known results on radii to results on inclinations. This will help us later on to discuss further the BOIP (bowl of integer property) and inclinations. In addition, we will discuss Apollonian packing, dual Apollonian packing and super Apollonian packing in connection with inclinations.

## 6.1 Some connections between Theorems 4.2 and 4.3

We recall that Theorem 4.2 deals with $n+1$ spheres in $G^n$ having mutual inclination $\gamma \neq 1$, and an orthogonal sphere $\Sigma$ to all of them. Theorem 4.3 deals with $n+2$ spheres in $G^n$ having mutual inclination $\gamma \neq 1$. First note that $\lambda_{n+2}$ is a notation of two different quantities in Theorem 4.2 and Theorem 4.3. In order to avoid confusion, we first change the notation in

Theorem 4.2 and replace $\lambda_{n+2}$ appearing there by $\sigma$. Hence $\sigma$ is now the inclination of the orthogonal sphere with the reference sphere. $\lambda_{n+2}$ is (as before) the inclination of the $n+2^{th}$ sphere with the reference sphere. Hence, with the new notation we have instead of (4.11),

$$\left(\Sigma_{k=1}^{n+1}\lambda_k\right)^2 - \left(n+\frac{1}{\gamma}\right)\Sigma_{k=1}^{n+1}\lambda_k^2 = \left(n+\frac{1}{\gamma}\right)(1-\gamma)(\sigma^2-1). \tag{4.11'}$$

We have

**Theorem 6.1.** *Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+2}$ be $n+2$ spheres in $G^n$ having mutual inclination $\gamma \neq 0$, $\gamma \neq 1$, $n+\frac{1}{\gamma} \neq 0$. Let $\Sigma_0$ be orthogonal to each of $\{\Sigma_j\}_{j=1}^{n+1}$. Let $\Sigma$ be a sphere of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$ with $\Sigma$ respectively. Denote by $\sigma$ the inclination of $\Sigma_0$ with $\Sigma$. Then*

$$\lambda_{n+2} = \frac{\Sigma_{j=1}^{n+1}\lambda_j \pm \sigma\sqrt{\left(n+1+\frac{1}{\gamma}\right)\left(n+\frac{1}{\gamma}\right)(1-\gamma)}}{n+\frac{1}{\gamma}}. \tag{6.1}$$

*Proof.* It will be convenient to denote

$$a = \Sigma_{k=1}^{n+1}\lambda_k, \quad b = \Sigma_{k=1}^{n+1}\lambda_k^2. \tag{6.2}$$

>From Theorem 4.3 and (4.21), we have with the notation of (6.2),

$$(a+\lambda_{n+2})^2 - \left(n+1+\frac{1}{\gamma}\right)(b+\lambda_{n+2}^2) = \left(n+1+\frac{1}{\gamma}\right)(\gamma-1).$$

Solving this quadratic equation for $\lambda_{n+2}$ we get

$$\lambda_{n+2} = \frac{a \pm \sqrt{\left(n+1+\frac{1}{\gamma}\right)\left[a^2 - \left(n+\frac{1}{\gamma}\right)b + \left(n+\frac{1}{\gamma}\right)(1-\gamma)\right]}}{n+\frac{1}{\gamma}}. \tag{6.3}$$

>From $(4.11')$ we get

$$a^2 - \left(n+\frac{1}{\gamma}\right)b + \left(n+\frac{1}{\gamma}\right)(1-\gamma) = \left(n+\frac{1}{\gamma}\right)(1-\gamma)\sigma^2.$$

Substituting this expression in (6.3) we have

$$\lambda_{n+2} = \frac{1}{n+\frac{1}{\gamma}}\left[a \pm \sqrt{\left(n+1+\frac{1}{\gamma}\right)\left(n+\frac{1}{\gamma}\right)(1-\gamma)\sigma^2}\,\right],$$

which is another form of (6.1). This ends the proof of Theorem 6.1.

We now specialize (6.1) for a particular case.

Let $n = 2$ and $\gamma = -1$ (tangency case). We get,

$$\lambda_4 = \frac{\lambda_1 + \lambda_2 + \lambda_3 \pm \sigma\sqrt{(3-1)(2-1)\cdot 2}}{2-1}$$

or

$$\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3 \pm 2\sigma. \tag{6.4}$$

Also, for this particular case (i.e., $n = 1, \gamma = -1$) we have for $\sigma$ (from (4.11$'$), $(\lambda_1 + \lambda_2 + \lambda_3)^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2 + 2(\sigma^2 - 1)$. Hence

$$\sigma^2 = \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 1. \tag{6.5}$$

>From (6.4) and (6.5) we have

$$\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3 \pm 2\sqrt{\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 + 1}. \tag{6.6}$$

Using our standard procedure to "pass" to radii, we use, as usual, Lemma 4.1. Then from (6.6),

$$\frac{\lambda_4}{\lambda_1} = 1 + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1} \pm 2\sqrt{\frac{\lambda_2}{\lambda_1} + \frac{\lambda_3}{\lambda_1} + \frac{\lambda_2\lambda_3}{\lambda_1^2} + \frac{1}{\lambda_1^2}},$$

and letting $u \to \infty$ in Lemma 4.1

$$\frac{R_1}{R_4} = 1 + \frac{R_1}{R_2} + \frac{R_1}{R_3} \pm 2\sqrt{\frac{R_1}{R_2} + \frac{R_1}{R_3} + \frac{R_1^2}{R_2R_3}}$$

and dividing by $R_1$,

$$\frac{R_1}{R_4} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \pm 2\sqrt{\frac{1}{R_1R_2} + \frac{1}{R_1R_3} + \frac{1}{R_2R_3}}. \tag{6.7}$$

Here (6.7) is the complex form of Descartes 4 circle theorem. Similarly, from (6.5), passing to radii and denoting by $\rho$, we get the radius of the orthogonal sphere $\Sigma_0$,

$$\frac{1}{\rho} = \pm\sqrt{\frac{1}{R_1R_2} + \frac{1}{R_1R_3} + \frac{1}{R_2R_3}}. \tag{6.8}$$

We now point out an important result that follows very easily from Theorem 6.1.

**Corollary 6.1.** *Let $\Sigma_1, \Sigma_2, \ldots, \Sigma_{n+1}$ be $n+1$ spheres in $G^n$, having mutual inclination $\gamma$, $\gamma \neq 0$ $\gamma \neq 1$, $\gamma \neq \frac{-1}{n}$. Let $\Sigma^1_{n+2}, \Sigma^2_{n+2}$ be two additional spheres, each having inclination $\gamma$ with $\{\Sigma_j\}_{j=1}^{n+2}$. Let $\Sigma$ be a sphere of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+1}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+1}$ with $\Sigma$. Also denote by $\lambda^1_{n+2}, \lambda^2_{n+2}$ the inclinations of $\Sigma^1_{n+2}, \Sigma^2_{n+2}$ with $\Sigma$ respectively. Then*

$$\lambda^1_{n+2} + \lambda^2_{n+2} = 2\frac{\Sigma_{j=1}^{n+1}\lambda_j}{n+\frac{1}{\gamma}}. \tag{6.9}$$

*Proof.* The result follows at once from (6.1), adding the two posssible solutions for $\lambda_{n+2}$.

Denoting by $\{x_j\}_{j=1}^{n+1}$ the "bends" (i.e., reciprocal of radii) of $\{\Sigma_j\}_{j=1}^{n+1}$ respectively, and by $x^j_{n+2}, j = 1, 2$, the bends of $\Sigma^j_{n+2}, j = 1, 2$ respectively, we get from (6.9)

$$x^1_{n+2} + x^2_{n+2} = 2\frac{1}{n+\frac{1}{\gamma}}\left(\Sigma_{j=1}^{n+1}x_j\right). \tag{6.10}$$

This is the complex form of a known result in the real case (cf. [13]).

## 6.2 Matrix approach and inclinations for Apollonian packing in $R^2$

We now specialize our discussion for the real case. Moreover, we take $\gamma = -1$, i.e., the tangency case, and also $n = 2$. This is the setting for creating the Apollonian packing. We start by recalling a few basic known facts. Let $\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3$ be four mutually tangent (generalized) spheres in $R^2$. We also assume that their interiors are disjoint. Given $\Sigma_0, \Sigma_1, \Sigma_2$, then in addition to $\Sigma_3$ there is another tangent sphere to $\{\Sigma_j\}_{j=0}^{2}$, say, $\Sigma_{-1}$ (again, with disjoint interior). Given $\Sigma_1, \Sigma_2, \Sigma_3$, we have similarly, in addition to $\Sigma_0$, another sphere, say $\Sigma_4$, also tangent to $\Sigma_1, \Sigma_2, \Sigma_3$, and having disjoint interior with each of them. We may say that after putting $\Sigma_4$ in the "hole" created by $\{\Sigma_j\}_{j=1}^{3}$, three new "holes" are created and we can "put" three additional spheres in these holes. If we continue with this procedure for ever, we get the *Apollonian packing* in $R^2$ (see Figure 11).

In the situation described in Figure 11, we have for the straight lines, $\Sigma_0, \Sigma_1$, that the bends of both of them are zero, i.e., $x_0 = x_1 = 0$. Denoting the bends of $\Sigma_j$ by $x_j$, we now describe a known technique (see [14]) of operating with three different matrices in connection with "filling" the three "holes" as described above. To be more specific, let $\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4$ be the

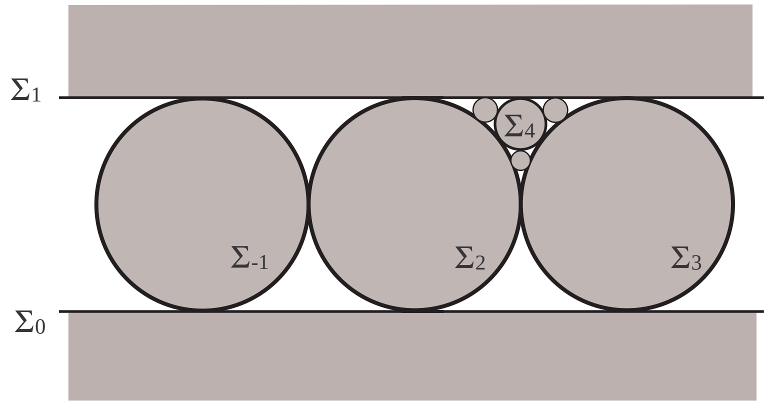


Figure 11: A special case of Appolonian Packing.

spheres described as above. Thus we may say that $\Sigma_4$ "fills" the "hole" created by $\Sigma_1, \Sigma_2, \Sigma_3$. Three new holes are created, i.e., by $\{\Sigma_1, \Sigma_2, \Sigma_4\}$, $\{\Sigma_1, \Sigma_3, \Sigma_4\}, \{\Sigma_2, \Sigma_3, \Sigma_4\}$. To simplify notation, we denote by $\Sigma_5$ the sphere which "fills" the "hole" for each of the three cases. Thus, we arrive at the following three options:

$$\begin{aligned}
(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) &\rightarrow (\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5) \\
(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) &\rightarrow (\Sigma_1, \Sigma_2, \Sigma_4, \Sigma_5) \\
(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) &\rightarrow (\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5).
\end{aligned}$$

We first analyze the case $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \rightarrow (\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5)$. Since $\{\Sigma_1, \Sigma_3, \Sigma_4\}$ are mutually tangent to each other, and $\Sigma_1, \Sigma_5$ are the two options to "complete" this set to four mutually disjoint four spheres, we get from (6.7) (or (6.10) for $n = 2$ and $\gamma = -1$) that

$$x_2 + x_5 = 2(x_1 + x_3 + x_4). \tag{6.11}$$

This may be written as a matrix form, namely,

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{6.12}$$

Similarly, the case $\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\} \to \{\Sigma_1, \Sigma_2, \Sigma_4, \Sigma_5\}$ may be described by

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{6.13}$$

and the third case, namely, $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \to (\Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5)$ may be described by

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 2 & 2 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}. \tag{6.14}$$

The three cases will be called the "Ring Lemma Case" for (6.12), the "Knife Case" for (6.13), and the "Spiral Case" for (6.14) (cf. [4] for motivation of these names).

A similar description to the above may be given with the aid of the "dual Apollonian packing". We then also get three matrices, but with nonnegative entries, which is sometimes more convenient. The "dual Apollonian packing" is simply the Apollonian packing created from a given Apollonian packing, by taking all orthogonal spheres. This means creating, for each three mutually tangent spheres in the given Apollonian packing, the orthogonal sphere to all three. Denoting these three spheres by $\Sigma_1, \Sigma_2, \Sigma_3$ and their bends by $x_1, x_2, x_3$ respectively, and the bend of the orthogonal sphere by $\sigma_{(123)}$, we get from (6.7) and (6.8),

$$x_4 = x_1 + x_2 + x_3 + 2\sigma_{(123)}. \tag{6.15}$$

Also,

$$x_3 = x_1 + x_2 + x_4 - 2\sigma_{(124)} \tag{6.16}$$

(with an obvious notation).

>From (6.15) and (6.16) we at once get

$$\sigma_{(124)} = x_1 + x_2 + \sigma_{(123)} \tag{6.17}$$

(see Figure 12).

$\Sigma_5$ and $\Sigma_3$ are both tangent to $\Sigma_1, \Sigma_2, \Sigma_4$.
$\Sigma_5$ has a smaller radius, hence $x_5 > x_3$.
Thus
$x_5 = x_1 + x_2 + x_4 + 2\sigma_{(124)}$
$x_3 = x_1 + x_2 + x_4 - 2\sigma_{(124)}$.

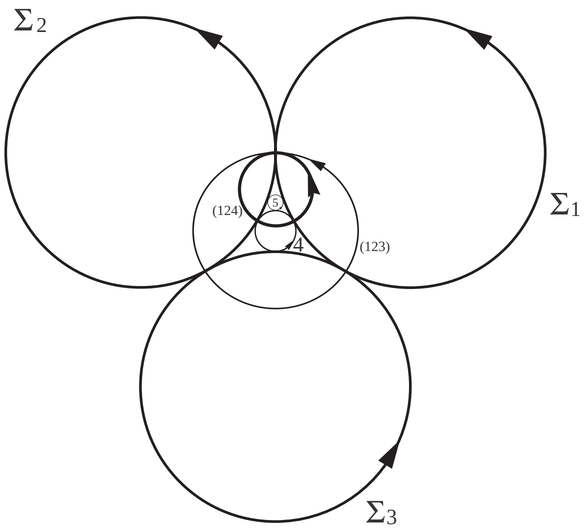


Figure 12: Dual Appolonian Packing.

In view of (6.15) and (6.17) we get

$$\begin{pmatrix} x_1 \\ x_2 \\ x_4 \\ \sigma_{(124)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \sigma_{(123)} \end{pmatrix}. \tag{6.18}$$

This replaces the description (6.13) for the "Knife Case". As said above, the fact that there are no negative entries in this matrix is sometimes an advantage. Similarly, for the "Ring Lemma Case"

$$\begin{pmatrix} x_1 \\ x_3 \\ x_4 \\ \sigma_{(134)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \sigma_{(123)} \end{pmatrix}. \tag{6.19}$$

and for the "Spiral Case"

$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \\ \sigma_{(234)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \sigma_{(123)} \end{pmatrix}. \tag{6.20}$$

(For reference of (6.18), (6.19) and (6.20) cf. [2] or [14]).

It is our aim now to show the complete analogy between the above and the discussion for inclinations. Indeed, using (6.6) (or (6.9) for $n = 2$ and $\gamma = -1$)

$$\lambda_4^1 + \lambda_4^2 = 2(\lambda_1 + \lambda_2 + \lambda_3). \tag{6.21}$$

Thus, instead of (6.12), we now have the more general relation

$$\begin{pmatrix}\lambda_1\\ \lambda_3\\ \lambda_4\\ \lambda_5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 2 & -1 & 2 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \tag{6.12'}$$

for the "Ring Lemma Case".
For the "Knife Case", instead of (6.13), we get

$$\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_4\\ \lambda_5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 0 & 1\\ 2 & 2 & -1 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \tag{6.13'}$$

For the "Spiral Case", instead of (6.14), we have

$$\begin{pmatrix}\lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\end{pmatrix} = \begin{pmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ -1 & 2 & 2 & 2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\end{pmatrix} \tag{6.14'}$$

We now turn to the other description (namely, the one involved with the orthogonal packing). With an obvious notation, we get from (6.4),

$$\lambda_4 = \lambda_1 + \lambda_2 + \lambda_3 + 2\lambda_{(123)}; \tag{6.22}$$

$$\lambda_3 = \lambda_1 + \lambda_2 + \lambda_4 - 2\lambda_{(123)}. \tag{6.23}$$

>From (6.22) and (6.23) we get

$$\lambda_{(124)} = \lambda_1 + \lambda_2 + \lambda_{(123)}. \tag{6.24}$$

Thus (6.22), (6.23) and (6.24) are generalizations of (6.15), (6.16) and (6.17) respectively. (The choice of sign is, again, explained by Figure 12.) Indeed, one may use Lemma 4.1 and the fact that in the limiting case we get the results for radii.

Hence, for the "Knife Case", from (6.24) we get

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_{(124)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_{(123)} \end{pmatrix}. \tag{6.25}$$

(In the limiting case, if the center of the "sphere of reference" tends to infinity, we get (6.18) from (6.25).) Similarly, for the "Ring Lemma Case",

$$\begin{pmatrix} \lambda_1 \\ \lambda_3 \\ \lambda_4 \\ \lambda_{(134)} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_{(123)} \end{pmatrix} \tag{6.26}$$

and for the "Spiral Case"

$$\begin{pmatrix} \lambda_2 \\ \lambda_3 \\ \lambda_4 \\ \lambda_{(234)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 2 \\ 0 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_{(123)} \end{pmatrix}. \tag{6.27}$$

Here (6.26) and (6.27) are generalizations of (6.19) and (6.20) respectively.

We end this section by answering a question posed in [4], where the significance of eigenvalues and eigenvectors of the matrices introduced above was investigated.

We consider the "Ring Lemma Case" and the matrix appearing in (6.12) or (6.12′) representing it. It is known (cf. [4]) that the eigenvalues and eigenvectors of this matrix are $\{\tau, (0,1,\tau,\tau^2)\}$ $\{\frac{1}{\tau}, (0,\tau^2,\tau,1)\}$, $\{-1, (0,-1,-1,-1)\}$, and $\{1, (1,-1,-1,-1)\}$ where $\tau^2 - 3\tau + 1 = 0$.

In [4] the eigenvalues $\tau$ and $\frac{1}{\tau}$ are explained and the same is true for the respective eigenvectors. This was done with the aid of the relation given in (6.12), i.e., by investigating the geometry of the dynamic of changes of the values of radii. Now we are able to explain the other two eigenvalues (and their respective eigenvectors) with the aid of the more general relation (6.12′).

Let $\{\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4\}$ be any "quad" (i.e., four mutually tangent spheres with disjoint interiors). We now take in particular $\Sigma$, the sphere of reference to be $\Sigma_1$. Then

$$\lambda_1 = \lambda(\Sigma, \Sigma_1) = 1, \lambda_j = \lambda(\Sigma, \Sigma_j) = -1, \ j = 2, 3, 4.$$

Since,

$$\begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & -1 & 2 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}.$$

This means that $\lambda_5 = \lambda(\Sigma, \Sigma_5) = -1$ (in view of (6.12)).

Figure 13 explains the situation where we have chosen $\Sigma = \Sigma_1$ to be a straight line. Of course, in view of the invariance of inclinations, there is nothing special in this choice, and we can take an arbitrary quad.

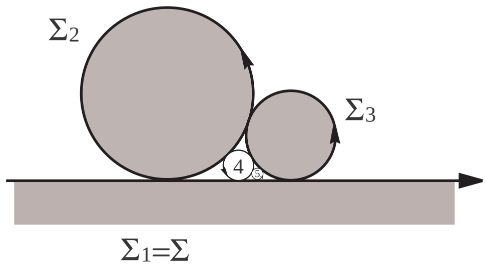


Figure 13: "Ring Lenna" case.

