HAIT Journal of Science and Engineering C, Volume 4, Issues 1-2, pp. 233-262

Moment Banach spaces: Theory and applications

Eugene Ostrovsky[1] and Leonid Sirota[2,*]

[1]*Department of Mathematics, Hadas Ltd., 16 Galgaley Haplada St., Herzelia Pituach 46722, Israel*
[2]*Bar Ilan University, Ramat Gan 59200, Israel*
[*]*Corresponding author: sirota@zahav.net.il*

Received 1 May 2006, revised 19 November 2006, accepted 21 November 2006

Abstract

In this article we introduce and investigate some new Banach spaces, so-called moment spaces, and consider applications to the Fourier series, singular integral operators, theory of martingales.

2000 Mathematics Subject Classification: Primary (1991) 37B30, 33K55, Secondary (2000) 34A34, 65M20, 42B25

Key words: Banach, Orlicz, Lorentz, Marcinkiewicz, moment and rearrangement invariant spaces, martingales, singular operators, Fourier series and transform.

1 Definitions. Simple Properties

Let (X, Σ, μ) be a measurable space with non-trivial measure $\mu: \ \exists A \in \Sigma, \mu(A) \in (0, \mu(X))$. We will assume that either $\mu(X) = 1$, or $\mu(X) = \infty$ and that the measure μ is $\sigma-$ finite and diffuse: $\forall A \in \Sigma, 0 < \mu(A) < \infty \ \exists B \subset A, \mu(B) = \mu(A)/2$. Define as usually for all the measurable function $f: X \to R^1$

$$|f|_p = \left(\int_X |f(x)|^p \ \mu(dx) \right)^{1/p}, \ p \geq 1;$$

$L_p = L(p) = L(p; X, \mu) = \{f, |f|_p < \infty\}$. Let $a = const \geq 1, b = const \in (a, \infty]$, and let $\psi = \psi(p)$ be some positive continuous on the *open* interval (a, b) function, such that there exists a measurable function $f : X \to R$ for which

$$\psi(p) = |f|_p, \ p \in (a, b).$$

Note that the function $p \to p \cdot \log \psi(p), \ p \in (a, b)$ is convex.

The set of all such a functions we will denote $\Psi : \ \Psi = \Psi(a, b) = \{\psi(\cdot)\}$. The functions are described below.

Theorem 0. *Let the measure* μ *be diffuse. The function* $\nu(p), \ p \in (a, b)$ *belongs to the set* Ψ *if and only if there exist a two functions* $\Lambda_1(p), \ \Lambda_2(p)$, *such that* $\nu^p(p) = \Lambda_1(p) + \Lambda_2(p)$, *where* $\Lambda_1(p)$ *is absolute monotonic on the interval* (a, b) *and* $\Lambda_2(p)$ *is relative monotonic on the interval* (a, b) : $\forall k = 0, 1, 2, \ldots$

$$\forall p \in (a, b) \ \Rightarrow \Lambda_1^{(k)}(p) \geq 0, \ (-1)^k \Lambda_2^{(k)}(p) \geq 0.$$

Proof. Let $\nu(\cdot) \in \Psi$, then $\exists f : X \to R, \ \nu^p(p) =$

$$\int_X |f(x)|^p \ \mu(dx) = \int_X \exp(p \log |f(x)|)) \mu(dx) = \Lambda_1(p) + \Lambda_2(p),$$

where

$$\Lambda_1(p) = \int_{\{x: |f(x)| \geq 1\}} \exp(p \log |f(x)|) \ \mu(dx), \ \Lambda_1^{(k)}(p) \geq 0;$$

$$\Lambda_2(p) = \int_{\{x: |f(x)| < 1\}} \exp(p \log |f(x)|) \ \mu(dx), \ (-1)^k \Lambda_2^{(k)}(p) \geq 0.$$