Of course we can continue this process:

$$(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \to (\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5, ) \to (\Sigma_1, \Sigma_4, \Sigma_5, \Sigma_6), \dots .$$

All corresponding vectors of inclinations:

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4), (\lambda_1, \lambda_3, \lambda_4, \lambda_5), (\lambda_1, \lambda_4, \lambda_5, \lambda_6), \dots$$

are all equal to $(1, -1, -1, -1)$ as $\Sigma_4, \Sigma_5, \Sigma_6, \dots,$ are all tangent to $\Sigma = \Sigma_1$ with
disjoint interiors, which implies $\lambda(\Sigma_j, \Sigma) = -1,\ j = 2, 3, \dots.$ This explains $\{1, (1, -1, -1, -1)\}$. To explain the remaining eigenvalue and eigenvector $\{-1, (0, -1, 1, -1)\}$, multiply by $a$ to get another eigenvector, namely, $(0, -a, a, -a)$. To motivate the choice of $a$, we use (4.21) for $n = 2, \gamma = -1$, to get $(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)^2 - 2(\lambda_1^2 + \lambda_2^2 + \lambda_3^2 + \lambda_4^2) = -4$ and for our vector $(0, -a, a, -a)$ this is $(0 - a + a - a)^2 - 2(a^2 + a^2 + a^2) = -4$ or $a^2 = \frac{4}{5}$. Hence, $a = \pm\frac{2}{\sqrt{5}}$. Thus (since $\lambda_1 = \lambda(\Sigma, \Sigma_1) = 0$), this means that $\Sigma$ is orthogonal to $\Sigma_1$ and intersects with $\Sigma_2, \Sigma_3, \Sigma_4, \dots$ in an angle $\pm\varphi$, as $\cos\varphi = \pm\frac{2}{\sqrt{5}}$.

We now use, again, the model described in Figure 13. It is well known that if the radius of $\Sigma_3$ is chosen to be 1, then the radius of $\Sigma_2$ is $\tau$, where $\tau$ satisfies $\tau^2 - 3\tau + 1 = 0$, i.e., the eigenvalue described above (cf. [4]). Also it is known that all tangency points among $(\Sigma_2, \Sigma_3), (\Sigma_3, \Sigma_4), \ldots$ are on the perpendicular line to $\Sigma_1$ ([4, p. 521]). We now choose $\Sigma$ to be this perpendicular line. It is an easy calculation to show that $\lambda(\Sigma, \Sigma_2) = \frac{-2}{\sqrt{5}}$, $\lambda(\Sigma, \Sigma_3) = \frac{2}{\sqrt{5}}$, $\lambda(\Sigma, \Sigma_4) = \frac{-2}{\sqrt{5}}, \ldots$ (see Figure 14).

$\cos\varphi = -\frac{h_j}{R_j}$
$\sin\varphi = \frac{\tau-1}{\tau+1} = \frac{1}{\sqrt{5}}$
$\tau = \frac{3+\sqrt{5}}{2}$
$\cos\varphi = \frac{2}{\sqrt{5}}$

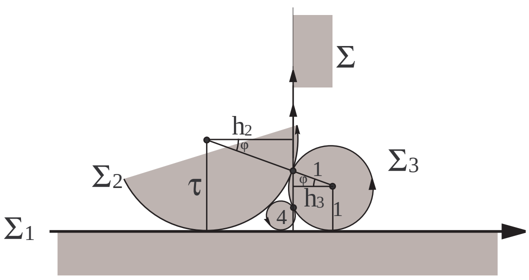


Figure 14:

Note that if the line $\Sigma$ is chosen as in Figure 14, then $h$ is positive for $\Sigma_2, \Sigma_4, \ldots$ and negative for $\Sigma_3, \Sigma_5, \ldots$ where $\lambda(\Sigma, \Sigma_j) = \lambda_j = -\frac{h_j}{R_j}$, $j = 2, 3, 4, \ldots$.

Summing up, we have for $\{\Sigma_j\}_1^4$ and $a = \frac{2}{\sqrt{5}}$,

$$(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (0, -a, a, -a) \to (0, a, -a, a) = (\lambda_1, \lambda_3, \lambda_4, \lambda_5).$$

This ends our discussion of eigenvalues and eigenvectors of the matrix associated with the "Ring Lemma Case".

Next, we will be interested in particular with some aspects of Apollonian packing in $R^1$. But before, we want to make some remarks concerning packing in $R^n$ and $G^n$. We point out that the discussion about matrices may be generalized to $R^n$ and even to $G^n$. Of course, the disccussion in $G^n$ is formal, since it is hard to give a geometrical reasoning to "packing" in $G^n$. But we definitely may speak about "chains" of spheres like in $R^2$. We restrict ourselves to one case, namely, the "Ring Lemma Case", just as an example to what can be done also in all the cases treated above. Hence consider, as in Corollary 6.1, $\{\Sigma_j\}_{j=1}^{n+2}$ spheres in $G^n$, having mutual inclination $\gamma$, and $\{\Sigma_j\}_{j=1}^{n+1}$, $\Sigma_{n+3}$ having also mutual inclination $\gamma$, provided $\gamma \neq 0, \gamma \neq 1, \gamma \neq -\frac{1}{n}$. Then, from (6.9) we have $\lambda_2 + \lambda_{n+3} = \frac{2}{n+\frac{1}{\gamma}}\left(\Sigma_{j=3}^{n+2}\lambda_j + \lambda_1\right)$,

$$\begin{pmatrix} \lambda_1 \\ \lambda_3 \\ \vdots \\ \lambda_{n+3} \end{pmatrix} = \begin{pmatrix} 1 & 0 & & \cdots & & 0 \\ 0 & 0 & 1 & & 0 & \cdots & 0 \\ & & & \vdots & & & \\ 0 & 0 & & \cdots & & & 1 \\ \frac{2}{n+\frac{1}{\gamma}}, & -1 & & \cdots & & & \frac{2}{n+\frac{1}{\gamma}} \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{n+2} \end{pmatrix}. \tag{6.28}$$

Instead of three matrices in $R^2$ representing the three cases, "Knife", "Ring" and "Spiral", we now have $n+1$ matrices in $G^n$. We will not push further this line of reasoning.

## 6.3 Apollonian packing in $R^1$

First, we recall the concept of inclination in $R^1$. It is sometimes more convenient to consider "balls", i.e., segments in $R^1$ rather than "spheres", i.e., pairs of points. Hence, let two "balls", $\Sigma_1, \Sigma_2$ with radii $R^1, R^2$ respectively, be given, such that the distance between their centers is $d$. Then the inclination between $\Sigma_1$ and $\Sigma_2$ is,

$$\lambda(\Sigma_1, \Sigma_2) = \frac{R_1^2 + R_2^2 - d^2}{2R_1R^2} \tag{6.29}$$

(see Figure 15).

In case the two segments intersect, we may talk about the "angle" $\varphi$ between these two segments determined by $\cos\varphi = \lambda(\Sigma_1, \Sigma_2)$. As usual, $R < 0$ for the "ball" $\Sigma = \{x, |x-a| < R\}$ means the complement of $\bar{\Sigma}$ in $\hat{R}$, i.e., $\hat{R}\backslash\bar{\Sigma}$.

We check first Descartes "three circles theorem" and show that it actually degenerates. Indeed, the theorem reduces to $\left(\frac{1}{R_1} + \frac{1}{R_2} - \frac{1}{R_3}\right)^2 = \left(n + \frac{1}{\gamma} + 1\right)\left(\frac{1}{R_1^2} + \frac{1}{R_2^2} + \frac{1}{R_3^2}\right)$ for $n = 1$, and $\gamma = -1$ (in the case of tangency). Hence

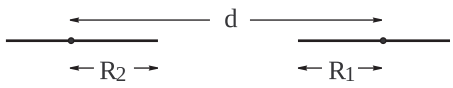


Figure 15: Packing of the real line.

$\left(\frac{1}{R_1}+\frac{1}{R_2}-\frac{1}{R_3}\right)=\frac{1}{R_1^2}+\frac{1}{R_2^2}+\frac{1}{R_3^2}$ or $\frac{1}{R_1R_2}=\frac{1}{R_1R_3}+\frac{1}{R_2R_3}$. Indeed, $R_3=R_1+R_2$, and thus, this is a trivial identity (see Figure 16).

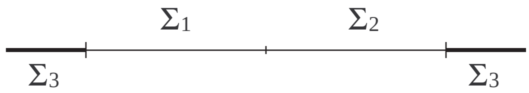


Figure 16: Degenerate case (tangent).

In order to understand better why this is a degenerate case, note that for $x_j=\frac{1}{R_j}$, $j=1,2$, we have to solve $(x_1+x_2+x)^2=\left(2+\frac{1}{\gamma}\right)(x_1^2+x_2^2+x^2)$. Hence, if $\gamma=-1$ (the tangency case), we get only *one* solsution. But, as we see, for other values of $\gamma$ we get *two* solutions, which makes the situation very similar to what occurs in $R^n$ (or $G^n$) for $n\geq 2$. Solving for $x$ we get,

$$x=\frac{x_1+x_2\pm\sqrt{\Delta}}{1+\frac{1}{\gamma}},\Delta=\left(2+\frac{1}{\gamma}\right)\left[(x_1+x_2)^2-\left(1+\frac{1}{\gamma}\right)(x_1^2+x_2^2)\right]. \quad (6.30)$$

As an example fo "Apollonian packing" in $R^1$, such that all $\{R_j\}$ become positive, consider the case $1+\frac{1}{\gamma}<0$. Then for $a=x_1+x_2,\ b=x_1^2+x_2^2$,

$$a^2>b>b\left(1+\frac{1}{\gamma}\right),\quad \Delta>0. \quad (6.31)$$

Hence, for the choice $x=\left(x_1+x_2+\sqrt{\Delta}\right)\cdot\frac{1}{1+\frac{1}{\gamma}}$ we get $x>0$ and thus $R_3=\frac{1}{x}>0$. It is possible to continue in this way, and to create infinite "Apollonian packings" (see Figure 17).

An interesting question might be to investigate the size of the residual set (say, the Hausdorff dimension). Similar questions in $R^2$ are very deep and not yet completely solved (cf. [4]).

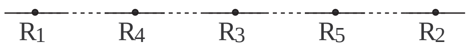


Figure 17: Non degenerate case (inclination).

In the next section we will be interested in the BOIP for $R^2$ and $R^3$. Also, we will discuss the BOIP and its connection to reflecting the Apollonian packing in any of its spheres.

# 7 "Bowl of Integers" property

## 7.1 Inclinations, Apollonian and dual Apollonian packing

As mentioned already in the introduction, Soddy was apparently the first to discover this property [5]. It was rediscovered many times later on. We now recall what we mean by this property. Suppose we are given a quad in $R^2$, say, $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$. Assume further that its bends $x_j = \frac{1}{R_j}$, $1 \leq j \leq 4$ are all integers. Then creating the Apollonian packing from this quad, one gets *all* bends as integers. This is a surprising fact, but very easy to prove. Indeed, consider for instance the "Ring Lemma Case". Since $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \to (\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5)$ we have $x_2 + x_5 = 2(x_1 + x_3 + x_4)$ and thus if $\{x_j\}_{j=1}^4$ are integers, then obviously $x_5$ is an integer as well. Thus, by the same reasoning, *all* bends of this particular Apollonian packing are integers. For $R^3$ we get a similar situation. Indeed, consider for instance, $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5) \to (\Sigma_1, \Sigma_3, \Sigma_4, \Sigma_5, \Sigma_6)$. Then we have for the bends $\{x_j\}_{j=1}^6$, $x_2 + x_6 = \frac{2}{n+\frac{1}{\gamma}}\ \Sigma_{j=3}^5 x_j = \Sigma_{j=3}^5 x_j$ as $n = 3$ and $\gamma = -1$.

Thus again, if $\{x_j\}_{j=1}^5$ are integers, *all* bends of this particular Apollonian packing will be integers. (In fact, one may consider the case $n+\frac{1}{\gamma} = 1$ or $n + \frac{1}{\gamma} = 2$ even for higher values of $n$, i.e., for $R^n$, $n > 3$. We then get the same result again, i.e., *all* bends will be integers provided the starting $n+2$ bends are such.) Now, if we make a Möbius transformation, of course the radii are changed, and the BOIP may be destroyed. It is a nice fact that we have a similar situation for the inclinations (meaning the BOIP). But contrary to the previous case of radii, inclinations are *not* affected by a Möbius transformation, as inclinations are invariant under such maps. We start with a known theorem of Boyd [9]. Our proof is much simpler.

**Theorem 7.1.** *(Boyd). Given any Apollonian packing in $R^n$, for $n = 2$ or $n = 3$, all mutual inclinations between any two spheres of the packing are odd numbers.*

*Proof.* We first recall (as Boyd mentioned in his paper quoted above) that the special "spiral" case was proved earlier by Coxeter [8].

Since, as noted above, the inclination remains invariant under a Möbius map, we can choose a convenient setting. Hence, consider the situation described in Figure 18.

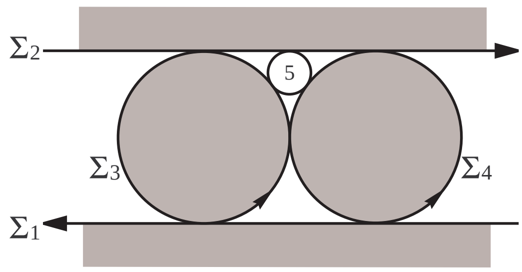


Figure 18: Mutual inclination (odd numbers).

Let $\Sigma = \Sigma_1$ be the sphere of reference. Then

$$\lambda_1 = \lambda(\Sigma, \Sigma_1) = \ \ 1, \qquad \lambda_2 = \lambda(\Sigma, \Sigma_2) \ = -1,$$
$$\lambda_3 = \lambda_3(\Sigma, \Sigma_3) = -1, \qquad \lambda_4 = \lambda_4(\Sigma, \Sigma_4) = -1.$$

Thus, using $\lambda_5 + \lambda_1 = 2(\lambda_2 + \lambda_3 + \lambda_4)$, we get $\lambda_5 = 2 \cdot (-3) - 1 = -7$. More generally, by induction, we have $(\Sigma_{i_j}, \Sigma_{i_2}, \Sigma_{i_3}, \Sigma_{i_4}) \to (\Sigma_{i_1}, \Sigma_{i_3}, \Sigma_{i_4}, \Sigma_{i_5})$ for two quads: $\lambda_{i_2} + \lambda_{i_5} = 2(\lambda_{i_1} + \lambda_{i_3} + \lambda i_4)$ and thus, assuming $\lambda_{i_1}, \lambda_{i_2}, \lambda_{i_3}, \lambda_{i_4}$ to be odd integers, it follows that the same is true for $\lambda_{i_5}$.

For $n = 3$ the proof is very similar. We take two parallel planes $\Sigma_1$ and $\Sigma_2$ and three spspheres $\Sigma_3, \Sigma_4, \Sigma_5$ between them and all are mutually tangent. Then for $\Sigma = \Sigma_1$, $\lambda_1 = \lambda(\Sigma, \Sigma_1) = 1$, $\lambda_j = \lambda(\Sigma, \Sigma_j) = -1$ for $j = 2, 3, 4, 5$. Thus from $\lambda_1 + \lambda_6 = \lambda_2 + \lambda_2 + \lambda_4 + \lambda_5$ we get $\lambda_6 = -4 - 1 = -5$. The passage from $n$ to $n + 1$ is very similar and is omitted.

Next, we prove a similar theorem for the dual packing.

**Theorem 7.2.** *Given an Apollonian packing in $R^2$, consider the dual (i.e., the orthogonal) Apollonian packing. If $L$ is any sphere in the original Apollonian packing and $K$ is any sphere in the dual Apollonian packing, then the inclination $\lambda = \lambda(K, L)$ between the two spheres is an even integer.*

*Proof.* Let $L$ be any sphere in the original Apollonian packing and $K$ any sphere in the dual one. Our aim is to show that $\lambda(L, K)$ is even. We take $K$ as a sphere of reference. Denote it by $\Sigma_1 = \Sigma$. If $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ is a quad, then $\lambda(\Sigma, \Sigma_1) = \lambda(\Sigma_1, \Sigma_1) = 1$, $\lambda(\Sigma, \Sigma_j) = -1$ for $j = 2, 3, 4$. Let $\Sigma_{(123)}$ be orthogonal to the set $\Sigma_1, \Sigma_2, \Sigma_3$. Then, obviously, $\lambda(\Sigma_{(123)}, \Sigma) = 0$. The given sphere $K$ is orthogonal to three spheres of the original Apollonian packing. Denote these three spheres by $(\Sigma_n, \Sigma_{n+1}, \Sigma_{n+2})$ and $K$ by $\Sigma_{(n,n+1,n+2)}$. Operating with the three matrices described in (6.25), (6.26) and (6.27) (i.e., "Knife", "Ring" and "Spiral"), we can reach in a finite number of steps, say $n-1$, from $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_{(123)})$ to $(\Sigma_n, \Sigma_{n+1}, \Sigma_{n+2}, \Sigma_{n,n+1,n+2})$. We now proceed with the induction process. The first "move" is already explained above. Indeed, $\lambda(\Sigma_{(124)}, \Sigma) = 0$ is an even number. As explained above we reach the final stage by a finite number of "moves", say, $n-1$. Assume, as an example, that the second "move" is with the "spiral" matrix, i.e., the situation described in (6.27). We then have, $\lambda(\Sigma_{(234)}, \Sigma) = \lambda(\Sigma_2, \Sigma) + \lambda(\Sigma_3, \Sigma) + \lambda(\Sigma_{(123)}, \Sigma)$. But $\lambda(\Sigma_{(123)}, \Sigma) = \lambda(\Sigma_{(123)}, \Sigma_1) = 0$ and $\lambda(\Sigma_2, \Sigma) = \lambda(\Sigma_3, \Sigma) = 1$, as $\Sigma = \Sigma_1$ and thus $(\Sigma, \Sigma_2, \Sigma_3, \Sigma_4)$ is a quad. Hence $\lambda(\Sigma_{(234)}, \Sigma) = 2$ which is an even number as claimed. More generally, the $m^{\text{th}}$ "move" is very similar. Assuming that $\lambda(\Sigma_{(m,m+1,m+2)}, \Sigma)$ is even by the induction assumption, and using Theorem 7.1, we get that $\lambda(\Sigma_{(m,m+2,m+3)}, \Sigma)$ is an even number too, as a sum of two odd numbers and an even number. (In the notation above, we used the "Ring" case, $(m, m+1, m+2, m+3) \to (m, m+2, m+3, m+4)$ with an obvious notation. Of course, the reasoning is identical for the other two options.) Thus showing the assertion from $m$ to $m+1$, we now have that at the final $n^{\text{th}}$ stage, $\lambda(\Sigma_{(n,n+1,n+2)}, \Sigma) = \lambda(K, L)$ is an even number as claimed.

**Remark**. The situation in $R^3$ is different. Indeed, going back to Theorem 6.1, we get from (6.1) for $n = 3$ and the tangency case, i.e., $\gamma = -1$, that $\lambda_4 = \left(\Sigma_{j=1}^{3}\lambda_j \pm \sigma\sqrt{(3+1-1)(3-1)(1+1)}\right)\frac{1}{3-1}$ or $\lambda_4 = \frac{1}{2}[\lambda_1 + \lambda_2 + \lambda_3] \pm \sigma\sqrt{3}$. In view of Theorem 7.1, $\{\lambda_j\}_{j=1}^{4}$ are odd integers and thus $\sigma = \lambda(K, L)$, $K$ a sphere of the original Apollonian packing in $R^3$, and $L$ a sphere in its dual Apollonian packing, is an irrational number of the form $(2\lambda_4 - \lambda_1 - \lambda_2 - \lambda_3)\frac{1}{\sqrt{3}}$ (except for the trivial case of orthogonality, i.e., $\sigma = 0$).

## 7.2 Reflected Apollonian packing and inclination

Given an Apollonian packing, we have seen that it is worthwhile to consider the dual (orthogonal) Apollonian packing. We can consider also other Apollonian packings created from the given one. Indeed, it is possible to reflect the Apollonian packing in any of its members. We first recall some known facts about this procedure (cf. [4]), and then relate the results to the BOIP. To make the discussion easier and more geometrical, we confine ourselves to $R^n$, rather than to $G^n$.

Hence, let $\Sigma_1$ be a sphere in $R^n$, $\Sigma$ be a sphere of reference and $\Sigma_0$ another sphere. We aim to reflect $\Sigma_1$ with respect to $\Sigma_0$ and find how the inclination $\lambda(\Sigma_1, \Sigma)$ is changed after the reflection. We further denote by $\Sigma_1^*$ the reflection of $\Sigma_1$ with respect to $\Sigma_0$. Our aim is to confirm that in the special case of tangency between $\Sigma_1$ and $\Sigma_0$ (from outside),

$$\lambda(\Sigma, \Sigma_1^*) = \lambda(\Sigma, \Sigma_1) + 2\lambda(\Sigma, \Sigma_0). \tag{7.1}$$

First we consider the situation without the limitations of tangency. Because of the invariance property of the inclination, there is no loss of generality to assume that $\Sigma_0$ is a plane.

We now introduce some notations:
$h_1$ = the distance of the center of $\Sigma_1$ from $\Sigma_0$
$h$ = the distance of the center of $\Sigma$ from $\Sigma_0$
$d$ = the distance of the centers of $\Sigma$ and $\Sigma_1$
$d^*$ = the distance of the centers of $\Sigma$ and $\Sigma_1^*$
$R$ = the radius of $\Sigma$
$R_1$ = the radius of $\Sigma_1$.

We have (see Figure 19)

$$\lambda(\Sigma, \Sigma_0) = \frac{-h}{R}, \lambda(\Sigma_1, \Sigma_0) = \frac{-h_1}{R_1}. \tag{7.2}$$

Also,

$$d^2 = a^2 + (h - h_1)^2, (d^*)^2 = a^2 + (h + h_1)^2 \tag{7.3}$$

and

$$\lambda(\Sigma, \Sigma_1) = \frac{R^2 + R_1^2 - d^2}{2RR_1}, \lambda(\Sigma, \Sigma_1^*) = \frac{R^2 + R_1^2 - (d^*)^2}{2RR_1}. \tag{7.4}$$

Hence, combining (7.3) and (7.4), we get,

$$\lambda(\Sigma, \Sigma_1) - \lambda(\Sigma, \Sigma_1^*) = \frac{2hh_1}{RR_1}. \tag{7.5}$$

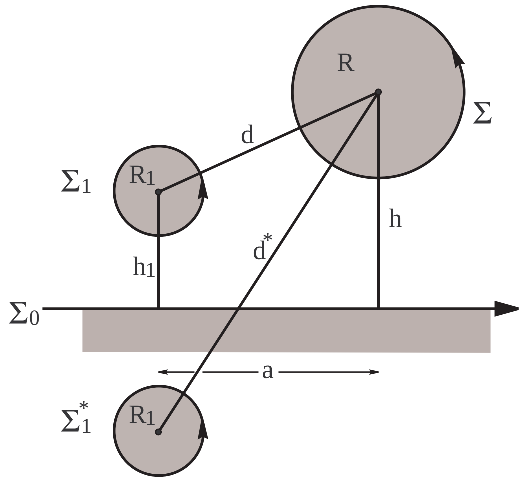


Figure 19: Reflected Appolonian Packing.

Since $\frac{h}{R} \cdot \frac{h_1}{R_1} = \left(\frac{-h}{R}\right)\left(\frac{-h_1}{R_1}\right)$ we deduce from (7.5) with the aid of (7.2),

$$\lambda(\Sigma, \Sigma_1^*) = \lambda(\Sigma, \Sigma_1) - 2\lambda(\Sigma, \Sigma_0)\lambda(\Sigma_1, \Sigma_0). \tag{7.6}$$

Specializing now to the case of tangency of $\Sigma_1$ (from outside) to $\Sigma_0$, we get in particular that $h_1 = R_1, \lambda(\Sigma_1, \Sigma_0) = -1$, and thus (7.1) follows from (7.6). We now make use of the invariance property of the quantities appearing in (7.1). Hence consider now the case where $\Sigma_0$ is a sphere of radius $\rho$, and we use Lemma 4.1, as usual, to pass from inclinations to radii by letting the center of $\Sigma$ tend to infinity. Denoting by $R$ the radius of $\Sigma_1$ and by $R^*$ the radius of the reflection of $\Sigma_1$ with respect to $\Sigma_0$, we then easily get,

$$\frac{1}{R^*} = \frac{1}{R} + \frac{2}{\rho}. \tag{7.7}$$

Comparing (7.6) and (7.7) we see that again, as in previous cases, we have a complete analogy between the case of inclinations and the case of radii.

We now restrict ourselves to $R^2$. Also, to make things clear, we specialize ourselves to the "Knife" case, having in mind that "Ring" and "Spiral" cases are similar. Hence, let $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ be a quad in $R^2$ and $\Sigma$ be a sphere of reference in $R^2$. $\{\lambda_j\}_{j=1}^4$ will denote the inclinations of $\{\Sigma_j\}_{j=1}^4$ and $\Sigma$ respectively. Since we are interested in the "Knife" case, we consider $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4) \to (\Sigma_1, \Sigma_2, \Sigma_4, \Sigma_5)$ and we recall (6.13′) for $\lambda_j = \lambda(\Sigma, \Sigma_j), 1 \le j \le 5$:

$$\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_4 \\ \lambda_5 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 2 & 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}.$$

It is our aim now to show that operating with the transpose of this matrix, we get information about reflection of the quad $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4)$ with respect to $\Sigma_4$. This somewhat surprising fact was already pointed out in [4]. Indeed, we have

$$\begin{pmatrix} \lambda_1 + 2\lambda_4 \\ \lambda_2 + 2\lambda_4 \\ -\lambda_4 \\ \lambda_3 + 2\lambda_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix}. \tag{7.8}$$

Comparing with (7.1) we see that the new vector on the left of (7.8) actually describes the new inclinations after reflection. Here $\Sigma_0 = \Sigma_4$ (as we reflect with respect to $\Sigma_4$). Hence from (7.1) we get for our case

$$\lambda(\Sigma, \Sigma_j^*) = \lambda(\Sigma, \Sigma_j) + 2\lambda(\Sigma, \Sigma_4), j = 1, 2, 3. \tag{7.1'}$$

(Note that (7.1′) does not apply for $j = 4$ as $\Sigma_4$ is not tangent to itself from the outside!)

Since the reflection of $\Sigma_4$ with respect to itself changes the sign of the radius, we get that

$$\lambda(\Sigma, \Sigma_4^*) = -\lambda(\Sigma, \Sigma_4). \tag{7.9}$$

Denoting $\lambda_j^* = \lambda(\Sigma, \Sigma_j^*), j = 1, 2, 3, 4$ using (7.8), we get from (7.1′) and (7.9),

$$\begin{pmatrix} \lambda_1^* \\ \lambda_2^* \\ \lambda_4^* \\ \lambda_3^* \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} \tag{7.10}$$

(see Figure 20).

Since $(\Sigma_1^*, \Sigma_2^*, \Sigma_3^*, \Sigma_4^*)$ is a quad, note that $(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) = (\lambda_1 + 2\lambda_4, \lambda_2 + 2\lambda_4,$
$-\lambda_4, \lambda_3 + 2\lambda_4)$ must satisfy (4.21) for $n = 2, \gamma = -1$ or $\big((\lambda_1 + 2\lambda_4) + (\lambda_2 + 2\lambda_4) + (\lambda_3 + 2\lambda_4) - \lambda_4\big)^2 - 2\big(\Sigma_{j=1}^3 (\lambda_j + 2\lambda_4)^2 + \lambda_4^2\big) = -4$ provided $\{\lambda_j\}_1^4$ satisfy (4.21).

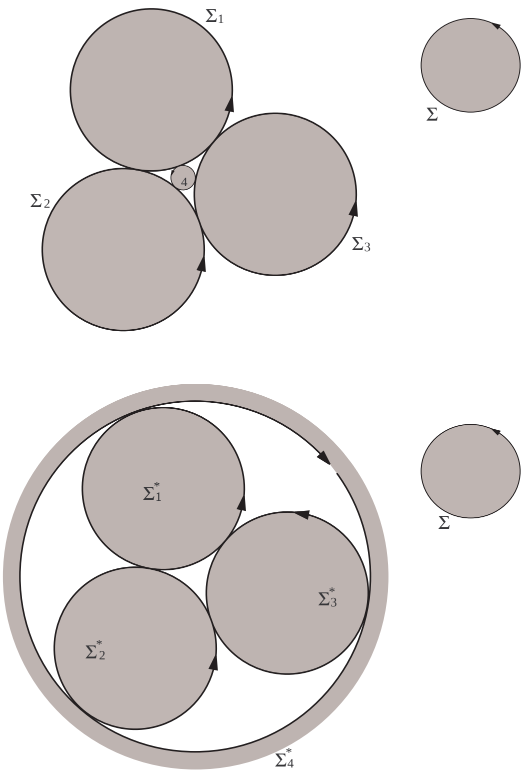


Figure 20: BOIP and reflection.

We leave it to the reader to check this simple calculation. Specializing to the radii, (7.8) is reduced to

$$\begin{pmatrix} x_1 + 2x_4 \\ x_2 + 2x_4 \\ -\ x_4 \\ x_3 + 2x_4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}, \tag{7.11}$$

where $\{x_j\}_{j=1}^4$ are the bends $\{R_j^{-1}\}_{j=1}^4$.

We also have a similar remark in the special case, namely, the Descartes four circle theorem must be satisfied by $(x_1 + 2x_4, x_2 + 2x_4, -x_4, x_3 + 2x_4)$, or,

$$2\big[\Sigma_{j=1}^3 (x_j + 2x_4)^2 + x_4^2\big] = \big[(x_1 + 2x_4) + (x_2 + 2x_4) + (x_3 + 2x_4) - x_4\big]^2,$$

provided $2\Sigma_{j=1}^4 x_j^2 = \left(\Sigma_{j=1}^4 x_j\right)^2$. This is easily checked to be valid.

We are now in a position to relate the BOIP to reflection.

**Theorem 7.3.** *Let $A$ be an Apollonian packing in $R^2$, and let $A^D$ be its dual Apollonian packing. Let $\Sigma_0$ be any sphere in $A$. Reflect $A$ and $A^D$ with respect to $\Sigma_0$ to create two new Apollonian packings. Consider $\Sigma_j \in A, j = 1, 2$ and their reflections with respect to $\Sigma_0$ denote by $\Sigma_j^*$ respectively. Then the inclination $\lambda(\Sigma_1, \Sigma_2^*)$ is an odd integer. Consider any $\Sigma_1^D \in A^D$ and its reflection $(\Sigma_1^D)^*$. Then $\lambda(\Sigma_1(\Sigma_1^D)^*)$ is an even integer.*

*Proof.* The proof is an easy corollary of Theorems 7.1 and 7.2, and (7.1). We have, by Theorem 7.1, that $\lambda(\Sigma_1, \Sigma_2)$ is an odd number. The same is true for $\lambda(\Sigma_1, \Sigma_0)$.

$>$From (7.1) we get

$$\lambda(\Sigma_1, \Sigma_2^*) = \lambda(\Sigma_1, \Sigma_2) + 2\lambda(\Sigma_1, \Sigma_0).$$

Hence $\lambda(\Sigma_1, \Sigma_2^*)$ is an odd integer. To prove the second assertion, we use Theorem 7.2. We have that $\lambda(\Sigma_1, \Sigma_1^D)$ is an even integer. Applying, again, the relation (7.1), we have

$$\lambda(\Sigma_1, (\Sigma_1^D)^*) = \lambda(\Sigma_1, \Sigma_1^D) + 2\lambda(\Sigma_1, \Sigma_0).$$

Hence $\lambda(\Sigma_1, (\Sigma_1^D)^*)$ is even as the sum of two even numbers. This ends the proof of Theorem 7.3.

Note that in the limiting case we get a BOIP for radii in view of (7.7). Indeed, if we are given an Apollonian packing that contains a quad with four bends that are integers, then not only all bends of the Apollonian packing are integers, but all bends of reflected Apollonian packings, and repeated reflections with respect to each of the spheres will give a similar result. We can now state

**Theorem 7.4.** *Given any Apollonian packing in $R^2$, having a quad with four bends that are all integers, we have necessarily the BOIP for the dual Apollonian packing and any new Apollonian packing created by a finite number of reflections with any of the spheres in the construction.*

*Proof.* To complete the proof, there is only one thing to check. Since by the above discussion there is nothing left to prove concerning the reflections, we are left with the following assertion to prove: *Given a quad having four bends that are integers, then necessarily the bends of the orthogonal spheres are also integers* (see Figure 21).

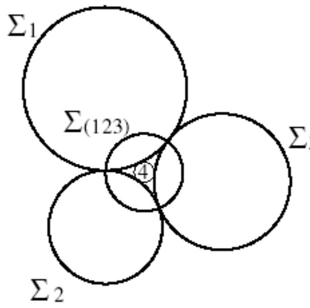


Figure 21: Preservation of BOIP under reflection.

But (see (6.7) and (6.8)

$$x_4 = x_1 + x_2 + x_3 \pm 2\sigma,$$

where $\sigma$ is the bend of the orthogonal sphere, $\Sigma_{(123)}$, to $(\Sigma_1, \Sigma_2, \Sigma_3)$. Hence, what is left to show in order to confirm that $\sigma$ is an integer, is that $\Sigma_{j=1}^{3} x_j - x_4$ is an even integer. Indeed, $\left(\Sigma_{j=1}^{3} x_j\right) - x_4 = \left(\Sigma_{j=1}^{4} x_j\right) - 2x_4$ implies that it is enough to show that $\Sigma_{j=1}^{4} x_j$ is an even number. Instead we show the equivalent thing, namely, that $\left(\Sigma_1^4 x_j\right)^2$ is an even number. But this follows at once from Descartes' four circle theorem, namely, $\left(\Sigma_{j=1}^{4} x_j\right)^2 = 2\Sigma_{j=1}^{4} x_j^2$. This ends the proof of Theorem 7.4.

The situation in $R^3$ is different. In fact, we have already seen that for $n = 3$ and $\gamma = -1$ (i.e., the tangency case) we get from (6.1), $\lambda_5 = (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \pm 2\sqrt{3}\sigma)\frac{1}{2}$. Hence, except for the trivial case $\sigma = 0$, if $\{\lambda_j\}_{j=1}^{5}$ are integers, then $\sigma$ is *never* an integer. In fact, it is an irrational number of a specific form, namely, $\frac{m}{\sqrt{3}}$ for some integer $m$. On the other hand, a similar result for reflection still holds. We omit the details.

We end this section by discussing the nature of the matrices associated with reflection in $R^3$. It turns out that the nice property of the transposed matrices that we have shown in $R^2$, is not valid any more in $R^3$. This makes the property in $R^2$ even more mysterious and hard to motivate. Hence, suppose we are given $(\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4, \Sigma_5)$, a set of five spheres in $R^3$, mutually tangent, and with disjoint integers. Let $\Sigma$ be a sphere of reference in $R^3$ and denote by $\{\lambda_j\}_{j=1}^{5}$ the inclinations of $\Sigma$ with $\{\Sigma_j\}_{j=1}^{5}$ respectively. Suppose we consider the "move" which is similar to the "Ring Lemma Case" in $R^2$.

Then $\lambda_2+\lambda_6=\lambda_1+\lambda_3+\lambda_4+\lambda_5$ and the matrix description is

$$\begin{pmatrix}\lambda_1\\ \lambda_3\\ \lambda_4\\ \lambda_5\\ \lambda_6\end{pmatrix}=\begin{pmatrix}1&0&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\\1&-1&1&1&1\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\end{pmatrix}.$$

Hence, it turns out that instead of the last line, we have to take the vector $(2,-1,2,2,2)$ and then take the transpose. Indeed, doing that, we get

$$\begin{pmatrix}\lambda_1^*\\ \lambda_5^*\\ \lambda_2^*\\ \lambda_3^*\\ \lambda_4^*\end{pmatrix}=\begin{pmatrix}\lambda_1+2\lambda_5\\ -\lambda_5\\ \lambda_2+2\lambda_5\\ \lambda_3+2\lambda_5\\ \lambda_4+2\lambda_5\end{pmatrix}=\begin{pmatrix}1&0&0&0&2\\0&0&1&0&-1\\0&1&0&0&2\\0&0&1&0&2\\0&0&0&1&2\end{pmatrix}\begin{pmatrix}\lambda_1\\ \lambda_2\\ \lambda_3\\ \lambda_4\\ \lambda_5\end{pmatrix}.$$

$\{\lambda_j^*\}_{j=1}^5$ are the values of the inclinations of the $\{\Sigma_j^*\}_{j=1}^5$ with respect to $\Sigma$, where $\Sigma_j^*$ is the reflection of $\Sigma_j$ with respect to $\Sigma_5$ for each $j, 1\le j\le 5$. Obviously $\{\Sigma_j^*\}_{j=1}^5$ must satisfy the mutual inclination theorem. Indeed, putting $n=3$ and $\gamma=-1$ in (4.21), we have to check $\left(\Sigma_{k=1}^5\lambda_k^*\right)^2-3\Sigma_{k=1}^5(\lambda_k^*)^2=-6$, provided $\left(\Sigma_{k=1}^5\lambda_k\right)^2-3\Sigma_{k=1}^5(\lambda_k)^2=-6$. We leave it to the reader to check that this is, indeed, correct.

Our aim in the next section is to give a "translation" of inclination results proved earlier, to radii results. We have done that already by taking a limiting process, namely, sending the center of the reference sphere $\Sigma$ to infinity. We now take a different procedure.

# 8 Further "translation" of inclination results to results on radii

## 8.1 The linear theorem

Instead of a limiting case, we now take fixed spheres with mutual inclination $\gamma$ and let the sphere of reference $\Sigma$ be arbitrary. We then equate coefficients and find more detailed information about radii and coordinates of the given fixed spheres. To make things more clear we start "translation" of the linear theorem, i.e., Theorem 4.1. Hence, as in Theorem 4.1, let $n+1$ spheres $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma$, $\gamma\ne 1,\gamma\ne-\frac{1}{n}$,

$\Sigma_{n+2}$ be another sphere in $G^n$, with inclination $\mu$ with each of $\Sigma_j, 1 \leq j \leq n+1$, satisfying (4.1). $\Sigma$ is the sphere of reference. We denote by $a_j = (a_{j_1}, \ldots, a_{jn})$ the centers of $\{\Sigma_j\}_{j=1}^{n+1}$ respectively, $\{R_j\}_{j=1}^{n+1}$ are the radii of $\{\Sigma_j\}_{j=1}^{n+1}$, $r, y$ denote the radius and center of $\Sigma$ where $y = (y_1, \ldots, y_n)$. $\rho, c$ denote the radius and center of $\Sigma_{n+2}$ for $c = (c_1, \ldots, c_n)$. With these notations we have

$$\lambda_j = \frac{r^2 + R_j^2 - (a_j - y)^2}{2rR_j}, 1 \leq j \leq n+1. \tag{8.1}$$

$$\lambda_{n+2} = \frac{r^2 + \rho^2 - (c-y)^2}{2r\rho}. \tag{8.2}$$

By Theorem 4.1 we have from (4.2),

$$\Sigma_{j=1}^{n+1}\{\frac{r^2 + R_j^2 - (a_j - y)^2}{2rR_j}\} = (n+1)\mu(\frac{r^2 + \rho^2 - (c-y)^2)}{2r\rho}).$$

As explained above, we fix $\{\Sigma_j\}_{j=1}^{n+2}$ and let $\Sigma$ move freely, or in other words, we consider $r$ and $\{y_1, \ldots, y_n\}$ as $n+1$ free parameters.

Cancelling $r$ from both sides in the above equality, we have,

$$\Sigma_{j=1}^{n+1}\frac{r^2 + R_j^2 - a_j^2 - y^2 + 2(a, y)}{R_j} = (n+1)\mu\frac{r^2 + \rho^2 - (c-y)^2}{\rho}.$$

Equating coefficients on both sides, we start with the free term. Then

$$\Sigma_{j=1}^{n+1}\big(R_j - \frac{a_j^2}{R_j}\big) = (n+1)\mu\big(\rho - \frac{c^2}{\rho}\big). \tag{8.3}$$

Equating the coefficient of $r^2$,

$$\Sigma_{j=1}^{n+1}\frac{1}{R_j} = \frac{(n+1)\mu}{\rho}. \tag{8.4}$$

Similarly, equating the coefficient of $y^2$, we get again the relation (8.4). Equating the coefficient of $\{y_k\}_{k=1}^n$ we have (using $(a_j, y) = \Sigma_{k=1}^n a_{jk}y_k, (c, y) = \Sigma_{k=1}^n c_k y_k$),

$$\Sigma_{j=1}^{n+1}\frac{a_{jk}}{R_j} = \frac{(n+1)\mu c_k}{\rho}, k = 1, 2, \ldots, n. \tag{8.5}$$

Summing up we have

**Theorem 8.1.** *Let $n+1$ spheres $\Sigma_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma, \gamma \neq 1, \gamma \neq \frac{-1}{n}$. Denote by $\{R_j\}_{j=1}^{n+1}$ their radii and by $a_j = (a_{j_1}, \dots, a_{jn})$ their centers, respectively. Let $\Sigma_{n+2}$ be another sphere in $G^n$ having inclination $\mu$ with each of $\{\Sigma_j\}_{j=1}^{n+1}$ where $\mu^2 = \frac{1+\gamma n}{1+n}$. Denote by $\rho$ and $c = (c_1, \dots, c_n)$ the radius and center of $\Sigma$. Then we have (8.3), (8.4) and (8.5).*

It is of interest to give another proof of Theorem 8.1. For this aim we need two simple observations. We put these observations as lemmas, since they will be needed also at a later stage.