Inversely, assume that $\nu^p(p) = \Lambda_1(p) + \Lambda_2(p), \ \Lambda_1^{(k)}(p) \geq 0$, $(-1)^{(k)} \Lambda^{(k)}(p) \geq 0$. It follows from Bernstein's theorem that

$$\Lambda_1(p) = \int_R \exp(pt) \mu_1(dt), \ \Lambda_2(p) = \int_R \exp(pt) \mu_2(dt),$$

where $\mu_1, \ \mu_2$ are a Borel measures on the set R such that $\mu_1\{(-\infty, 0)\} = 0, \ \mu_2\{(0, \infty)\} = 0$ and

$$\forall p \in (a, b) \ \Rightarrow \Lambda_1(p) < \infty, \ \Lambda_2(p) < \infty.$$

Therefore

$$\nu^p(p) = \int_{-\infty}^{\infty} \exp(pt)(\mu_1(dt) + \mu_2(dt)).$$

Since the measure μ is diffuse, there exists a (measurable) function $\eta : X \to R$ such that

$$\nu^p(p) = \int_X \exp(p\eta(x)) \ \mu(dx).$$

Thus, for $f(x) = \exp(\eta(x))$ we obtain:

$$|f|_p^p = \int_X \exp(p\eta(x))\mu(dx) = \nu^p(p), \ |f|_p = \nu(p).$$

Corollary 1. Note that if $\psi_1(\cdot) \in \Psi(a,b), \ \psi_2(\cdot) \in \Psi(c,d), \ \max(a,c) < \min(b,d)$, then $\psi_1(\cdot) \cdot \psi_2(\cdot) \in \Psi(\max(a,c), \ \min(b,d))$. Indeed, if

$$\psi_1(p) = |f_1|_p, \ \psi_2(p) = |f_2|_p,$$

and the functions $f_1, \ f_2$ are independent, then we have at $p \in (a,b) \cap (c,d)$

$$\psi_1(p) \cdot \psi_2(p) = |f_1 \cdot f_2|_p.$$

We extend the set Ψ as follows:

$$EX\Psi \stackrel{def}{=} EX\Psi(a,b) = \{\nu = \nu(p)\} =$$

$$\{\nu : \exists \psi(\cdot) \in \Psi : 0 < \inf_{p \in (a,b)} \psi(p)/\nu(p) \le \sup_{p \in (a,b)} \psi(p)/\nu(p) < \infty\},$$

$$U\Psi \stackrel{def}{=} U\Psi(a,b) = \{\psi = \psi(p), \forall p \in (a,b) \ \Rightarrow \psi(p) > 0\},$$

the function $p \to \psi(p), \ p \in (a,b)$ is continuous, and such that there exist a limits $\psi(a+0), \psi(b-0) \in (0,\infty]$; and we define formally for convenience $\psi(a) = \psi(a+0), \ \psi(b) = \psi(b-0)$.

Hereafter $a = const \ge 1, \ b \in (a, \infty]$.

For this case we define at $b = \infty$ $\psi(b-0) = \lim_{p \to \infty} \psi(p) \in (0, \infty]$.

Definition 1. Let $\psi(\cdot) \in U\Psi(a,b)$. The space $G(\psi) = G(X,\psi) = G(X,\psi,\mu) = G(X,\psi,\mu,a,b)$ consist on all the measurable functions $f : X \to R$ with finite norm

$$||f||G(\psi) \stackrel{def}{=} \sup_{p \in (a,b)} \left[|f|_p / \psi(p) \right].$$

If $\psi(a) < \infty$ and $\psi(b) < \infty$, then the correspondent $G(\psi)$ space is isimorphic to the direct sum $L(a) + L(b)$. In the "subcase" $b < \infty$ this space is equivalent to the Orlicz's space $Or(X, \Sigma, \mu; \Phi)$ with the Orlicz's function $\Phi(u) = |u|^a + |u|^b$.

Therefore, we will assume further that either $\psi(a) = \infty$ *or* $\psi(b) = \infty$, *or both the cases:* $\psi(a) = \psi(b) = \infty$. Briefly, $\min(\psi(a), \psi(b)) = \infty$.

The spaces $G(\psi)$, $\psi \in U\Psi$ are non - trivial: arbitrary bounded $\sup_x |f(x)| < \infty$ measurable function $f : X \to R$ with finite support: $\mu(supp\ |f|) < \infty$ belongs to arbitrary space $G(\psi)$, $\forall \psi \in U\Psi$.