**Lemma 8.1.** *Let $\Sigma$ be a plane in $G^n, \Sigma = \{z, (z, \alpha) = \beta\}$ for $\alpha^2 \neq 0$. Let $\Sigma_u$ be a sphere in $G^n$ where $\Sigma_u = \{z - ua)^2 = R^2\}$ and $a, R \in \mathbb{C}$, $a^2 \neq 0$, $R \neq 0$. Then $\frac{\lambda(\Sigma_0, \Sigma_u)}{a^2u^2} \to 0$ as $u \to \infty$.*

*Proof.* The proof is immediate. Indeed, we have

$$\frac{\lambda(\Sigma_0, \Sigma_u)}{u^2a^2} = \frac{(ua, \alpha) - \beta}{R\alpha u^2 a^2} \to 0 \quad \text{as} \quad u \to \infty.$$

We also have

**Lemma 8.2.** *Let $\Sigma = \{z, (z - a)^2 = R^2\}$ be a sphere in $G^n$. Let $\Sigma_k$ be the plane $z_k = 0$ for some $k$, $1 \leq k \leq n$. Then for $a = (a_1, a_2, \dots, a_k, a_{k+1}, \dots, a_nj)$*

$$\lambda(\Sigma, \Sigma_k) = \frac{a_k}{R}, \quad 1 \leq k \leq n. \tag{8.6}$$

*Proof.* The proof follows at once. Indeed, the equation of $\Sigma_k$ is $(z, \alpha) = 0$ for $\alpha = (0, 0, \dots, 1, 0, \dots, )$ where 1 stands at the $k^{\text{th}}$ plane. Hence

$$\lambda(\Sigma, \Sigma_k) = \frac{(a, \alpha)}{R\alpha} = \frac{\Sigma_{j=1}^n a_j \alpha_j}{R\sqrt{\Sigma_1^n \alpha_j^2}} = \frac{a_k \alpha_k}{R\alpha} = \frac{a_k}{R},$$

Since $\alpha = \alpha^2 = 1$.

Using the above two lemmas, we can now give another independent proof of Theorem 8.1. We have to show (8.3), (8.4) and (8.5). (8.4) is an immediate corollary of Theorem 4.1. Indeed, we use the limitation process (Lemma 4.1).

To prove (8.3) we separate the two cases: $\rho^2 - c^2 \neq 0$, $R_j^2 - a_j^2 \neq 0$ for all $1 \leq j \leq n+1$, and the other case is a possibility where some (or all) of them are zero.

We now reflect $\{\Sigma_j\}_{j=1}^{n+2}$ with respect to the unit sphere in $G^n$. Then the new radii are $\frac{R_j}{a_j^2 - R_j^2}$ for $1 \leq j \leq n+1$ and $\frac{\rho}{c^2 - \rho^2}$. Thus, (8.3) follows at

once from (8.4) and invariance property of inclination under reflection. In the other case, namely, $a_j^2 - R_j^2 = 0$, or $c^2 - \rho^2 = 0$, we have to extend (8.4) for the limiting case $R_j = \infty$ for some $j$ or $\rho = \infty$, or both. For this aim we go back to Theorem 4.1 and use it in addition to Lemma 4.1 and Lemma 8.1. Thus we start with (4.2), i.e., $\Sigma_{k=1}^{n+1}\lambda_k = (n+1)\mu\lambda_{n+2}$. For $\Sigma$, the sphere of reference, we now take
$\Sigma = \{z, (z-ua)^2 = R^2, a^2 \neq 0, u \text{ a parameter}\}$.
If $\Sigma_k$ is a sphere, $\Sigma_k = \{z, (z-a_k)^2 = R_k^2\}$, then $\lambda_k = \frac{R_k^2+R^2-(ua-a_k)^2}{2R_kR}$.
If $\Sigma_k$ is a plane, $\Sigma_k = \{z, (z, \eta_k) = \beta_k\}$, then $\lambda_k = \frac{(ua,\eta_k)-\beta_k}{\eta_k R}$.
For the case of a sphere , $\frac{\lambda_k}{-u^2a^2} \to \frac{1}{2R_kR}$ as $u \to \infty$.
For the case of a plane, $\frac{\lambda_k}{-u^2a^2} \to 0$ as $u \to \infty$.
If we agree to consider a plane as a sphere with radius $\infty$ (or bend zero), we may say that $\frac{\lambda_k}{-n^2a^2} \to \frac{1}{2R_kR}$ also for the case of a plane. In any case, the result (8.4) (and thus also (8.3)) is extended for the case where some of the spheres (or reflected spheres) may be planes. It is left to prove (8.5). But this follows at once from $\Sigma_{k=1}^{n+1}\lambda_j = (n+1)\mu\lambda_{n+2}$ where the sphere of reference is $\Sigma_k = \{z, z_k = 0\}$ for some $k$, $1 \le k \le n$. Indeed, by Lemma 8.2 we have

$$\lambda_j = \frac{a_{jk}}{R_j}, 1 \le j \le n+1, \lambda_{n+2} = \frac{C_k}{\rho}.$$

This ends the alternative proof of Theorem 8.1.

## 8.2 Translation of Theorems 4.2 and 4.3

We start with Theorem 4.2. Hence let $\Sigma_j = \{z, (z-a_j)^2 = R_j^2\}_{j=1}^{n+2}$ where $\{\Sigma_j\}_{j=1}^{n+2}$ satisfy the conditions of Theorem 4.2. Let $\Sigma$ be a sphere of reference,
$\Sigma = \{z, (z-y)^2 = r^2\}, y = (y_1, \ldots, y_2)$. Thus

$$\lambda_j = \lambda(\Sigma_j, \Sigma) = \frac{R_j^2 + r^2 - (a_j - y)^2}{2R_jr} = \frac{R_j^2 - a_j^2 + (r^2 - y^2) + 2(a_j, y)}{2R_jr}.$$

It will be convenient to put the notations:

$$r^2 - y^2 = 2y_{n+1}, \quad r^2 = y^2 + 2y_{n+1} = \Sigma_{k=1}^n y_k^2 + 2y_{n+1}; \tag{8.7}$$

$$\frac{R_j^2 - a_j^2}{2R_j} = P_{j0}, \quad 1 \le j \le n+2; \tag{8.8}$$

$$\frac{1}{R_j} = P_{j,n+1}, \quad 1 \le j \le n+2; \tag{8.9}$$

$$\frac{a_{jk}}{R_j} = P_{jk}, \quad 1 \leq j \leq n+2, \quad a_j = (a_{j1}, \ldots, a_{jn}) \quad 1 \leq k \leq n. \tag{8.10}$$

We have from (8.7)-(8.10) and the expression for $\lambda_j$,

$$\lambda_j = \frac{1}{\gamma}\left[P_{j0} + P_{jn+1}y_{n+1} + \Sigma_{k=1}^{n} P_{jk}y_k\right] = \frac{1}{\gamma}\left[P_{j0} + \Sigma_{k=1}^{n+1} P_{jk}y_k\right], \; 1 \leq j \leq n{+}2.$$

>From Theorem 4.2 we get (multiplying by $\gamma^2 = 2y_{n+1} + y^2)$),

$$\begin{aligned} &\big(\Sigma_{j=1}^{n+1}\big[P_{j0} + \Sigma_{k=1}^{n+1} P_{jk}y_k\big]\big)^2 - \big(n + \frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\big[P_{j0} + \Sigma_{k=1}^{n+1} P_{jk}y_k\big]^2 \\ = \quad &\big(n + \frac{1}{\gamma}\big)(1-\gamma)\big\{\big(P_{n+2} + \Sigma_{k=1}^{n+1} P_{n+2,k}y_k\big)^2 - \big(2y_{n+1} + \Sigma_{k=1}^{n} y_k^2\big)\big\}. \end{aligned}$$

We now change to homogeneous coordinates, i.e., we replace $y_k$ by $\frac{y_k}{y_0}$ for $1 \leq k \leq n$. Then

$$\begin{aligned} &\big[\Sigma_{j=1}^{n+1}\Sigma_{k=0}^{n+1}(P_{jk}y_k)\big]^2 - \big(n + \frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\big(\Sigma_{k=0}^{n+1} P_{jk}y_k\big)^2 \\ = \quad &\big(n + \frac{1}{\gamma}\big)(1-\gamma)\big\{\big(\Sigma_{k=0}^{n+1} P_{n+2,k}y_k\big)^2 - 2y_{n+1}y_0 - \Sigma_{k=1}^{n} y_k^2\big\}. \end{aligned}$$

Thus

$$\begin{aligned} &\Sigma_{k,\ell=0}^{n+1}\big(\Sigma_{j=1}^{n+1} P_{jk}\Sigma_{j=1}^{n+1} P_{j\ell}\big)y_k y_\ell - \big(n + \frac{1}{\gamma}\big)\Sigma_{k,\ell=0}^{n+1}\big(\Sigma_{j=1}^{n+1} P_{jk}P_{j\ell}\big)y_k y_\ell \\ = \quad &\big(n + \frac{1}{\gamma}\big)(1-\gamma)\big\{\big(\Sigma_{k,\ell=0}^{n+1} P_{n+2,k}P_{n+2,\ell}y_k y_\ell - 2y_{n+1}y_0 - \Sigma_{k=1}^{n} y_k^2\big\}. \end{aligned}$$

Since the sphere of reference is arbitrary, it follows that the center's coordinates and radius of this sphere may be considered as free parameters. Hence it follows at once that $(y_0, y_1, \ldots, y_{n+1})$ may be considered as free parameters and we can equate coefficients on both sides. We differentiate between various cases.

(I) $\quad 1 \leq k, \ell \leq n, \quad k \neq \ell$

$$\begin{aligned} &\Sigma_{j=1}^{n+1} P_{jk}\Sigma_{j=1}^{n+1} P_{j\ell} - \big(n + \frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1} P_{jk}P_{j\ell} \\ &\qquad = \big(n + \frac{1}{\gamma}\big)(1-\gamma)P_{n+2,k}P_{n+2,\ell}. \end{aligned}$$

>From (8.10) we get

$$\begin{aligned} &\Sigma_{j=1}^{n+1}\frac{a_{jk}}{R_j}\Sigma_{j=1}^{n+1}\frac{a_{j\ell}}{R_j} - \big(n + \frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{a_{jk}a_{j\ell}}{R_j^2} \\ &\qquad = \big(n + \frac{1}{\gamma}\big)(1-\gamma)\frac{a_{n+2,k}a_{n+2,\ell}}{\gamma_{n+2}^2}, \quad k \neq \ell \\ &\qquad\qquad 1 \leq k, \ell \leq n. \end{aligned} \tag{8.11}$$

(II) $\quad 1 \le k \le n, \quad \ell = n+1$

$$\Sigma_{j=1}^{n+1} P_{jk} \Sigma_{j=1}^{n+1} P_{j,n+1} - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} P_{jk} P_{j,n+1}$$
$$= \big(n + \frac{1}{\gamma}\big)(1-\gamma) P_{n+2,k} P_{n+2,n+1}.$$

$>$From (8.9) and (8.10) we have

$$\Sigma_{j=1}^{n+1} \frac{a_{jk}}{R_j} \Sigma_{j=1}^{n+1} \frac{1}{R_j} - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} \frac{a_{jk}}{R_j^2} \tag{8.12}$$
$$= \frac{(n + \frac{1}{\gamma})(1-\gamma) a_{n+2,k}}{R_{n+2}^2}, \quad 1 \le k \le n.$$

(III) $\quad k = \ell, \quad 1 \le k \le n$

$$\big(\Sigma_{j=1}^{n+1} P_{jk}\big)^2 - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} P_{jk}^2 = \big(n + \frac{1}{\gamma}\big)(1-\gamma)(P_{n+2,k}^2 - 1).$$

Using (8.10) we have

$$\big(\Sigma_{j=1}^{n+1} \frac{a_{jk}}{R_j}\big)^2 - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} \frac{a_{jk}^2}{R_j^2} \tag{8.13}$$
$$= \big(n + \frac{1}{\gamma}\big)(1-\gamma)\big(\frac{a_{n+2,k}^2}{R_{n+2}^2} - 1\big), \quad 1 \le k \le n.$$

(IV) $\quad k = \ell = n+1$

$$\big(\Sigma_{j=1}^{n+1} P_{j,n+1}\big)^2 - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} P_{j,n+1}^2 = \big(n + \frac{1}{\gamma}\big)(1-\gamma) P_{n+2,n+1}^2$$

and using (8.9),

$$\big(\Sigma_{j=1}^{n+1} \frac{1}{R_j}\big)^2 - \big(n + \frac{1}{\gamma}\big) \Sigma_{j=1}^{n+1} \frac{1}{R_j^2} = \big(n + \frac{1}{\gamma}\big)(1-\gamma) \frac{1}{R_{n+2}^2}. \tag{8.14}$$

This ends all cases arising from $1 \le k, \ell \le n+1$. We are left with cases involved with $k$ or $\ell$ (or both) zero.

(V) $\quad k = \ell = 0$

$$\big(\Sigma_{j=1}^{n+1} P_{j0}\big)^2 - \big(n + \frac{1}{\gamma}\big)(1-\gamma) P_{n+2,0}^2.$$

Putting the value of $P_{j0}$ by (8.8),

$$\begin{aligned}\big(\Sigma_{j=1}^{n+1}\frac{R_j^2-a_j^2}{R_j}\big)^2-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\big(\frac{R_j^2-a_j^2}{R_j}\big)^2 \\ =\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big(\frac{R_{n+2}^2-a_{n+2}^2}{R_n+2}\big).\end{aligned}\tag{8.15}$$

(VI) $\quad \ell=0,\ \ 1\le k\le n$

$$\Sigma_{j=1}^{n+1}P_{jk}\Sigma_{j=1}^{n+1}P_{j0}-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}P_{jk}P_{j0}=\big(n+\frac{1}{\gamma}\big)(1-\gamma)P_{n+2,k}P_{n+2,0},$$

and using (8.8) and (8.10),

$$\begin{aligned}\big(\Sigma_{j=1}^{n+1}\frac{a_{jk}}{R_j}\big)\Sigma_{j=1}^{n+1}\big(\frac{R_j^2-a_j^2}{R_j}\big)-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{a_{jk}}{R_j^2}(R_j^2-a_j^2) \\ =\big(n+\frac{1}{\gamma}\big)(1-\gamma)\frac{a_{n+2,k}}{R_{n+2}^2}(R_{n+2}^2-a_{n+2}^2).\end{aligned}\tag{8.16}$$

(VII) $\quad \ell=0,\ \ k=n+1$

$$\begin{aligned}\Sigma_{j=1}^{n+1}P_{j,n+1}\Sigma_{j=1}^{n+1}P_{j0}-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}P_{j,n+1}P_{j0} \\ =\big(n+\frac{1}{\gamma}\big)(1-\gamma)[P_{n+2,n+1}P_{n+1,0}-1]\end{aligned}$$

and, from (8.8) and (8.9),

$$\begin{aligned}\Sigma_{j=1}^{n+1}\frac{1}{R_j}\Sigma_{j=1}^{n+1}\big(\frac{R_j^2-a_j^2}{2R_j}\big)-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{2R_j^2}(R_j^2-a_j^2) \\ =\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big[\frac{1}{2R_{n+2}^2}(R_{n+2}^2-a_{n+2}^2)-1\big].\end{aligned}\tag{8.17}$$

Analysis of the above seven cases shows that there are in fact two different groups. If $k=\ell$, (i.e., cases III, IV and V), the results are elementary consequences from the previous theorem. The situation is different for the other four cases. In fact, the information in these four other cases leads us to guess a new interesting more general theorem that will be proved at a later stage. To be more specific we start with case (III). We first choose $\Sigma=\{z,z_k=0\}$ for some $k, 1\le k\le n$, as the sphere of reference, and we use Lemma 8.2. Then we have $\lambda_j=\lambda(\Sigma_j,\Sigma)=\frac{a_{jk}}{R_j}, 1\le j\le n+2$, and we get from (8.13) $\left(\Sigma_{j=1}^{n+1}\lambda_j\right)^2 - \left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\lambda_j^2 = \left(n+\frac{1}{\gamma}\right)(1-\gamma)(\lambda_{n+2}^2-1)$, i.e., we are back in Theorem 4.2. Continuing with case IV, i.e., (8.14), we see that we have the well known radii version of Theorem 4.2.

Case V is similar to case IV. Indeed, (8.15) is again the radii version of Theorem 4.2, but after reflection with respect to the unit sphere in $G^n$. (Compare with Theorem 8.1 and its second proof.)

We now consider case I. Using Lemma 8.2 for $\Sigma = \{z, z_k = 0\}$, $\Sigma' = \{z, z_\ell = 0\}$ we have from (8.11),

$$\left(\Sigma_{j=1}^{n+1}\lambda_j\right)\Sigma_{j=1}^{n+1}(\lambda'_j) - \left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\lambda_j\lambda'_j = \left(n+\frac{1}{\gamma}\right)(1-\gamma)\lambda_{n+2}\lambda'_{n+2}$$
$$\lambda_j = \lambda(\Sigma_j, \Sigma), \quad \lambda'_j = \lambda(\Sigma_j, \Sigma'), \quad 1 \le j \le n+2. \quad \text{(8.11')}$$

Note that $\Sigma$ is orthogonal to $\Sigma'$.

Comparing (8.11′) with Theorem 4.2, one is naturally led to guess that one more general relation might exist, namely,

$$\begin{aligned} &\Sigma_{j=1}^{n+1}\lambda_j\Sigma_{j=1}^{n+1}\lambda'_j - \left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\lambda_j\lambda'_j \\ = \quad &\left(n+\frac{1}{\gamma}\right)(1-\gamma)\left(\lambda_{n+2}\lambda'_{n+2} - \lambda(\Sigma,\Sigma')\right) \end{aligned} \tag{8.18}$$

for $\lambda_j = \lambda(\Sigma_j, \Sigma)$, $\lambda'_j = \lambda(\Sigma_j, \Sigma')$, $1 \le j \le n+2$. Indeed, if $\Sigma = \Sigma'$, we are back in Theorem 4.2, as $\lambda(\Sigma, \Sigma') = 1$. If $\Sigma$ is orthogonal to $\Sigma'$, we have (8.11′).

It turns out that this guess fits the other three remaining cases as well. Indeed, the results of cases II, VI and VII follow from (8.11′). Later on we present (8.18) in the form of a theorem. In the meantime we present the above more particular results.

**Theorem 8.2.** *Let $n+1$ spheres $\Sigma_j = \{z, (z-a_j)^2 = R_j^2\}$, $1 \le j \le n+1$ be given in $G^n$. Assume that they have a mutual inclination $\gamma, \gamma \ne 0, \gamma \ne 1, \gamma \ne -\frac{1}{n}$. Let $\Sigma_{n+2} = \{z_j(z-a_n)^2 = R_{n+2}^2\}$ be another sphere in $G^n$ which is orthogonal to each of $\{\Sigma_j\}_{j=1}^{n+2}$. Denote $a_j = (a_{j1}, a_{j2}, \dots, a_{jn})$ for $1 \le j \le n+2$. Also denote by $\{R'_j\}_{j=1}^{n+2}$ the radii of reflected spheres $\{\Sigma'_j\}_{j=1}^{n+2}$ with respect to the unit sphere in $G^n$. Then (8.11) (or, alternatively, the more general form (8.11′)) is satisfied.*

*Putting $\lambda_j = \frac{a_{jk}}{R_j}$ for some $k$, $1 \le k \le n$, we also have*

$$\Sigma_{j=1}^{n+1}\lambda_j\Sigma_{j=1}^{n+1}\frac{1}{R_j} - \left(n-\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\frac{\lambda_j}{R_j} = \left(n+\frac{1}{\gamma}\right)(1-\gamma)\frac{\lambda_{n+2}}{R_{n+2}}. \quad \text{(8.12')}$$

*Also*

$$\Sigma_{j=1}^{n+1}\lambda_j\Sigma_{j=1}^{n+1}\frac{1}{R'_j}-\big(n-\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\lambda_j\frac{1}{R'_j}=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\lambda_{n+2}\cdot\frac{1}{R'_{n+2}}, \quad (8.16')$$

$$\Sigma_{j=1}^{n+1}\frac{1}{R_j}\Sigma_{j=1}^{n+1}\frac{1}{R'_j}-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{R_j}\frac{1}{R'_j}=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big[\frac{1}{R_{n+2}}\frac{1}{R'_{n+2}}+2\big]. \quad (8.17')$$

*Proof.* In fact, almost everything is proved above. We just note that $R'_j=\frac{a_j^2-R_j^2}{R_j}$. That is why the plus sign appears in (8.17′). We also note that $\frac{1}{R'_j}$ may be equal to zero (in the case $a_j^2-R_j^2=0$) (see Theorem 8.1 and its second proof). Note also that only four of the seven cases were presented in the statement of the theorem, as the other three cases (i.e., $k=\ell$) are most elementary (this was explained earlier) and essentially presented previously. As mentioned above, at a later stage we will prove a more general theorem that will imply Theorem 8.2. Indeed, this will imply an independent proof of (8.11′), (8.12′), (8.16′) and (8.17′).

Our next aim is to present a similar theorem to Theorem 8.2, arising from Theorem 4.3 instead of from Theorem 4.2. In other words, we now consider the case of $n+2$ spheres with mutual inclination $\gamma$ instead of $n+1$ spheres with mutual inclination $\gamma$, and an additional orthogonal sphere to these $n+1$ spheres. For this aim we start with Theorem 4.3, i.e., with (4.21) and imitate what we did earlier in order to prove Theorem 8.2. We use, again, the same ideas and the same notation, i.e., (8.7)-(8.10).

We then get

$$\Sigma_{k,\ell=0}^{n+1}\big(\Sigma_{j=1}^{n+2}P_{jk}\Sigma P_{j\ell}\big)y_ky_\ell-\big(n+1+\frac{1}{\gamma}\big)\Sigma_{k,\ell=0}^{n+1}\big(\Sigma_{j=1}^{n+2}P_{jk}P_{j\ell}\big)y_ky_\ell$$
$$=\big(n+1+\frac{1}{\gamma}\big)(\gamma-1)\big[2y_{n+1}y_0+\Sigma_{k=1}^{n}y_k^2\big].$$

As in the previous proof, here too we get seven different cases.

(I) $1\le k,\ell\le n,\quad k\ne\ell,$

$$\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\Sigma_{j=1}^{n+2}\frac{a_{j\ell}}{R_j}-\big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\frac{a_{j\ell}}{R_\ell}=0. \quad (8.19)$$

(II) $1\le k\le n,\quad \ell=n+1,$

$$\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\Sigma_{j=1}^{n+2}\frac{1}{R_j}-\big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\frac{1}{R_j}=0. \quad (8.20)$$

(III) $1 \le k, \ell \le n, \quad k = \ell,$

$$\big(\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\big)^2 - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{a_{jk}^2}{R_j^2} = \big(n+1+\frac{1}{\gamma}\big)(\gamma-1). \tag{8.21}$$

(IV) $k = \ell = n+1,$

$$\big(\Sigma_{j=1}^{n+1}\frac{1}{R_j}\big)^2 - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{1}{R_j^2} = 0. \tag{8.22}$$

(V) $k = 0, \quad \ell = 0,$

$$\big(\Sigma_{j=1}^{n+2}\big(\frac{R_j^2 - a_j^2}{R_j}\big)\big)^2 - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\big(\frac{R_j^2 - a_j^2}{R_j}\big)^2 = 0. \tag{8.23}$$

(VI) $\ell = 0, \quad 1 \le k \le n,$

$$\big(\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\big)\big(\Sigma_{j=1}^{n+2}\frac{R_j^2 - a_j^2}{R_j}\big) - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\big(\frac{R_j^2 - a_j^2}{R_j}\big) = 0. \tag{8.24}$$

(VII) $\ell = 0, \quad k = n+1,$

$$\begin{aligned} &\Sigma_{j=1}^{n+2}\big(\frac{R_j^2 - a_j^2}{2R_j}\big)\Sigma_{j=1}^{n+2}\frac{1}{R_j} - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{R_j^2 - a_j^2}{2R_j}\frac{1}{R_j} \\ = \ &\big(n+1+\frac{1}{\gamma}\big)(\gamma-1). \end{aligned} \tag{8.25}$$

Like before, cases III, IV and V are not essentially new. We present the four other cases in the form of a theorem.

**Theorem 8.3.** *Let $n+2$ spheres $\Sigma_j = \{z, (z-a_j)^2 = R_j^2\}, 1 \le j \le n+2$ be given in $G^n$. Assume that all spheres have mutual inclination $\gamma$, $\gamma \ne 1$. Denote $a_j = (a_{j1}, \ldots, a_{jn})$, $1 \le j \le n+2$. Also denote by $\{R_j'\}_{j=1}^{n+2}$ the radii of reflected spheres $\{\Sigma_j'\}_{j=1}^{n+2}$ with respect to the unit sphere. Then we have (8.19), (8.20), and*

$$\big(\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\big)\Sigma_{j=1}^{n+2}\frac{1}{R_j'} - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{a_{jk}}{R_j}\frac{1}{R_j'}, \quad 1 \le k \le n, \tag{8.24'}$$

$$\Sigma_{j=1}^{n+2}\frac{1}{R_j}\Sigma_{j=1}^{n+2}\frac{1}{R_j'} - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{1}{R_j}\frac{1}{R_j'}, = 2\big(n+1+\frac{1}{\gamma}\big)(1-\gamma). \tag{8.25'}$$

The proof is given above. Again, note that the sign in (8.18$'$) is reversed, as $R_j' = \frac{R_j}{a_j^2 - R_j^2}$. Like before, here too we have a more general result involved with two spheres of references instead of one, which will be presented later and will contain Theorem 8.3 as a special case.

# 9 Two spheres of references: Generalization of Theorems 4.2 and 4.3

## 9.1 Generalization of Theorem 4.2

As explained in the previous section, in view of the results there (i.e., Theorems 8.2 and 8.3), it is natural to conjecture a generalization of Theorems 4.2 and 4.3, where two spheres of reference replace the one sphere. We first start with an extension of Theorem 4.2.

**Theorem 9.1.** *Let $n+1$ spheres $\{\Sigma_j\}_{j=1}^{n+1}$ be given in $G^n$ with mutual inclination $\gamma \neq 0, \gamma \neq 1, \gamma \neq -\frac{1}{n}$. Let $\Sigma_{n+2}$ be another sphere which is orthogonal to each of $\{\Sigma_j\}_{j=1}^{n+1}$. Let $\Sigma, \Sigma'$ be two arbitrary spheres of reference. Denote by $\{\lambda_j\}_{j=1}^{n+2}$ and $\{\mu_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$ with $\Sigma, \Sigma'$ respectively. Also denote by $\lambda(\Sigma, \Sigma')$ the inclination between $\Sigma$ and $\Sigma'$. Then*

$$\begin{aligned}\big(\Sigma_{k=1}^{n+1}\lambda_k\big)\big(\Sigma_{k=1}^{n+1}\mu_k\big) - \big(n+\frac{1}{\gamma}\big)\Sigma_{k=1}^{n+1}\lambda_k\mu_k \\ = \big(n+\frac{1}{\gamma}\big)(1-\gamma)\big[\lambda_{n+2}\mu_{n+2} - \lambda(\Sigma,\Sigma')\big].\end{aligned} \tag{9.1}$$

*Proof.* The proof is very similar to the proof of the particular case $\Sigma = \Sigma'$, i.e., Theorem 4.2. The notations introduced in the proof of Theorem 4.2 will be used here and the same is true for the ideas of the proof. Hence we follow again the model of Coxeter [8] (as in Theorem 4.1 and Theorem 4.2) and consider the plane $\Sigma_{j=1}^{n+1} z_j = 1$ in $G^{n+1}$. We may assume by Theorem 3.2 that all radii of $\{\Sigma_j\}_{j=1}^{n+1}$ are equal. The common value is denoted by $R$. Their centers are located at $(1,0\cdots),(0,1,0\cdots),\ldots,(0,0\cdots1)$. As in (4.3) we have $\gamma = 1-\frac{1}{R^2}, 1-\gamma = \frac{1}{R^2}$. The radius of $\Sigma_{n+2}$ is again denoted by $\rho$, and its center is necessarily at $\big(\frac{1}{n+1},\ldots,\frac{1}{n+1}\big)$. As before, the center of $\Sigma$ is denoted by $(y_1,\ldots,y_{n+1})$ and its radius is denoted by $r$. For the other sphere of reference $\Sigma'$ we now put the notation $\tau$ for its radius and $t = (t_1,\ldots,t_{n+1})$ for its center. We recall the orthogonality condition (4.12), i.e., $\rho^2 + R^2 - \frac{4}{n+1} = 0$. We use, again, the notation $a = R^2 - \rho^2 - \frac{1}{n+1}, b = y^2 - r^2 - R^2$ (see (4.13)). In addition, we now put the notation

$$c = t^2 - \tau^2 - R^2. \tag{9.2}$$

Exactly as in the proof of Theorem 4.2 we now have (see (4.14) and (4.15)),

$$\lambda_k = (2y_k - 1 - b)\frac{1}{2rR}, \;\; \mu_k = (2t_k - 1 - c)\frac{1}{2\tau R}, \;\; 1 \le k \le n+1; \tag{9.3}$$

$$\lambda_{n+2} = \frac{-b-a}{2r\rho}, \quad \mu_{n+2} = \frac{-c-a}{2\tau\rho}. \tag{9.4}$$

We also have

$$\lambda(\Sigma, \Sigma') = \frac{r^2 + \tau^2 - y^2 - t^2 + 2(y,t)}{2r\tau}. \tag{9.5}$$

Using $\Sigma_{k=1}^{n+1} y_k = 1$, it follows easily from (9.3) that

$$\big(\Sigma_{k=1}^{n+1}\lambda_k\big)\big(\Sigma_{k=1}^{n+1}\mu_k\big) = \frac{((n+1)b + (n-1))((n+1)c + (n-1))}{4r\tau R^2} \tag{9.6}$$

(compare with (4.16) for $\lambda_k = \mu_k$ and $b = c$). Similarly,

$$\Sigma_{k=1}^{n+1}\lambda_k\mu_k = \frac{(n-3) + (n-1)b + (n-1)c + (n+1)bc + 4(t,y)}{4r\tau R^2} \tag{9.7}$$

(compare with (4.17) for $b = c$ and $(y,t) = y^2$). >From (9.4), using (4.19),

$$\lambda_{n+2}\mu_{n+2} = \frac{\big(b + \frac{1+\gamma}{1-\gamma}\big)\big(c + \frac{1+\gamma}{1-\gamma}\big)}{4r\tau p^2}. \tag{9.8}$$

Using $\frac{1}{R^2} = 1 - \gamma$, (9.5), (9.6), (9.7) and (9.8) we get that (9.1) is reduced to

$$\begin{aligned} & \big((n+1)b + (n-1)\big)\big((n+1)c + (n-1)\big) \\ & -\big(n + \frac{1}{\gamma}\big)\big[(n-3) + (n-1)b + (n-1)c + (n+1)bc + 4(y,t)\big] \\ = \quad & \big(n + \frac{1}{\gamma}\big)\Big[\frac{\big(b + \frac{1+\gamma}{1-\gamma}\big)\big(c + \frac{1+\gamma}{1-\gamma}\big)}{\rho^2} - 2\big(r^2 + \tau^2 - y^2 - t^2 + 2(y,t)\big)\Big]. \end{aligned}$$

>From $\rho^2 + R^2 - \frac{n}{n+1} = 0, \quad R^2 = \frac{1}{1-\gamma}$, we easily get

$$\frac{1}{\rho^2} = \frac{(n+1)(1-\gamma)}{-(n\gamma+1)}. \tag{9.9}$$

Also from $b = y^2 - r^2 - R^2, \;\; c = t^2 - \tau^2 - R^2$, and $R^2 = \frac{1}{1-\gamma}$, we have

$$b + c = t^2 + y^2 - r^2 - \tau^2 - \frac{2}{1-\gamma}. \tag{9.10}$$

Putting (9.9) and (9.10) in the above equation, we get

$$\begin{aligned}
&\big((n+1)b+(n-1)\big)\big((n+1)c+(n-1)\big)-\big(n+\frac{1}{\gamma}\big)\\
&\big[(n-3)+(n-1)b+(n-1)c+(n+1)bc+4(y,t)\big]\\
=\;&\big(n+\frac{1}{\gamma}\big)\Big[\frac{\big(b+\frac{1+\gamma}{1-\gamma}\big)\big(c+\frac{1+\gamma}{1-\gamma}\big)(n+1)(1-\gamma)}{-(n\gamma+1)}\\
&+2\big(b+c+\frac{2}{1-\gamma}-2(y,t)\big)\Big].
\end{aligned}$$

Comparing coefficients of $bc, b, c, (y,t)$, this is easily seen to be an identity for each $\gamma$. This ends the proof of the theorem.

### 9.2 Generalization of Theorem 4.3

**Theorem 9.2.** *Let $\{\Sigma_j\}_{j=1}^{n+2}$ be $n+2$ spheres in $G^n$ having mutual inclination $\gamma \neq 1$. Let $\Sigma, \Sigma'$ be two spheres of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}$, $\{\mu_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$ with $\Sigma$ and $\Sigma'$ respectively. Also, $\lambda(\Sigma,\Sigma')$ denotes the inclination between $\Sigma$ and $\Sigma'$. Then*

$$\begin{aligned}
\big(\Sigma_{k=1}^{n+2}\lambda_k\big)\big(\Sigma_{k=1}^{n+2}\mu_k\big)-\big(n+1+\frac{1}{\gamma}\big)\Sigma_{k=1}^{n+2}\lambda_k\mu_k \qquad (9.11)\\
=\big(n+1+\frac{1}{\gamma}\big)(\gamma-1)\lambda(\Sigma,\Sigma').
\end{aligned}$$

*Proof.* Theorem 9.2 follows from Theorem 9.1 in exactly the same way as Theorem 4.3 follows from Theorem 4.2. All considerations are identical and thus the details are omitted.

We note that, alternatively, we can give a direct proof as for Theorem 9.1 (the same is true for Theorem 4.3).

## 10 Complex approach to hyperbolic transformation

### 10.1 Poincaré extension and hyperbolic transformation

We use the notations introduced in section 5. We started there with the sphere $\{z, \Sigma_{j=1}^n z_j^2 = R^2,\ R = i\}$ and considered the "projected" sphere $\Sigma_{j=1}^n x_j^2 + (ix_0)^2 = i^2 = -1$, or $1+\Sigma_{j=1}^n x_j^2 = x_0^2$. In other words, the hyperboloid $1+\Sigma_{j=1}^n x_j^2 = x_0^2$ is viewed as a projected sphere in $G^{n+1}$. We then considered the sphere centered at $(\mu_1, \ldots, \mu_n, i\mu_0)$ with radius $R$, i.e., $\Sigma_{j=1}^n (x_j - \mu_j)^2 - (x_0 - \mu_0)^2 = R^2$ is another hyperboloid which is viewed as a projected sphere in $G^{n+1}$ from a sphere with center $(\mu_1, \ldots, \mu_n, i\mu_0)$ and a real radius $R$. We then found the condition of orthogonality of these two spheres (i.e., (5.7), or $R^2 - 1 = \mu^2 - \mu_0^2$ for $\mu^2 = \Sigma_{j=1}^n \mu_i^2$).

Later, in Lemma 5.1 we proved that if two such projected spheres (see (5.10) are both orthogonal to the sphere $S$ mentioned above, then the inclination between these two spheres is invariant under hyperbolic transformation of these two spheres (or hyperboloids) onto the hyperbolic space $\Delta^n$. Using the orthogonality conditions for both spheres, we showed (see (5.3)) that the inclination $\lambda$ satisfied $\lambda = \frac{1+(\mu,\eta)-\mu_0\eta_0}{\rho R}$, which is the same as the inclination between the images $\Sigma_{j=1}^n \big(y_j - \frac{\mu_j}{1+\mu_0}\big)^2 = \big(\frac{R}{1+\mu_0}\big)^2, \Sigma_{j=1}^n \big(y_j - \frac{\eta_j}{1+\eta_0}\big) = \big(\frac{\rho}{1+\eta_0}\big)^2$ (see (5.17) and (5.17$'$)).

We now propose a different way to look at these issues. Since $S$ is orthogonal to the two mentioned spheres, it is clear that if we use a Möbius transformation to map $S$ on to $z_{n+1} = 0$, the two spheres will be mapped onto two orthogonal spheres to $z_{n+1} = 0$. Indeed, this is assured by the invariance property of the inclination via a Möbius transformation. it is only natural to expect that the Poincaré extension of the two spheres mentioned above (i.e., (5.17) and (5.17$'$) are exactly these two orthogonal spheres. This is indeed correct, as we verify below. Of course, this gives a new proof of Lemma 5.1. But not less important, it gives a new look at hyperbolic transformation. Indeed, we may view this transformation as a Möbius transformation in $G^{n+1}$ that maps $S$ onto $z_{n+1} = 0$.

## 10.2 A Möbius map from $S$ onto $z_{n+1} = 0$

**Theorem 10.1.** *Define in $G^{n+1}$, $a = (0, \ldots, 0, -i)$ and*

$$a - z = 2\frac{(w-a)}{(w+a)^2}. \tag{10.1}$$

*Then the sphere $w^2 = -1 = i^2$ is mapped onto $z_{n+1} = 0$. Also define in $G^{n+1}$ the sphere*

$$(w - \tilde{\mu})^2 = R^2, \quad \tilde{\mu} = (\mu_1, \ldots, \mu_n, i\mu_0) \tag{10.2}$$
$$\tilde{\mu}^2 = \mu^2 - \mu_0^2 = \Sigma_{k=1}^n \mu_k^2 - \mu_0^2.$$

*Assume also that $R^2 + (i)^2 = R^2 - 1 = \tilde{\mu}^2$ (orthogonality condition). Then*

$$\Sigma_{j=1}^n \big(z_j - \frac{\mu_j}{1+\mu_0}\big)^2 = \big(\frac{R}{1+\mu_0}\big)^2 \ \text{ if } \ \mu_0 \neq -1, \tag{10.3}$$

$$\Sigma_{j=1}^n z_j\mu_j + 1 = 0 \quad \text{if} \quad \mu = -1. \tag{10.4}$$

*Proof.* To prove the first part, note that $a^2 = -1$. Hence, from $w^2 = -1$ we get $-1 = (w-a+a)^2 = (w-a)^2+a^2+2(w-a,a) = (w-a)^2-1+2(w-a,a)$. Hence $1 + 2\big(\frac{w-a}{(w-a)^2}, a\big) = 0$. From (10.1) this means that $1 + (a - z, a) = 0$ or $1 + (a, a) - (z, a) = 0$. But $(a, a) = a^2 = -1$ now yields $(z, a) = 0$ or $z_{n+1} = 0$. This ends the proof of the first part.

To prove the second part, we start with

$$(w - \tilde{\mu})^2 = R^2, \qquad \tilde{\mu} = (\mu_1, \ldots, \mu_n, i\mu_0), \qquad \tilde{\mu}^2 = \mu^2 - \mu_0^2.$$

$$(w - a + a - \tilde{\mu})^2 = R^2, \quad (w - a)^2 + (a - \tilde{\mu})^2 + 2(w - a, a - \tilde{\mu}) = R^2.$$

Hence, $(w-a)^2+2(w-a,a-\tilde{\mu}) = R^2-a^2-\tilde{\mu}^2+2(a,\tilde{\mu}) = R^2+1-\tilde{\mu}^2+2(a,\tilde{\mu})$ $= 2 + 2(a,\tilde{\mu})$ where the orthogonality condition $R^2 - \tilde{\mu}^2 = 1$ has been used. But $(a,\tilde{\mu}) = \mu_0$ as $a = (0, 0, \ldots, 0, -i)$ and $\tilde{\mu} = (\mu_1, \mu_2, \ldots, i\mu_0)$. Putting this in the above, it follows that $(w-a)^2+2(w-a,a-\tilde{\mu}) = 2(1+\mu_0)$. Thus

$$1 + 2\big(\frac{w - a}{(w - a)^2}, a - \tilde{\mu}\big) = \frac{2(1 + \mu_0)}{(w - a)^2}.$$

Using (10.1), this implies $1 + (a - z, a - \tilde{\mu}) = \frac{2(1+\mu_0)}{(w-a)^2}$. Again, by (10.1), we have $(a-z)^2 = \frac{4(w-a)^2}{w-a)^4} = 4(w-a)^2$. Hence, $1+(a-z,a-\tilde{\mu}) = \frac{(1+\mu_0)(a-z)^2}{2}$ or $1+(a,a-\tilde{\mu})-(z,a-\tilde{\mu}) = \frac{1+\mu_0}{2}(a-z)^2$. We have $(a,a-\tilde{\mu}) = a^2-(a,\tilde{\mu}) = -1-\mu_0$. This implies $-\mu_0-(z,a)+(z,\tilde{\mu}) = \frac{1+\mu_0}{2}(z-a)^2 = \frac{1+\mu_0}{2}(z^2-2(a,z)+a^2)$ or $-\mu_0-(z,a)+(z,\tilde{\mu}) = \frac{1+\mu_0}{2}(z^2-a)-(a,z)(1+\mu_0)$. Cancelling $-(z,a)$ on both sides and using $(z,\tilde{\mu}) = \Sigma_{k=1}^n z_k\mu_k+i\mu_0 z_{n+1}$, we have $-\mu_0+\Sigma_{k=1}^n z_k\mu_k+i\mu_0 z_{n+1} = \frac{1+\mu_0}{2}(z^2 - 1)$ $-\mu_0(a,z)$. But $(a,z) = -iz_{n+1}$, and thus $i\mu_0 z_{n+1}$ is cancelled with $-\mu_0(a,z)$. Hence $-\mu_0+\Sigma_{k=1}^n z_k\mu_k = \big(\frac{1+\mu_0}{2}\big)(z^2-1)$ or $-2\mu_0+2\Sigma_{k=1}^n z_k\mu_k = (1+\mu_0)z^2-1-\mu_0$. We get

$$(1 + \mu_0)z^2 + (\mu_0 - 1) - 2(z, \mu) = 0. \tag{10.5}$$

If $\mu_0 \neq -1$ we divide by $1 + \mu_0$ and then $z^2 = \frac{-2(z,\mu)}{1+\mu_0} = \frac{1-\mu_0}{1+\mu_0}$. Hence, $\big(z - \frac{\mu}{1+\mu_0}\big)^2 = \frac{1-\mu_0}{1+\mu_0} + \frac{\mu^2}{(1+\mu_0)^2} = \frac{1-\mu_0^2+\mu^2}{(1+\mu_0)^2}$. But $1 - \mu_0^2 + \mu^2 = 1 + \tilde{\mu}^2 = R^2$ by the orthogonality condition. Thus $\big(z-\frac{\mu}{1+\mu_0}\big)^2 = \big(\frac{R}{1+\mu_0}\big)^2$ which confirms (10.3). This ends the case $\mu_0 \neq -1$ in (10.5). If $\mu_0 = -1$, then (10.5) implies $-2 - 2(z, \mu) = 0$, which is (10.4). This ends the proof of the theorem.