We define also $B(\psi) = \{p : \ \psi(p) < \infty\}$ and recall that for arbitrary function $f : X \to R$ $supp\ f \stackrel{def}{=} \{x : f(x) \neq 0.\}$

We investigate in this paper some properties of moment spaces: the structure of some subspaces, non-separability, fundamental functions, conditions for convergence of sequences, martingales, Fourier series, and boundedness of singular operators.

Our results are some extensions and generalizations of papers [22], [23], [24], [25] etc.

Some preliminary results was partially announced in [5].

We consider now a very important for applications examples of $G(\psi)$ spaces. Let $a = const \geq 1, b = const \in (a, \infty]; \alpha, \beta = const$. Assume also that at $b < \infty$ $\min(\alpha, \beta) \geq 0$ and denote by h the (unique) root of equation

$$(h - a)^{\alpha} = (b - h)^{\beta},\ a < h < b;\ \zeta(p) = \zeta(a, b; \alpha, \beta; p) =$$

$$(p - a)^{\alpha},\ p \in (a, h);\ \zeta(a, b; \alpha, \beta; p) = (b - p)^{\beta},\ p \in [h, b);$$

and in the case $b = \infty$ assume that $\alpha \geq 0, \beta < 0$; denote by h the (unique) root of equation $(h - a)^{\alpha} = h^{\beta}, h > a$; define in this case

$$\zeta(p) = \zeta(a, b; \alpha, \beta; p) = (p - a)^{\alpha},\ p \in (a, h);\ p \geq h\ \Rightarrow \zeta(p) = p^{\beta}.$$

Note that at $b = \infty\ \Rightarrow \zeta(p) \asymp (p - a)^{\alpha}\ p^{-\alpha+\beta} \asymp \min\{(p - a)^{\alpha}, p^{\beta}\},\ p \in (a, \infty)$ and that at $b < \infty\ \Rightarrow \zeta(p) \asymp (p - a)^{\alpha}(b - p)^{\beta} \asymp \min\{(p - a)^{\alpha}, (b - p)^{\beta}\}, p \in (a, b)$. Here and further $p \in (a, b)\ \Rightarrow \psi(p) \asymp \nu(p)$ denotes that

$$0 < \inf_{p \in (a,b)} \psi(p)/\nu(p) \leq \sup_{p \in (a,b)} \psi(p)/\nu(p) < \infty.$$

We will denote also by the symbols $C_j, j \geq 1$ some "constructive" finite non - essentially positive constants. As usually, $I(A) = I(A, x) = I(x \in A) = 1, x \in A;\ I(A) = 0, x \notin A$.

Definition 2. The space $G = G_X = G_X(a, b; \alpha, \beta) = G(a, b; \alpha, \beta)$ consists on all measurable functions $f : X \to R^1$ with finite norm

$$||f||G(a, b; \alpha, \beta) = \sup_{p \in (a,b)} \left[|f|_p \cdot \zeta(a, b; \alpha, \beta; p)\right].$$

Corollary 2. As we know, the cases $\alpha \le 0$; $b < \infty, \beta \le 0$ and $b = \infty, \beta \ge 0$ are trivial for us and we will assume further that either $1 \le a < b < \infty, \min(\alpha, \beta) > 0$, or $1 \le a, b = \infty, \alpha \ge 0, \beta < 0$.

Lemma 1. *Let $\psi \in U\Psi$, $\psi(a) = \psi(b) = \infty, b < \infty$. There exist a two functions $\nu_1, \nu_2 \in U\Psi, \nu_1(a+0) \in (0, \infty), \nu_1(p) \sim \psi(p), p \to b-0; \nu_2(b-0) \in (0, \infty), \nu_2(p) \sim \psi(p), p \to a+0$ such that the space $G(\psi)$ may be represented as a direct sum*

$$G(\psi) = G(\nu_1) + G(\nu_2).$$

Proof. Indeed, if $f = f_1 + f_2$, $f_1 \in G(\nu_1)$, $\nu_1 \in U\Psi, \nu(a+0) \in (0, \infty)$; $f_2 \in G(\nu_2)$, $\nu_2 \in U\Psi, \nu_2(b-0) \in (0, \infty)$, then $f_1 \in G(\psi)$, $f_2 \in G(\psi)$, hence $f \in G(\psi)$.