Note that both generalized spheres in (10.3) and (10.4) are orthogonal to $z_{n+1} = 0$, as expected. Indeed, the sphere in (10.3) is centered at $(\mu_1, \mu_2, \ldots, \mu_n, 0)$, and thus is orthogonal to $z_{n+1} = 0$. The same is true for the plane $\Sigma_{j=1}^n z_j\mu_j = -1$ appearing in (10.4).

# 11 Some additional aspects of the hyperbolic space $\Delta^n$

## 11.1 Exponential hyperbolic radius

In section 5 (see also section 10) we pointed out that the limitation $R < 1$ is somewhat artificial. More specifically, $R = \tan h\beta$, where $\beta$ is the hyperbolic radius, implies that $R < 1$ (see (5.8)). On the other hand, there is not any limitation of this sort before applying the hyperbolic transformation.

This naturally raises the question whether one can extend the definition of $R$ to avoid this limitation. This is our first aim in the present section.

Let $\Sigma_{j=1}^n (x_j - a_j)^2 = \rho^2$ be a sphere in $R^n$. We denote, as usual, by $\rho' = \frac{\rho}{a^2-\rho^2}$ the radius of the sphere after inversion with respect to the unit sphere. We then have

**Definition 11.1.** *The "Exponential hyperbolic radius" $R$ is defined by*

$$\frac{1}{R} = \frac{1}{2}\Big(\frac{1}{\rho} - \frac{1}{\rho'}\Big). \tag{11.1}$$

To justify this definition, we now prove

**Theorem 11.1.** *Let $\Sigma_{j=1}^n (x_i - a_i)^2 = \rho^2$ be a sphere in $\Delta^n$. Denote by $\beta$ its hyperbolic radius. Also denote $R = \tan h\beta$. Then (11.1) is satisfied.*

*Proof.* We have for the hyperbolic radius $\beta$,

$$\beta = \frac{1}{2}\ell n\left[\frac{1+x}{1-x}\right]_{a-\rho}^{a+\rho} \quad \text{for} \quad a^2 = \Sigma_{j=1}^n a_j^2.$$

Then

$$\beta = \frac{1}{2}\ell n\left[\frac{1+(\rho+a)}{1-(\rho+a)} \cdot \frac{1-(a-\rho)}{1+(a-\rho)}\right] = \frac{1}{2}\ell n\frac{(1+\rho)^2 - a^2}{(1-\rho)^2 - a^2}.$$

Hence, for $R = \tan h\beta = \frac{e^{2\beta}-1}{e^{2\beta}+1}$ we get

$$R = \Big(\frac{(1+\rho)^2 - a^2}{(1-\rho)^2 - a^2} - 1\Big)\Big(\frac{(1+\rho)^2 - a^2}{(1-\rho)^2 - a^2} + 1\Big)^{-1}.$$

Hence,

$$R = \frac{(1+\rho)^2 - (1-\rho)^2}{(1+\rho)^2 + (1-\rho)^2 - 2a^2} = \frac{4\rho}{2(1+\rho^2) - 2a^2} = \frac{2\rho}{1+\rho^2-a^2}.$$

This implies

$$\frac{1}{R} = \frac{1+\rho^2-a^2}{2\rho} = \frac{1}{2}\big(\frac{1}{\rho} - \frac{(a^2-\rho^2)}{\rho}\big) = \frac{1}{2}\big(\frac{1}{\rho} - \frac{1}{\rho'}\big).$$

Hence (11.1) is confirmed and this ends the proof of the theorem.

It is only natural now to make

**Definition 11.2.** *Given the sphere $\Sigma_{j=1}^{n}(x_j - a_j)^2 = \rho^2$ in $R^n$, we define the hyperbolic radius of it by*

$$\beta = \frac{1}{2}\ell n\frac{1+R}{1-R} \tag{11.2}$$

*where $R$ is the exponential hyperbolic radius defined in (1.1).*

Note that if the sphere is in $\Delta^n$, then $\beta$ is the ordinary hyperbolic radius by (5.19) and Theorem 11.1. Indeed, $R = \tan h\beta$ is equivalent to (11.3).

We now prove the following generalization of the Mauldon hyperbolic inclination theorem (Theorem 5.1).

**Theorem 11.2.** *Let $n+2$ spheres $\{S_1, \dots, S_{n+2}\}$ be given in $R^n$. Denote by $\{\beta_j\}_{j=1}^{n+2}$ the (generalized) hyperbolic radii as defined above.*

*Assume further that $\{S_j\}_{j=1}^{n+2}$ have mutual inclination $\gamma \neq 0$, $\gamma \neq 1$. Then*

$$\big(\Sigma_{j=1}^{n+2}\frac{1}{\tan h\beta_j}\big)^2 - \big(n+\frac{1}{\gamma}+1\big)\Sigma_{j=1}^{n+2}\frac{1}{\tan h^2\beta_j} = \big(n+1+\frac{1}{\gamma}\big)(\gamma-1). \tag{11.3}$$

*Proof.* We have to show for $\frac{1}{R_j} = \frac{1}{2}\big(\frac{1}{\rho_j} - \frac{1}{\rho_j'}\big) = \frac{1}{\tan h\beta_j}$ that

$$\begin{aligned}\big(\Sigma_{j=1}^{n+2}\frac{1}{2}\big(\frac{1}{\rho_j} - \frac{1}{\rho_j'}\big)\big)^2 &- \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\big(\frac{1}{2}\big(\frac{1}{\rho_j} - \frac{1}{\rho_j'}\big)\big)^2 \\ &= \big(n+1+\frac{1}{\gamma}\big)(\gamma-1).\end{aligned} \tag{11.4}$$

For this aim we use Theorem 4.4 and Theorem 8.3. >From (4.22) we get

$$\big(\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\big)^2 - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{1}{\rho_j^2} = 0, \tag{11.5}$$

$$\big(\Sigma_{j=1}^{n+2}\frac{1}{\rho_j'}\big)^2 - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{1}{(\rho_j')^2} = 0. \tag{11.6}$$

>From (8.18′) we have

$$\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\Sigma_{j=1}^{n+2}\frac{1}{\rho'_j}-\big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\frac{1}{\rho'_j}=2\big(n+1+\frac{1}{\gamma}\big)(1-\gamma). \qquad (11.7)$$

>From (11.5),

$$\begin{aligned}
&\frac{1}{4}\big(\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\big)^2+\frac{1}{4}\big(\Sigma_{j=1}^{n+2}\frac{1}{\rho'_j}\big)^2-\frac{1}{2}\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\Sigma_{j=1}^{n+2}\frac{1}{\rho'_j}\\
= \;& \frac{(n+1+\frac{1}{\gamma})}{4}\left[\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\frac{1}{\rho_j^2}+\Sigma_{j=1}^{n+2}\frac{1}{(\rho'_j)^2}-2\Sigma_{j=1}^{n+2}\frac{1}{\rho_j\rho'_j}\right]\\
= \;& \big(n+1+\frac{1}{\gamma}\big)(\gamma-1).
\end{aligned}$$

>From (11.6) and (11.7) this is reduced to

$$-\frac{1}{2}\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\Sigma_{j=1}^{n+2}\frac{1}{\rho'_j}+\frac{\big(n+1+\frac{1}{\gamma}\big)}{2}\Sigma_{j=1}^{n+2}\frac{1}{\rho_j}\frac{1}{\rho'_j}=\big(n+1+\frac{1}{\gamma}\big)(\gamma-1).$$

But this is equivalent to (11.7) and thus the confirmation of (11.3) is established, which ends the proof of Theorem 11.2.

It is worthwhile to point out that the above method gives not only a generalization of Theorem 5.1 (i.e., the Mauldon complex hyperbolic inclination theorem), but also an alternative proof of it.

We now have

**Theorem 11.3.** *Let $n+1$ spheres $\{S_j\}_{j=1}^{n+1}$ be given in $R^n$ with mutual inclination $\gamma \neq 0$, $\gamma \neq 1$, $\gamma \neq -\frac{1}{n}$. Let $S_{n+2}$ be another sphere in $R^n$ which is orthogonal to each of $\{S_j\}_{j=1}^{n+1}$. Denote by $\{\beta_j\}_{j=1}^{n+2}$ the (generalized) hyperbolic radii of $\{S_j\}_{j=1}^{n+2}$. Then*

$$\begin{aligned}
\big(\Sigma_{j=1}^{n+1}\frac{1}{\tan h\beta_j}\big)^2 \;&-\; \big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{\tan h^2\beta_j} \qquad (11.8)\\
&=\; \big(n+\frac{1}{\gamma}\big)(1-\gamma)\big(\frac{1}{\tan h^2\beta_{n+2}}-1\big).
\end{aligned}$$

*Proof.* The proof is very similar to the proof of the previous theorem. We use (8.14) (replacing the notation $R_j$ by $\rho_j$),

$$\big(\Sigma_{j=1}^{n+1}\frac{1}{\rho_j}\big)^2-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{\rho_j^2}=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\frac{1}{\rho_{n+2}^2}, \qquad (11.9)$$

$$\left(\Sigma_{j=1}^{n+1}\frac{1}{\rho'_j}\right)^2-\left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\frac{1}{(\rho'_j)^2}=\left(n+\frac{1}{\gamma}\right)(1-\gamma)\frac{1}{(\rho'_{n+2})^2}. \tag{11.10}$$

We also use Theorem 8.2. From (8.17′) we get

$$\begin{aligned}\Sigma_{j=1}^{n+1}\frac{1}{\rho_j}\Sigma_{j=1}^{n+1}\frac{1}{\rho'_j} &- \left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\frac{1}{\rho_j}\frac{1}{\rho'_j} \\ &= \left(n+\frac{1}{\gamma}\right)(1-\gamma)\left[\frac{1}{\rho_{n+2}}\frac{1}{\rho'_{n+2}}+2\right].\end{aligned} \tag{11.11}$$

In view of (11.8) we have to show for $\frac{1}{R_j}=\frac{1}{2}\left(\frac{1}{\rho_j}-\frac{1}{\rho'_j}\right)=\frac{1}{\tan h\beta_j}$ that

$$\begin{aligned}\Sigma_{j=1}^{n+1}\left(\frac{1}{2}\left(\frac{1}{\rho_j}-\frac{1}{\rho'_j}\right)\right)^2 &- \left(n+\frac{1}{\gamma}\right)\Sigma_{j=1}^{n+1}\left(\frac{1}{2}\left(\frac{1}{\rho_j}-\frac{1}{\rho'_j}\right)\right)^2 \\ &= \left(n+\frac{1}{\gamma}\right)(1-\gamma)\left[\left(\frac{1}{2}\left(\frac{1}{\rho_{n+2}}-\frac{1}{\rho'_{n+2}}\right)\right)^2-1\right].\end{aligned}$$

Putting (11.9) and (11.10) in the above, we are left with

$$\begin{aligned}-\frac{1}{2}\Sigma_{j=1}^{n+1}\frac{1}{\rho_j}\Sigma_{j=1}^{n+1}\frac{1}{\rho'_j} &+ \frac{(n+\frac{1}{\gamma})}{2}\Sigma_{j=1}^{n+1}\frac{1}{\rho_j}\frac{1}{\rho'_j} \\ &= \left(n+\frac{1}{\gamma}\right)(1-\gamma)\left[-\frac{1}{2}\frac{1}{\rho_{n+1}}\frac{1}{\rho'_{n+2}}-1\right].\end{aligned}$$

But this is equivalent to (11.11) and thus (11.8) is confirmed. This completes the proof of the theorem.

## 11.2 An extension of Theorem 11.3

We start with a slightly different approach to the concept of exponential hyperbolic radius.

Let $S=\{y,\Sigma_{j=1}^{n}(y_j-a_j)^2-\rho^2\}$. We define $\{\eta_j\}_1^n,\eta_0,R$ to satisfy

$$\rho=\frac{R}{1+\eta_0},\quad a_j=\frac{\eta_j}{1+\eta_0},\quad \eta^2-\eta_0^2=R^2-1,\quad 1\le j\le n. \tag{11.12}$$

Compare this with (5.7).

We have

$$\begin{aligned}a^2=\frac{\eta^2}{(1+\eta_0)^2},\quad \eta^2-\eta_0^2 &= \rho^2(1+\eta_0)^2-1,\\ a^2(1+\eta_0)^2-\eta_0^2 &= \rho^2(1+\eta_0)^2-1,\\ a^2(1+\eta_0)^2 &= \rho^2(1+\eta_0)^2+(\eta_0^2-1).\end{aligned}$$

Cancelling by $1+\eta_0$ we deduce that $a^2(1+\eta_0) = \rho^2(1+\eta_0)+\eta_0-1$. Hence

$$\eta_0 = \frac{1+a^2-\rho^2}{1-a^2+\rho^2}, \quad 1+\eta_0 = \frac{2}{1+\rho^2-a^2}. \tag{11.13}$$

>From (11.12) and (11.13) we have

$$R = \frac{2\rho}{1+\rho^2-a^2}, \quad \frac{1}{R} = \frac{1}{2}\left(\frac{1}{\rho}-\frac{1}{\rho'}\right), \tag{11.14}$$

where $\rho'$ denotes, as usual, the radius of the inverse sphere, i.e., $\rho' = \frac{\rho}{a^2-\rho^2}$.

We conclude that $R$ is the exponential radius of $S$. Also note that

$$S = \left\{y, \Sigma_{j=1}^n \left(y_j - \frac{\eta_j}{1+\eta_0}\right)^2 = \left(\frac{R}{1+\eta_0}\right)^2\right\}.$$

(Compare this with the discussion in section 5, in particular (5.1)).

Now consider $n+1$ spheres in $\mathrm{R}^n$,

$$S_k = \left\{y, \Sigma_{j=1}^n (y_j - a_{kj})^2 - \rho_k^2\right\}, \quad 1 \le k \le n+1$$

having mutual inclination $\gamma$. Also, consider an orthogonal sphere $S_{n+2}$ to $\{S_k\}_{k=1}^{n+1}$ where

$$S_{n+2} = \left\{y, \Sigma_{j=1}^n (y_j - a_{n+2,j})^2 = \rho_{n+2}^2\right\}.$$

In addition, consider a sphere of reference $S$ where $S = \{y, \Sigma_{j=1}^n (y_j - a_j)^2 = \rho^2\}$. Similarly to (11.12) we define $\{\mu_{kj}\}$, $\{R_k\}$ as follows:

$$\rho_k = \frac{R_k}{1+\mu_{k0}}, \quad a_{kj} = \frac{\mu_{kj}}{1+\mu_{k0}}, \qquad 1 \le k \le n+2, \quad 1 \le j \le n. \tag{11.15}$$

In addition to (11.13) and (11.14) we also have similarly,

$$\mu_{k0} = \frac{1+a_k^2-\rho_k^2}{1-a_k^2-\rho_k^2}, \quad 1+\mu_{k0} = \frac{2}{1+\rho-k^2-a_k^2}, \quad 1 \le k \le n+2, \tag{11.16}$$

$$R_k = \frac{2\rho_k}{1+\rho_k^2-a_k^2}, \quad \frac{1}{R_k} = \frac{1}{2}\left(\frac{1}{\rho_k}-\frac{1}{\rho_k'}\right). \tag{11.17}$$

Hence $R_k = \tan h\beta_k$ are the hyperbolic exponential radii of the spheres $S_k$, $1 \le k \le n+2$. Denote, as usual, by $\lambda_k$, the inclination of $S_k$ to the sphere of reference $S$. By an identical computation to the one in section 5 we have

$$\lambda_k = \frac{1+(\mu_k,\eta)-\mu_{k0}\eta_0}{R_k R}, \quad 1 \le k \le n+2. \tag{11.18}$$

We now use Theorem 4.2 and substitute $\{\lambda_k\}_{k=1}^{n+2}$ in (4.11) to get

$$\big(\Sigma_{k=1}^{n+1}\big(\frac{1+(\mu_k,\eta)-\mu_{k0}\eta_0}{R_kR}\big)\big)^2-\big(n+\frac{1}{\gamma}\big)\Sigma_{k=1}^{n+1}\big(\frac{1+(\mu_k,\eta)-\mu_{k0}\eta_0}{R_kR}\big)^2$$
$$=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big[\big(\frac{1+(\mu_{n+2},\eta)-(\mu_{n+2,0},\eta)}{R_{n+2}R}\big)^2-1\big].$$

Multiplying by $\mathrm{R}^2$ and using $R^2=1+\eta^2-\eta_0^2$ from (11.12),

$$\big(\Sigma_{k=1}^{n+1}\big(\frac{1+(\mu_k,\eta)-\mu_{k0}\eta_0}{R_k}\big)\big)^2-\big(n+\frac{1}{\gamma}\big)\Sigma_{k=1}^{n+1}\big(\frac{1+(\mu_k,\eta)-\mu_{k0}\eta_0}{R_k}\big)^2$$
$$=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big[\big(\frac{1+(\mu_{n+2},\eta)-(\mu_{n+2,0})}{R_{n+2}}\big)^2-(1+\eta^2-\eta_0^2)\big].$$

We can now compare coefficients of $\{\eta_k\}_{k=0}^{n}$ (see section 8). Doing this we may derive similar relations that appear in Theorem 8.2. Details are omitted. It is worthwhile to point out that comparing the free coefficient, we get

$$\Sigma_{k=1}^{n+1}\frac{1}{R_k^2}-\big(n+\frac{1}{\gamma}\big)\big(\Sigma_{k=1}^{n+1}\frac{1}{R_k}\big)=\big(n+\frac{1}{\gamma}\big)(1-\gamma)\big(\frac{1}{R_{n+2}^2}-1\big).$$

This gives an alternative proof of Theorem 11.3.

In the next section, instead of one orthogonal sphere, we consider a set of orthogonal spheres. This will give a natural generalization to some of the previous theorems.

# 12 Set of orthogonal spheres: The algebraic approach

## 12.1 The algebraic approach: Clifford's formula

In our paper we mainly have taken the geometrical approach. For the topics that we consider in this section, the algebraic approach is very useful and we will use it. Clifford discovered a special case of the Darbeux-Frobenius Formula for "Poly spherical coordinates" (see [9] for a detailed discussion of the above connections).

**Clifford's theorem**. *Let $\{\Sigma_j\}_{j=1}^{n+2}$ be $n+2$ spheres in $G^n$. Let $\Sigma,\Sigma'$ be two spheres of reference in $G^n$. Denote by $\lambda(\Sigma,\Sigma')$ the inclination between $\Sigma$ and $\Sigma'$. Also denote by $\{\lambda_j\}_{j=1}^{n+2}$ and $\{\mu_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$*

*with $\Sigma$ and $\Sigma'$ respectively. In addition $\{\gamma_{jk}\}_{j,k=1}^{n+2}$ will denote the inclination between $\Sigma_j$ and $\Sigma_k$ respectively. Then*

$$\det\begin{pmatrix}\lambda(\Sigma,\Sigma') & \lambda^\tau\\ \mu & \Delta\end{pmatrix}=0 \tag{12.1}$$

*where $\lambda^\tau=(\lambda_1,\dots,\lambda_{n+2})$, $\mu^\tau(\mu_1,\dots,\mu_{n+2})$ and $\Delta=(\gamma_{jk})_{j,k=1}^{n+2}$.*

Originally Clifford's theorem was proved for the real case (i.e., $R^n$ instead of $G^n$). But the proof is identical. We shall also need the following

**Lemma 12.1.** *Let $u,\{\lambda_j\}_{j=1}^{n+2},\{\mu_j\}_{j=1}^{n+2}$ be complex numbers. Let $\Delta$ be a matrix $(\gamma_{jk})_{j,k=1}^{n+2}$ which is symmetric such that $\det\Delta\neq 0$. Assume also*

$$\det\begin{pmatrix}u & \lambda^\tau\\ \mu & \Delta\end{pmatrix}=0 \tag{12.2}$$

*for $\lambda^\tau=(\lambda_1,\dots,\lambda_{n+2})$, $\mu^\tau=(\mu_1,\dots,\mu_{n+2})$. Then*

$$u=\lambda^T\Delta^{-1}\mu \tag{12.3}$$

*where $\Delta^{-1}$ is the inverse matrix of $\Delta$.*

The proof of Lemma 12.1 is exactly the same as in [9, p. 306] and is omitted.

Using the above we now have

**Theorem 12.1.** *Let $\{\Sigma_j\}_{j=1}^{n+1}$ be $n+1$ spheres in $G^{n+k}$, $n\geq 1$, $k\geq 0$, having a mutual inclination $\gamma\neq\frac{-1}{n}$, $\gamma\neq 1$, $\gamma\neq 0$. Assume further that $\{\Sigma_j\}_{j=n+2}^{n+k+2}$ is another set of spheres in $G^{n+k}$ that are mutually orthogonal and, in addition, each of them is orthogonal to all spheres $\{\Sigma_j\}_{j=1}^{n+1}$. Let $\Sigma,\Sigma'$ be two spheres of reference in $G^{n+k}$. Denote by $\lambda(\Sigma,\Sigma')$ the inclination between $\Sigma$ and $\Sigma'$. Also denote by $\{\lambda_j\}_{j=1}^{n+k+2}$, $\{\mu_j\}_{j=1}^{n+k+2}$ the inclinations $\{\Sigma_j\}_{j=1}^{n+k+2}$ with $\Sigma,\Sigma'$ respectively. Then*

$$\begin{aligned}&\Sigma_{j=1}^{n+1}\lambda_j\Sigma_{j=1}^{n+1}\mu_j-\big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\lambda_j\mu_j\\ &\qquad=(1-\gamma)(n+\frac{1}{\gamma}\big)\big(\Sigma_{j=n+2}^{n+k+2}\lambda_j\mu_j-\lambda(\Sigma,\Sigma')\big).\end{aligned} \tag{12.4}$$

*Proof.* We first note that the above theorem is a generalization of Theorem 9.1. Indeed, putting $k=0$ in (12.4), we get (9.1).

Since we follow the notations of Boyd ([9, p.306-7]), it will be convenient to prove the above theorem by replacing $n$ with $n+1$.

>From Clifford's theorem stated earlier, by using (12.1) and replacing $n+2$ with $n+k+3$, we get

$$\det\begin{pmatrix}\lambda(\Sigma,\Sigma') & \tilde{\lambda}^{T} \\ \tilde{\mu} & \Delta_0\end{pmatrix} = 0, \tag{12.5}$$

where $\tilde{\lambda}^T = (\lambda_1, \lambda_2, \ldots, \lambda_{n+k+3})$, $\tilde{\mu}^T = (\mu_1, \mu_2, \ldots, \mu_{n+k+3})$. Using the given conditions about inclinations, we have $\lambda(\Sigma_j, \Sigma_p)$, $j \neq p$, is equal to $\gamma$, $1 \leq j,\ p \leq n+2$. Obviously $\lambda(\Sigma_j, \Sigma_j) = 1$ for $1 \leq j \leq n+k+3$. In addition, using the orthogonality conditions, we have $\lambda(\Sigma_j, \Sigma_p) = 0$ for $1 \leq j \leq n+k+$ 3,
$n+3 \leq p \leq n+k+3$. Hence,

$$\Delta_0 = \begin{pmatrix}\Delta & 0 \\ 0 & \tilde{I}\end{pmatrix}, \tag{12.6}$$

where

$$\Delta = \begin{pmatrix}1 & \gamma & \cdots & \gamma \\ \gamma & 1 & & \gamma \\ \vdots & & \ddots & \vdots \\ \gamma & \gamma & \cdots & 1\end{pmatrix}, \quad = \quad \gamma J + (1-\gamma) I \tag{12.7}$$

where $J$ is a $(n+2) \times (n+2)$ matrix, all of whose entries are 1; $I, \tilde{I}$ are identity matrices; and 0 is a zero matrix. Combining (12.5) and (12.6) we get

$$\det\left(\begin{array}{ccc:c}
\lambda(\Sigma,\Sigma') & \lambda_1, \ldots, \lambda_{n+2} & \vdots & \lambda_{n+3}, \ldots, \lambda_{n+k+3} \\
\mu_1 & & \vdots & \\
\vdots & \Delta & \vdots & 0 \\
\mu_{n+2} & & \vdots & \\
\cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots & \cdots\cdots\cdots\cdots\cdots \\
\mu_{n+3} & & \vdots & \\
\vdots & 0 & \vdots & \tilde{I} \\
\mu_{n+k+3} & & \vdots &
\end{array}\right) = 0. \tag{12.8}$$

We now multiply the $n+3$ row by $\lambda_{n+3}$ and subtract from the first row. We then multiply the $n+4$ row by $\lambda_{n+4}$ and, again, subtract from the first row. Continuing in this way for $n+3 \le j \le n+k+3$, as can be easily seen, we finally get

$$\det \begin{pmatrix} u & \lambda_1\lambda_2,\ldots,\lambda_{n+2} & \vdots & 0 & 0\cdots 0 \\ \mu_1 & & \vdots & & \\ \vdots & \Delta & \vdots & & 0 \\ \mu_{n+2} & & \vdots & & \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots & \cdots & \cdots & \cdots\cdots \\ \mu_{n+3} & & \vdots & & \\ \vdots & & \vdots & 0 & \tilde{I} \\ \mu_{n+k+3} & & \vdots & & \end{pmatrix} = 0, \tag{12.9}$$

where $u = \lambda(\Sigma, \Sigma') - \Sigma_{j+n+3}^{n+k+3}\lambda_j\mu_j$. But this leads at once to

$$\det \begin{pmatrix} u & \lambda^T \\ \mu & \Delta \end{pmatrix} = 0. \tag{12.10}$$

We now use Lemma 12.1 for this specific value of $u$. Hence from (12.3) and (12.7)

$$u = \lambda^T, \quad \Delta^{-1}\mu, \tag{12.11}$$

where $u = \lambda(\Sigma, \Sigma') - \Sigma_{j=n+3}^{n+k+3}\lambda_j\mu_j$, and $\Delta = \gamma J + (1-\gamma)I$.

We now find specifically $\Delta^{-1}$. In view of $J^2 = (n+2)J$ it is not difficult to guess that $\Delta^{-1} = \frac{1}{1-\gamma}(bJ + I)$ for some $b$. Indeed, $\Delta^{-1}\Delta = I$ leads to

$$\begin{aligned} \frac{1}{1-\gamma}(bJ+I)(\gamma J + (1-\gamma)I) &= \frac{1}{1-\gamma}(b\gamma J^2 + \gamma J + (1-\gamma)bJ + (1-\gamma)I) \\ &= \frac{1}{1-\gamma}(b\gamma(n+2)J + \gamma J + (1-\gamma)bJ + (1-\gamma)I). \end{aligned}$$

Solving for $b$, we have $b\gamma(n+2) + \gamma + (1-\gamma)b = 0$, or $b = \frac{-1}{\left(n+1+\frac{1}{\gamma}\right)}$. Thus

$$\Delta^{-1} = \frac{1}{(1-\gamma)\left(n+1+\frac{1}{\gamma}\right)}\left(-J + \left(n+1+\frac{1}{\gamma}\right)I\right). \tag{12.12}$$

>From (12.11) and (12.12) we get

$$u = \frac{\lambda^T}{(1-\gamma\left(n+1+\frac{1}{\gamma}\right)}\left(-J + \left(n+1+\frac{1}{\gamma}\right)I\right)\mu.$$

Hence

$$(1-\gamma)\big(n+1+\frac{1}{\gamma}\big)u = -\lambda^T J\mu + \big(n+1+\frac{1}{\gamma}\big)\lambda^T I\mu.$$

It is easy to check that

$$\lambda^T J\mu = \big(\Sigma_{j=1}^{n+2}\lambda_j\big)\big(\Sigma_{j=1}^{n+2}\mu_j\big), \quad \lambda^T I\mu = \Sigma_{j=1}^{n+2}\lambda_j\mu_j$$

Thus we get

$$(1-\gamma)\big(n+1+\frac{1}{\gamma}\big)u = -\big(\Sigma_{j=1}^{n+2}\lambda_j\big)\big(\Sigma_{j=1}^{n+2}\mu_j\big) + \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\lambda_j\mu_j.$$

But $u = \lambda(\Sigma,\Sigma') - \Sigma_{j=n+3}^{n+k+3}\lambda_j\mu_j$ yields

$$\begin{aligned}\Sigma_{j=1}^{n+2}\lambda_j\Sigma_{j=1}^{n+2}\mu_j - \big(n+1+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+2}\lambda_j\mu_j &= (1-\gamma)\big(n+1+\frac{1}{\gamma}\big)(-u)\\ &= (1-\gamma)\big(n+1+\frac{1}{\gamma}\big)\big(\Sigma_{j=n+3}^{n+k+3}\lambda_j\mu_j - \lambda(\Sigma,\Sigma')\big).\end{aligned}$$

Replacing $n+1$ by $n$, we get (12.4).

It remains to check the condition $\gamma \neq -\frac{1}{(n+1)}$ (where again $n+1$ replaces $n$). We prove this by induction on the number of orthogonal spheres. If there is only one orthogonal sphere, the condition is already proved in the special case of Theorem 9.1. For the general case, let $n+k+3$ spheres be given in $G^{n+1}$. The $n+k+3$ sphere is orthogonal to all other spheres $\{\Sigma_j\}_{j=1}^{n+k+2}$. Hence, by a Möbius transformation, we may assume that this sphere is the plane $z_{n+1}=0$. Now, reversing the Poincaré extension, we reduce the number of orthogonal spheres by 1 and we may look at $G^n$ instead of $G^{n+1}$. Using the induction assumption ends the proof of the condition.

This ends the proof of the theorem.

## 12.2 Comparison between the algebraic and geometrical methods

The above proof of Theorem 12.1 gives a new approach also to the special case of Theorem 9.1. Hence, for Theorem 9.1 we have two independent proofs, except for the condition $\gamma \neq -\frac{1}{n}$, which does not seem to follow from algebraic considerations very easily. It is worthwhile to note that the geometrical proof of Theorem 9.1 may be extended to the more general case of Theorem 12.1. Indeed, we may use an induction process as for the proof of the condition $\gamma \neq \frac{-1}{n+1}$ given above. Indeed, using the symmetry of the situation, it is enough to restrict the proof for the case $\lambda_{n+k+3} = \mu_{n+k+3} = 0$. Using a Möbius transformation, we may assume that the $n+k+3$ sphere is the plane $z_{n+1} = 0$. But the assumption $\lambda_{n+k+3} = \mu_{n+k+3} = 0$ implies that not only all spheres $\{\Sigma_j\}_{j=1}^{n+k+2}$ are orthogonal to $\Sigma_{n+k+3}$, but also $\Sigma$ and $\Sigma'$. Thus it is possible to use the Poincaré extension idea (in the reverse direction) and complete the induction process, giving an alternative proof of Theorem 12.1. We also note that the linear theorem (Theorem 4.1) was proved only geometrically.

## 12.3 Further radii results: The hyperbolic "translation"

As in the previous cases, we can "translate" the inclination theorem proved above to radii results. We take the special case $\Sigma = \Sigma'$ in Theorem 12.1. We also use Lemma 4.1 in the standard way. Thus we get

**Theorem 12.2.** *Let $\{S_j\}_{j=1}^{n+1}$ be $n+1$ spheres in $G^{n+k}$, $n \geq 1$, $k \geq 0$, having a mutual inclination $\gamma$, $\gamma \neq -\frac{1}{n}, \gamma \neq 1, \gamma \neq 0$. Assume further that $\{\Sigma_j\}_{j=n+2}^{n+k+2}$, is another set of spheres that are mutually orthogonal and, in addition, each of them is orthogonal to all spheres $\{\Sigma_j\}_{j=1}^{n+1}$. Denote by $\{R_j\}_{j=1}^{n+k+2}$ the radii of these spheres respectively. Then*

$$\big(\Sigma_{j=1}^{n+1}\frac{1}{R_j}\big)^2 - \big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{R_j^2} = (1-\gamma)\big(n+\frac{1}{\gamma}\big)\Sigma_{j=n+2}^{n+k+2}\frac{1}{R_j^2}. \tag{12.13}$$

Next, we give the following generalization of Theorem 11.3.

**Theorem 12.3.** *Let $n+1$ spheres $\{S_j\}_{j=1}^{n+1}$ be given in $R^{n+k}$ with mutual inclination $\gamma$, $\gamma \neq 0$, $\gamma \neq 1$, $\gamma \neq \frac{1}{n}$. Let $\{S_j\}_{j=n+2}^{n+k+2}, k \geq 0$ be another set of spheres in $R^{n+k}$ that are mutually orthogonal and, in addition, each one of them is orthogonal to all other spheres $\{S_j\}_{j=1}^{n+1}$. Denote by $\{\beta_j\}_{j=1}^{n+k+2}$ the (generalized) hyperbolic radii of $\{S_j\}_{j=1}^{n+k+2}$ respectively. Then*

$$\begin{aligned}\big(\Sigma_{j=1}^{n+1}\frac{1}{tanh\beta_j}\big)^2 - \big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{tanh^2\beta_j} \\ = \big(n+\frac{1}{\gamma}\big)(1-\gamma)\big(\Sigma_{j=n+2}^{n+k+2}\frac{1}{tanh^2\beta_j}-1\big).\end{aligned} \tag{12.14}$$

*Proof.* We can prove it in a similar way for the particular case of Theorem 11.3. Alternatively we can use the technique introduced in Theorem 5.1, namely, using transformation from $G^{n+1}$ to $R^n$. We omit the details that are very similar to previous considerations.

### 12.4 Mutual orthogonal spheres

We now consider in particular $n+2$ spheres that are mutually orthogonal, i.e., in other words, we consider the case of mutual inclination $\gamma$ where $\gamma = 0$. We have

**Theorem 12.4.** *Let $\{\Sigma_j\}_{j=1}^{n+2}$ be a set of mutually orthogonal spheres in $G^n$. Assume that $\Sigma, \Sigma'$ are two spheres of reference in $G^n$. Denote by $\{\lambda_j\}_{j=1}^{n+2}, \{\mu_j\}_{j=1}^{n+2}$ the inclinations of $\{\Sigma_j\}_{j=1}^{n+2}$ to $\Sigma, \Sigma'$ respectively. Also denote by $\lambda(\Sigma, \Sigma')$ the inclination between $\Sigma$ and $\Sigma'$. Then*

$$\Sigma_{j=1}^{n+2}\lambda_j\Sigma_{j=1}^{n+2}\mu_j = \lambda(\Sigma, \Sigma'). \tag{12.15}$$

*Proof.* We use Clifford's theorem quoted in section 12.1. In our case, using $\gamma = 0$, we have $\Delta = I$. Using Lemma 12.1, we get (12.15) from (12.3) where $u = \lambda(\Sigma, \Sigma')$.

**Remark**. "Translation" for radii give for $n+2$ mutually orthogonal spheres $\{S_j\}_{j=1}^{n+2}$ having radii $\{R_j\}_{j=1}^{n+2}$ respectively, so that

$$\Sigma_{j=1}^{n+2}\frac{1}{R_j^2} = 0. \tag{12.16}$$

In particular for $G^2$ we get for the two planes $z_1 = 0,\ \ z_2 = 0$ and the two spheres having the radii $R$, $iR$ and centered at the origin that

$$\Sigma_{j=1}^{4}x_j^2 = 0, \quad x_1 = 0, \quad x_2 = 0, \quad x_3 = \frac{1}{R}, \quad x_4 = \frac{1}{iR},$$

where $\{x_j\}_{j=1}^{4}$ are the "bends" $\{\frac{1}{R_j}\}_{j=1}^{4}$. (The two planes $z_1 = 0$ and $z_2 = 0$ may be considered as spheres with infinite radii.) Obviously, we have a similar situation in $G^n$.

## 13 Transformations of the unit sphere: Points at infinity

### 13.1 The unit sphere and half plane

Let $a = (0, \dots, 1) \in G^n$. Consider the map $w = a + \frac{2(z-a)}{(z-a)^2}$. We show that this is a map from $w^2 = 1$ onto $z_n = 0$. Indeed,

$$w^2 = \big((w-a)+a\big)^2 = (w-a)^2 + a^2 + 2(w-a, a).$$

Hence $w^2 = 1$ implies that $(w-a)^2 + 2(w-a, a) = 0$. Thus $1 + \big(2\frac{(w-a)}{(w-a)^2}, a\big) = 0$. But $(w-a)^2 = \frac{4(z-a)^2}{(z-a)^4} = \frac{4}{(z-a)^2}$. We get $1 + \big(\frac{(z-a)^2}{2}(w-a), a\big) = 0$. Putting $w - a = \frac{2(z-a)}{(z-a)^2}$ by the given initial transformation, it follows that $1 + (z-a, a) = 0$. Hence $(z, a) = 0$ as $a^2 = 1$. But this is equivalent to $z_n = 0$, and thus our assertion is proved. We point out the fact that this map implies that $z = a + \frac{2(w-a)}{(w-a)^2}$ as may be shown easily. Hence $z^2 = 1$ is mapped onto $w_n = 0$ by the same transformation.

We now have

**Theorem 13.1.** *Given the map $w - a = \frac{2(z-a)}{(z-a)^2}$ from the unit sphere $z^2 = 1$ onto $w_n = 0$, we have*

$$\frac{(dw)^2}{w_n^2} = \frac{4(dz)^2}{(1-z^2)^2}. \tag{13.1}$$

*Proof.* We have $dw_j = \frac{2dz_j}{(z-a)^2} - \frac{2(z_j-a_j)}{(z-a)^4} d\big((z-a)^2\big)$, $1 \le j \le n$, where $w_j = a_j + \frac{2(z_j-a_j)}{(z-a)^2}$, $1 \le j \le n$. Hence $\frac{1}{4}(dw)^2 = \frac{(dz)^2}{(z-a)^4} + \frac{\Sigma_{j=1}^n (z_j-a_j)^2}{(z-a)^8}\big(d(z-a)^2\big)^2$
$- 2\Sigma_{j=1}^n \frac{(z_j-a_j, dz_j)}{(z-a)^6}\big(d(z-a)^2\big)$. But $\frac{\Sigma_{j=1}^n (z_j-a_j)^2}{(z-a)^8} = \frac{(z-a)^2}{(z-a)^8} = \frac{1}{(z-a)^6}$. Also, $d\big((z-a)^2\big) = d\big(\Sigma(z_j-a_j)^2\big) = 2\Sigma_{j=1}^n (z_j - a_j, dz_j)$. Hence, the two last expressions are cancelled and we have

$$\frac{1}{4}(dw)^2 = \frac{(dz)^2}{(z-a)^4}. \tag{13.2}$$

We now calculate $(1-z^2)^2$. We have

$$\begin{aligned} z^2 &= a^2 + \frac{4(a, w-a)}{(w-a)^2} + \frac{4(w-a)^2}{(w-a)^4} \\ &= 1 + \frac{4(a,w) - 4a^2 + 4}{(w-a)^2} \\ &= 1 + \frac{4(a,w)}{(w-a)^2}. \end{aligned}$$

Hence $z^2 - 1 = \frac{4(a,w)}{(w-a)^2} = \frac{4w_n}{(w-a)^2}$ or

$$(1-z^2)^2 = \frac{16w_n^2}{(w-a)^4}. \tag{13.3}$$

But $(w-a)^2 = \frac{4(z-a)^2}{(z-a)^4} = \frac{4}{(z-a)^2}$ or $(w-a)^4 = \frac{16}{(z-a)^4}$. Putting this in (12.3) we get

$$w_n^2 = \frac{(1-z^2)^2}{(z-a)^4}. \tag{13.4}$$

Dividing (13.2) by (13.4) implies (13.1), and thus the proof is done.

We next prove the invariance of the hyperbolic "metric" for the unit "disc". For this we need the following

**Lemma 13.1.** *The hyperboloc "metric" on the "upper" half plane in $G^n$ is invariant under a self map.*

*Proof.* We have to show for the mapping of the upper half plane onto itself that $\frac{dw^2}{w_n^2}$ remains invariant.