Inversely, let $\psi \in U\Psi, \psi(a) = \psi(b) = \infty$. Let p_0 be some number inside the interval (a, b) such that

$$\psi(p_0) \stackrel{def}{=} C \in (\min \psi(p), \infty).$$

Define

$$\nu_1(p) = \psi(p) \cdot I(p \in (a, p_0)) + C \cdot I(p \in [p_0, b)),$$

$$\nu_2(p) = C \cdot I(p \in (a, p_0)) + \psi(p) \cdot I(p \in [p_0, b)).$$

If $f \in G(\psi)$, then

$$f(x) = f(x) I(|f(x)| \ge 1) + f(x) I(|f(x)| < 1) = f_1 + f_2.$$

It fillows from Tchebychev's inequality: $\mu\{x : |f(x)| \ge 1\} < |f|_p < \infty$ for some $p \in (a, b)$; therefore $f_1 \in G(\nu_1)$; and since $\forall q > p, A \in \Sigma$

$$\int_A |f_2|^q \mu(dx) \le \int_A |f_2|^p \mu(dx),$$

we obtain $f_2 \in G(\nu_2)$.

It is evident by virtue of Liapunov's inequality that in the bounded case $\mu(X) = 1$ $G(\psi) = G(\nu_1)$.

We denote by $G^o = G^o_X(\psi)$, $\psi \in U\Psi$ the closed subspace of $G(\psi)$, consisting on all the functions f, satisfying the following condition:

$$\lim_{p \to a+0} |f|_p / \psi(p) = \lim_{p \to b-0} |f|_p / \psi(p) = 0,$$

in the case $\psi(a) = \infty$, $\psi(b) = \infty$;

$$\lim_{p \to b-0} |f|_p / \psi(p) = 0$$

in the case $\psi(a) < \infty, \ \psi(b) = \infty$;

$$\lim_{p \to a+0} |f|_p / \psi(p) = 0$$

in the case $\psi(a) = \infty, \ \psi(b) < \infty$; and by $GB = GB(\psi)$ the closed span in the norm $G(\psi)$ the set of all the bounded measurable functions with finite support: $\mu(supp \ |f|) < \infty$.

We prove now that G^o is closed subspace of the space G. Let $f_n : X \to R$ be some sequence of a functions such that $f_n \in G^o, \ ||f_n - f||G(\psi) \to 0$ as $n \to \infty$. Let us denote $\delta(n) = ||f_n - f||G(\psi); \delta(n) \to 0, \ n \to \infty$. It follows from the direct definition of $G(\psi)$ spaces that for all $p \in B(\psi)$

$$|f|_p / \psi(p) \le |f_n|_p + \delta(n).$$

Let $\epsilon \in (0, 1)$ be a given. There exists a value n for wich $\delta(n) < \epsilon/2$. Further, as long as $f_n \in G^o(\psi)$, there exists a value $M = M(n) > 1$ such that for all values p satisfying a condition $\psi(p) > M$ we have: $|f_n|_p / \psi(p) < \epsilon/2$. Following, if $\psi(p) > M$, then $|f|_p / \psi(p) < \epsilon$ and $f \in G^o$.

Another definition: for a two functions $\nu_1(\cdot), \ \nu_2(\cdot) \in U\Psi$ we will write $\nu_1 << \nu_2$, iff

$$\lim_{p \to a+0} \nu_1(p) / \nu_2(p) = \lim_{p \to b-0} \nu_1(p) / \nu_2(p) = 0$$

in the case $\nu_2(a + 0) = \nu_2(b - 0) = \infty$ etc.

If for some $\nu_1(\cdot), \nu_2(\cdot) \in U\Psi, \ \nu_1 << \nu_2$ and $||f||G(\nu_1) < \infty$, then $f \in G^0(\nu_2)$. Moreover, if there exists a sequence of a functions f_n, f_∞ such that for some $\nu_1 \in G(\psi, a, b)$

$$\forall p \in (a, b) \ \Rightarrow |f_n - f_\infty|_p \to 0, n \to \infty$$

and $\sup_{n \le \infty} ||f_n||G(\nu_2) < \infty$, then $||f_n - f_\infty||G(\nu_1) \to 0$.