First let $w = Az$ for a complex number $A$. This yields for $w = (w_1, \ldots, w_n)$ and $z = (z_1, \ldots, z_n)$ the self map from $z_n = 0$ onto $w_n = 0$. Obviously, $w_n = Az_n$, and thus $\frac{(dw)^2}{w_n^2} = \frac{A^2(dz)^2}{A^2 z_n^2}$ and the invariance property is proved for this case.

We now consider the translation by $\beta = (\alpha_1, \ldots, \alpha_{n-1}, 0)$. This means that for $w = z + \beta$ we have $w_n = z_n$ and thus, obviously, the invariance proeprty is proved for this case as well.

It is left to show the invariance property for the inversion $w = \frac{z}{z^2}$.

We have $w_j = \frac{z_j}{z^2}$, $1 \le j \le n$. Hence $dw_j = \frac{dz_j}{z^2} - \frac{1}{z^4} z_j d(z^2)$, $1 \le j \le n$. We get $(dw)^2 = \frac{(dz)^2}{z^4} - \frac{2\Sigma_{j=1}^n (z_j, dz_j)}{z^6} d(z^2) + \frac{\Sigma_{j=1}^n (z_j)^2}{z^8} \big(d(z^2)\big)^2$. But $\Sigma_{j=1}^n \frac{(z_j)^2}{z^8} = \frac{1}{z^6}$ and $d(z^2) = d(\Sigma z_j^2) = 2\Sigma_{j=1}^n (z_j, dz_j)$, and thus the two last terms are cancelled. This implies

$$(dw)^2 = \frac{(dz)^2}{z^4}. \tag{13.5}$$

Also from $w = \frac{z}{z^2}$, we have $w_n = \frac{z_n}{z^2}$ or

$$w_n^2 = \frac{z_n^2}{z^4}. \tag{13.6}$$

>From (13.5) and (13.6) we get the invariance property at once.

Summing up, for the general self map of the "upper" half plane onto itself, the invariance property is proved.

>From Theorem 13.1 and Lemma 13.1 we easily get

**Theorem 13.2.** *For the general self map of the unit sphere $z^2 = 1$ onto itself, the hyperbolic "metric" is invariant.*

*Proof.* Indeed, using (13.1) for $\hat{z} = Tz$, the mapping of the unit sphere onto itself, we have $\frac{(d\hat{w})^2}{\hat{w}_n^2} = \frac{4(d\hat{z})^2}{(1-\hat{z}^2)^2}$. >From the invariance property $\frac{(d\hat{w})^2}{\hat{w}_n^2} = \frac{(dw)^2}{w_n^2}$ that follows from Lemma 13.1, we get that $\frac{(d\hat{z})^2}{\left(1-(\hat{z})^2\right)^2} = \frac{(dz)^2}{(1-z^2)^2}$, and the proof is finished.

## 13.2 Stereographic projection in $G^n$

Let $b = (0, \ldots, 0, 1) \in G^{n+1}$. Denote $a = (z_1, \ldots, z_n, 0)$ and by $c$ its projection on the unit sphere in $G^{n+1}$. Then the two vectors $c-a$ and $b-a$ have the same direction. Hence there exists a complex $t$ such that $c = a+t(b-a)$. Also $c^2 = b^2 = 1$, since both are located on the unit sphere. Hence $c = a(1-t)+bt$ implies $1 = a^2(1-t)^2+t^2+2t(1-t)(a,b)$. Obviously $(a,b) = 0$ by definition. Thus $a^2 = \frac{1-t^2}{(1-t)^2} = \frac{1+t}{1-t}$. In other words

$$1 = \frac{a^2 - 1}{a^2 + 1}. \tag{13.7}$$

For $c = c_1, \ldots, c_j, \ldots, c_n, c_{n+1})$ and $c = a+t(b-a)$, we have $c_j = a_j+t(b_j - a_j) = tb_j + a_j(1-t)$, $0 \leq j \leq n+1$. But $b_j = 0$ for $1 \leq j \leq n$ leads to $c_j = a_j(1-t)$. Also $a_j = a_j$, $t = \frac{a^2-1}{a^2-1}$ and thus

$$c_j = \frac{2z_j}{1+z^2}, \quad 1 \leq j \leq n, \tag{13.8}$$

as $a^2 = \Sigma_{j=1}^n z_j^2 = z^2$. In addition $c_{n+1} = tb_{n+1} + a_{n+1}(1-t) = tb_{n+1} = t = \frac{z^2-1}{z^2+1}$. Thus we get

$$c = \Big(\frac{2z_1}{1+z^2}, \cdots, \frac{2z_n}{1+z^2}, \frac{z^2-1}{z^2+1}\Big) \tag{13.9}$$

is the image on the unit sphere of $a = (z_1, \ldots, z_n, 0)$. We now consider the inverse transformation, i.e., the map from points on the unit sphere in $G^{n+1}$ on to points on the plane $z_{n+1} = 0$. Hence, let $c = (c_1, c_2, \ldots, c_n, c_{n+1})$ be a point on the unit sphere in $G^{n+1}$. Denote its image by $a = (z_1, z_2, \ldots, z_n, 0)$. From the above discussion we now get (see (12.9) that $c_{n+1} = \frac{z^2-1}{z^2+1}$ or $z^2 = \frac{1+c_{n+1}}{1-c_{n+1}}$. Hence $c_j = \frac{2z_j}{1+z^2}$ implies that $c_j = \frac{2z_j}{1+\left(\frac{1+c_{n+1}}{1-c_{n+1}}\right)}$. Thus $c_j = \frac{2z_j(1-c_{n+1})}{2}$ or $z_j = \frac{c_j}{1-c_{n+1}}$.

Summing up we have

$$z_j = \frac{c_j}{1-c_{n+1}}, \quad 1 \leq j \leq n, \quad z_{n+1} = 0. \tag{13.10}$$

Note that if $z^2 = 0$ then $c_{n+1} = -1$ and $z_j = \frac{c_j}{2}$, $\quad 1 \le j \le n$. On the other hand, if $c_{n+1} = 1$, then $z_j$ are not finite in general for $1 \le j \le n$. We come to this issue later on in this section when we deal with "points at infinity".

We next show that angles are preserved under stereographic projection.

**Theorem 13.3.** *Under a stereographic projection from the unit sphere in $G^{n+1}$ onto the plane, $z_{n+1} = 0$ angles are preserved.*

*Proof.* We first recall the notation $\overrightarrow{dw} = (dw_1, \dots, dw_n, dw_{n+1})$ and $dw = \sqrt{\Sigma_{j=1}^n (dw_j)^2}$ and a similar notation for $\overrightarrow{dz}$ and $dz$. So let $w_j = \frac{2z_j}{1+z^2}$, $1 \le j \le n$, $\ w_{n+1} = \frac{z^2-1}{z^2+1}$ be the map from the plane on the unit sphere. We then have $dw_j = \frac{2dz_j}{1+z^2} - \frac{2z_j}{(1+z^2)^2} 2\Sigma_{k=1}^n z_k dz_k, \quad 1 \le j \le n$.
(Indeed, $d(z^2) = d\big(\Sigma_{k=1}^n z_k^2\big) = 2\Sigma_{k=1}^n z_k dz_k.$). Thus

$$dw_j = \frac{(1+z^2)^2 dz_j}{(1+z^2)^2} - \frac{4z_j}{(1+z^2)^2}(z, dz), \quad 1 \le j \le n,$$

or

$$(dw_j)^2 \quad = \quad \frac{4}{(1+z^2)^4}\big[(1+z^2)^2 (dz_j)^2 + 4z_j^2 (z, dz)^2$$
$$- \ 4(z_j, dz_j)(z, dz)(1+z^2)\big], \quad 1 \quad \le \quad j \le n.$$

Hence summation on $j$ gives

$$\Sigma_{j=1}^n (dw_j)^2 = \frac{4}{(1+z^2)^4}\big[(1+z^2)^2 (dz)^2 + 4z^2 (z, dz)^2 - 4(z, dz)^2 (1+z^2)\big].$$

Cancelling $4z^2(z, dz)^2$ we get

$$\Sigma_{j=1}^n (dw_j)^2 = \frac{4}{(1+z^2)^4}\big[(1+z^2)^2 (dz)^2 - 4(z, dz)^2\big]. \tag{13.11}$$

Also from $w_{n+1} = \frac{z^2-1}{z^2+1} = 1 - \frac{2}{z^2+1}$ we get

$$dw_{n+1} = \frac{2}{(1+z^2)^2} d(z^2) = \frac{2}{(1+z^2)^2} 2\Sigma_{j=1}^n z_j dz_j$$

or

$$(dw_{n+1})^2 = \frac{16}{(1+z^2)^4}(z, dz)^2. \tag{13.12}$$

From (13.11) and (13.12) we finally get

$$\Sigma_{j=1}^{n+1}(dw_j)^2 = (dw)^2 = \frac{4(dz)^2}{(1+z^2)^2}. \tag{13.13}$$

More generally, by a computation that is very similar to the above one and which is therefore omitted, we get for $d\tilde{w} = (d\tilde{w}_1, \ldots, d\tilde{w}_{n+1})$, $dw = (dw_1, \ldots, dw_{n+1})$,

$$\Sigma_{j=1}^{n+1} dw_j d\tilde{w}_j = \frac{4(\overrightarrow{dz}, \overrightarrow{d\tilde{z}})}{(1+z^2)^2} = (\overrightarrow{dw}, \overrightarrow{d\tilde{w}}) \tag{13.14}$$

Since, as we pointed out above, $dw = \sqrt{(dw_j)^2}$, we can write (13.13) in the equivalent form

$$dw = \frac{2dz}{1+z^2}, \quad d\tilde{w} = \frac{2d\tilde{z}}{1+z^2}. \tag{13.15}$$

Hence from (13.14) and (13.15) we get

$$\frac{(\overrightarrow{dw}, \overrightarrow{d\tilde{w}})}{dw d\tilde{w}} = \frac{(\overrightarrow{dz}, \overrightarrow{d\tilde{z}})}{dz d\tilde{z}}, \tag{13.16}$$

which is the desired result we claimed in Theorem 13.3.

It is also easy to show that for $u = (u_1, \ldots, u_n, u_{n+1})$, $v = (v_1, v_2, \ldots, v_{n+1})$ on the unit sphere on $G^{n+1}$ that are the images of $\alpha = (\alpha_1, \ldots, \alpha_{n,0})$, $\beta = (\beta_1, \ldots, \beta_{n,0})$ we get

$$(u-v)^2 = \Sigma_{j=1}^{n+1}(u_j - v_j)^2 = \frac{4(\alpha-\beta)^2}{(1+\alpha^2)(1+\beta^2)} = 4\Sigma_{j=1}^{n}\frac{(\alpha_j-\beta_j)^2}{(1+\alpha^2)(1+\beta^2)}.$$

The calculation is identical to the classical case in $R^n$, and thus it is omitted. We may say that the "distance" $|u-v|$ between two points on the sphere is equal to the expression $\frac{(\alpha-\beta)}{(1+|\alpha|^2)(1+|\beta|^2)}$. But, of course, this is just an imitation of the situation in $R^n$, and there is no real meaning to "distances" or "metric" here.

Similarly, one may show that spheres on the unit spheres in $G^{n+1}$ are mapped onto planes or spheres on $z_{n+1} = 0$, depending on whether the sphere passes through the north pole or not. Again, details are omitted as the computations are identical to those in $R^{n+1}$.

We now move onto the discussion of

## 13.3 Points of infinity

In section 13.2 we pointed out the delicate situation $c_{n+1} = 1$. We first recall the concept of inversion on the unit sphere in $G^{n+1}$. Let $z = (z_1, z_2, \ldots, z_n, 0)$ be a point on $z_{n+1} = 0$. Consider the image $c = (c_1, c_2, \ldots, c_n, c_{n+1})$ on the unit sphere. By (12.9) we have $c_j = \frac{2z_j}{1+z^2}, \quad 1 \le j \le n, \quad c_{n+1} = \frac{z^2-1}{z^2+1}$. Let $w$ be the inverse point of $z$ in the plane $z_{n+1} = 0$, with respect to the unit sphere. Since $w_j = \frac{z_j}{z^2}, \quad 1 \le j \le n$, for the image $c^*$ we get $c^* = (c_1^*, c_2^*, \ldots, c_n^*, c_{n+1}^*) = \left(\frac{2w_1}{1+w^2}, \ldots, \frac{2w_n}{1+w^2}, \frac{w^2-1}{w^2+1}\right)$. Hence $\frac{2w_j}{1+w^2} = \frac{\frac{2z_j}{z^2}}{1+\frac{1}{z^2}} = \frac{2z_j}{1+z^2}$, $1 \le j \le n$ or $c_j^* = c_j, \quad 1 \le j \le n$. Also

$$c_{n+1}^* = \frac{w^2-1}{w^2+1} = \frac{\frac{1}{z^2}-1}{\frac{1}{z^2}+1} = \frac{1-z^2}{1+z^2} = -c_{n+1}.$$

Summing up, we have that the inverse point of $(c_1, c_2, \ldots, c_n, c_{n+1})$ is $(c_1, c_2, \ldots, c_n, -c_{n+1})$.

In view of $c_{n+1} = \frac{z^2-1}{z^2+1}$, for $z^2 = 0$ we have $c_{n+1} = -1$. These may be considered the points at the "south pole". But $c_j = \frac{2z_j}{1+z^2}, \quad 1 \le j \le n$, and $z^2 = 0$ imply that $c^2 = 0$. To each point $(c_1, \ldots, c_n, -1)$, $\Sigma_{j=1}^n c_j^2 = 0$, we now define the inverse point $(c_1, \ldots, c_n, 1), \quad \Sigma c_j^2 = 0$. These are the points of the "north pole". In contrast to the case of $c^{n+1}$ with the usual topology $|z-w|^2 = \Sigma_{j=1}^n |z_j - w_j|^2$, we have here many infinity points and not only one. Indeed, in $\mathbb{C}^n$ the condition $|z|^2 = 0$ implies $z = 0$, i.e., $z_j = 0$, $1 \le j \le n$. Here we get all points satisfying $\Sigma_{j=1}^n z_j^2 = 0$. In other words, in contrast to the case of $\mathbb{C}^{n+1}$, each point in $G^{n+1}$ satisfying $\Sigma_{j=1}^n c_j^2 = 0$, $c_{n+1} = -1$, may be considered as a point of the south pole suitable to the point $(z_1, z_2, \ldots, z_n, 0)$ where, by (13.10), we get $z_j = \frac{c_j}{1-c_{n+1}} = \frac{c_j}{2}$, $1 \le j \le n, \ \ z_{n+1} = 0$. Each of these points has an inverse point which may be considered as a "point at infinity" and it is the image of $(c_1, c_2, \ldots, c_n, 1)$, i.e., the inverse point of the "south pole" point $(c_1, c_2, \ldots, c_n, -1)$. In this way we get a one-one map of the "extended" plane $z_{n+1} = 0$ onto the extended unit sphere in $G^{n+1}$. Of course, the property where inverse points are mapped onto inverse points by Möbius maps can be extended now also to points at infinity, by continuity arguments applied to the unit sphere in $G^{n+1}$.

The discussion of inverse points referred to in the introduction is now complete.

## 14 Spherical geometry

In this final section we give some results in spherical geometry that are very similar to previous results in the hyperbolic geometry that were presented earlier. Our aim is to generalize a result of Mauldon [7, Theorem 3.2].

**Theorem 14.1.** *Let $n+1$ spheres $\{S\}_{j0}^{n+1}$ be given in the spherical $n+k-1$ space having mutual inclination $\gamma$, $\gamma \neq 0$, $\gamma \neq 1$, $\gamma \neq \frac{-1}{n}$, and $k \geq 0$, $n \geq 2$. Let $\{S_{j0}\}_{n+2}^{n+k+1}$ be another set of spheres in the same spherical $n+k-1$ space, such that they are mutually orthogonal and, in addition, each one of them is orthogonal to all other spheres $\{S_{j0}\}_{j=1}^{n+1}$. Denote by $\{\beta_j\}_{j=1}^{n+k+1}$ the angular radii of the $\{S_{j0}\}_{j=1}^{n+k+1}$ respectively. Then*

$$\begin{aligned}(\Sigma_{j=1}^{n+1}\frac{1}{\tan\beta_j^2}) - \big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{\tan^2\beta_j} \\ = \big(n+\frac{1}{\gamma}\big)(1-\gamma)\big(\Sigma_{j=n+2}^{n+k+1}\frac{1}{\tan^2\beta_j}+1\big).\end{aligned} \tag{14.1}$$

*Proof.* We first note that if $k=0$, the set $\{S_{j0}\}_{n+2}^{n+k-1}$ is void or, in other words, there are no orthogonal spheres, and we are back to the case Mauldon considered in the above mentioned theorem. Also, as usual, we consider the complex case, i.e., the $n+k-1$ spheres $\{S_{j0}\}_{j=1}^{n+k+1}$ are located on the unit sphere in $G^{n+k}$. The definition of the angular radius in the complex case needs some explanation. We transform the orthogonal spheres $\{S_j\}_{j=1}^{n+k+1}$ to the unit sphere in $G^{n+k}$, such that the intersection of $S_j$ with the unit sphere is $S_{j0}$. Then the angular radius of $S_{j0}$ is defined as the radius of $S_j$ (see Figure 22).

We now turn to the proof of the theorem. We construct the spheres $\{S_j\}_{j=1}^{n+k+1}$ in $G^{n+k}$ as explained above. Considering the situation in $G^{n+k}$ we observe that we have the spheres $\{S_j\}_{j=1}^{n+1}$ arising from $\{S_{j0}\}_{j=1}^{n+1}$, and another set of spheres $\{S_j\}_{j=1}^{n+k+1}$ arising from $\{S_{j0}\}_{j=n+2}^{n+k+1}$. We now claim that after the extension, the inclinations are preserved. More specifically, we claim that the sets $\{S_j\}_{j=1}^{n+1}$ have mutual inclination $\gamma$, and the sets $\{S_j\}_{j=n+2}^{n+k+1}$ are mutually orthogonal, and that each one of them is orthogonal to each of the sets $\{S_j\}_{j=1}^{n+1}$. To justify this claim, we use a Möbius transformation in $G^{n+k}$ that maps the unit sphere in this space onto the plane $z_{n+k}=0$. By construction $\{S_j\}_{j=1}^{n+k+1}$ are all orthogonal to the unit sphere in $G^{n+k}$. Hence, as a result of the Möbius map described above, the images of these spheres are orthogonal to $z_{n+k}=0$. Also, as a corollary of the above Möbius

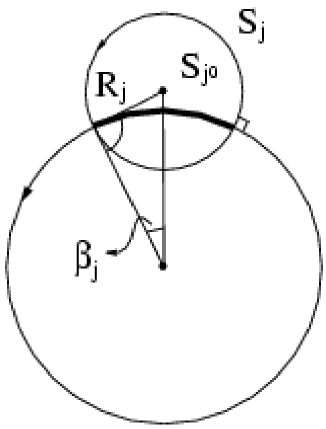


Figure 22: The angular radius.

map, the intersections of them with $z_{n+2} = 0$ are the images $\{S_{j0}\}_{j=1}^{n+k+1}$. The assertion about the invariance of inclination follows at once from the invariance property of inclination under a Möbius map and also under a Poincaré extension.

Using the above, we now see that the sets of $\{S_j\}_{j=1}^{n+1}$ have mutual inclination $\gamma$, since the same is true for $\{S_{j0}\}_{j=1}^{n+1}$. Also, $\{S_j\}_{j=n+2}^{n+k+1}$ are mutually orthogonal, and each one of them is orthogonal to each of the $\{S_j\}_{j=1}^{n+1}$. In addition, obviously, the unit sphere in $G^{n+k}$ is orthogonal to all spheres $\{S_j\}_{j=1}^{n+k+1}$. Hence, we can now use Theorem 12.2 for the set $\{S_j\}_{j=1}^{n+1}$ and the other set composed of $\{S_j\}_{j=n+2}^{n+k+1}$ and the unit sphere (i.e., there are $k+1$ spheres in the second set).

Since the radius of the unit sphere in $G^{n+k}$ is 1, from (12.13) we get

$$\big(\Sigma_{j=1}^{n+1}\frac{1}{R_j}\big)^2 - \big(n+\frac{1}{\gamma}\big)\Sigma_{j=1}^{n+1}\frac{1}{R_j^2} = (1-\gamma)\big(n+\frac{1}{\gamma}\big)\big(\Sigma_{j=n+2}^{n+k+1}\frac{1}{R_j^2}+1\big). \quad (14.2)$$

This is equivalent to (14.1) in view of $R_j = \tan\beta_j, \quad 1 \le j \le n+k+1$.

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