We consider now some important examples. Let $X = R, \ \mu(dx) = dx, 1 \le a < b < \infty, \gamma = const > -1/a, \ \nu = const > -1/b, \ p \in (a, b)$,

$$f_{a,\gamma} = f_{a,\gamma}(x) = I(|x| \ge 1) \cdot |x|^{-1/a} (|\log |x| \ |)^\gamma,$$

$$g_{b,\nu} = g_{b,\nu}(x) = I(|x| < 1) \cdot |x|^{-1/b} |\log x|^\nu,$$

$$h_m(x) = (\log |x|)^{1/m} I(|x| < 1), \ m = const > 0,$$

$$f_{a,b;\gamma,\nu}(x) = f_{a,\gamma}(x) + g_{b,\nu}(x), \ g_{a,\gamma,m}(x) = h_m(x) + f_{a,\gamma}(x),$$

$$\psi^p_{a,b;\gamma,\nu}(p) = 2(1-p/b)^{-p\nu-1}\ \Gamma(p\gamma+1) + 2(p/a-1)^{-p\gamma-1}\Gamma(p\nu+1),$$

$$\psi^p_{a,\gamma,m}(x) = 2(p/a-1)^{-p\gamma-1}\Gamma(p\gamma+1) + 2\Gamma((p/m)+1),$$

$\Gamma(\cdot)$ is usually Gamma - function.

We find by the direct calculation:

$$|f_{a,b;\gamma,\nu}|_p^p = \psi^p_{a,b;\gamma,\nu}(p);\ \ |g_{a,\gamma,m}|_p^p = \psi^p_{a,\gamma,m}(p).$$

Therefore,

$$\psi_{a,b;\gamma,\nu}(\cdot) \in \Psi(a,b),\ \ \psi_{a,\gamma,m}(\cdot) \in \Psi(a,\infty).$$

Further,

$$f_{a,b;\gamma,\nu}(\cdot) \in G(a,b;\gamma+1/a,\nu+1/b) \setminus G^o(a,b;\gamma+1/a,\nu+1/b),$$

$$g_{a,\gamma,m}(\cdot) \in G \setminus G^0(a,\infty;\gamma+1/a,-1/m),$$

and $\forall \Delta \in (0,1)\ f_{a,b,\alpha,\beta} \notin$

$$G(a,b;(1-\Delta)(\gamma+1/a),\nu+1/b)) \cup G(a,b;1/a,(1-\Delta)(\nu+1/b),$$

$$g_{a,\gamma,m}(\cdot) \in G \setminus G^o(a,\infty;\gamma+1/a;-1/m).$$

Another examples. Put

$$f^{(a,b;\alpha,\beta)}(x) = |x|^{-1/b} \exp\left(C_1|\log x|^{1-\alpha}\right) I(|x|<1) +$$

$$I(|x| \ge 1)|x|^{1/a} \exp\left(C_2(\log x)^{1-\beta}\right);$$

$1 \le a < b < \infty; \alpha,\beta = const \in (0,1)$. We have:

$$\log\left|f^{(a,b;\alpha,\beta)}(\cdot)\right|_p \asymp (p-a)^{1-1/\alpha} + (b-p)^{1-1/\beta},\ p \in (a,b).$$

Theorem 1. *The spaces* $G(\psi)$ *with respect to the ordinary operations and introdused norm* $||\cdot||G(\psi)$ *are Banach spaces.*

We need to prove only the completness of $G(\psi) -$ spaces. Denote

$$\epsilon(n,m) = ||f_n - f_m||G(\psi),\ \epsilon(n) = \sup_{m \ge n} \epsilon(m,n),$$

and assume $\lim_{n,m\to\infty}\epsilon(m,n) = 0$; then $\lim_{n\to\infty}\epsilon(n) = 0$. Let $p(i), i = 1,2,\ldots$ be the countable dense sequence of *all* the rational numbers of interval (a,b). We have from the direct definition of our spaces:

$$\forall i = 1,2,\ldots \ \Rightarrow |f_n - f_m|_{p(i)} \le \epsilon(n,m)\psi(p(i)).$$

We conclude that there exist a functions $f^{(i)},\ f^{(i)} \in L(p(i))$, such that

$$|f_n - f^{(i)}|_{p(i)} \le \epsilon(n)\psi(p(i)) \to 0,\ n \to \infty.$$

as long as the spaces $L(p(i))$ are complete. It is evident that

$$\mu\{x : \exists i :\ f^{(i)}(x) \ne f^{(1)}(x)\} = 0,$$

i.e. $f^{(i)}(x) = f^{(1)}(x)$ $\mu-$ almost everywhere. Hence $\forall i = 1,2,\ldots$

$$|f_n - f^{(1)}|_{p(i)} \le \epsilon(n)\psi(p(i)),$$

$$\forall p \in (a,b)\ \Rightarrow |f_n - f^{(1)}|_p \le \epsilon(n)\psi(p),$$

$$||f_n - f^{(1)}||G(\psi) = \sup_{p\in(a,b)} |f_n - f^{(1)}|_p/\psi(p) \le \epsilon(n) \to 0,$$

$n \to \infty$. This completes the proof of theorem 1.

Moreover, the spaces $G(\cdot)$ are rearrangement invariant (r.i.) spaces with the fundamental function

$$\phi(G,\delta) \stackrel{def}{=} \sup\{||I(A)||G,\ A \in \Sigma,\ \mu(A) \le \delta\};\ \delta \in (0,\infty).$$

We suppose further in this section that the measure μ is diffuse (still in the bounded case if $\mu(X) < \infty$), i.e. when $\mu(X) = 1$.

In this case, for the spaces $G(\psi),\ \psi(\cdot) \in U\Psi, B(\psi) = (a,b),\ b \le \infty$ we have:

$$\phi(G(\psi),\delta) = \sup_{p\in(a,b)} \left[\delta^{1/p}/\psi(p)\right].$$

Note that in the case $b < \infty$

$$\delta \le 1\ \Rightarrow C_1\delta^{1/a} \le \phi(G,\delta) \le C_2\delta^{1/b},$$

$$\delta > 1\ \Rightarrow C_3\delta^{1/b} \le \phi(G,\delta) \le C_4\delta^{1/a}.$$

Moreover, $\lambda \in (0,1)\ \Rightarrow$

$$\lambda^{1/b}\phi(G,\delta) \le \phi(G,\lambda\delta) \le \lambda^{1/a}\phi(G,\delta);$$

$$\lambda > 1 \ \Rightarrow \lambda^{1/b}\phi(G,\delta) \le \phi(G,\lambda\delta) \le \lambda^{1/a}\phi(G,\delta).$$

For instance, define in the case $b < \infty$ $\delta_1 = \exp(\alpha h^2/(h-a)), \ \delta \ge \delta_1 \ \Rightarrow$

$$p_1 = p_1(\delta) = \log\delta/(2\alpha) - \left[0.25\alpha^{-2}\log^2\delta - a\alpha^{-1}\log\delta\right]^{1/2},$$

$$\phi_1(\delta) = \delta^{1/p_1}(p_1 - a)^{\alpha};$$

$$\delta \in (0,\delta_1) \ \Rightarrow \phi_1(\delta) = \delta^{1/h}(h-a)^{\alpha};$$

$$\delta_2 = \exp(-h^2\beta/(b-h)), \ \delta \in (0,\delta_2) \ \Rightarrow$$

$$p_2 = p_2(\delta) = -|\log\delta|/2\beta + \left[\log^2(\delta/(4\beta^2)) + b|\log\delta|/\beta\right]^{1/2},$$

$$\phi_2(\delta) = \delta^{1/p_2(\delta)}(b - p_2(\delta))^{\beta};$$

$$\delta \ge \delta_2 \ \Rightarrow \phi_2(\delta) = \delta^{1/h}(b-h)^{\beta}.$$

We obtain after some calculations:

$$b < \infty \ \Rightarrow \phi(G(a,b;\alpha,\beta),\delta) = \max\left[\phi_1(\delta),\phi_2(\delta)\right].$$

Note that as $\delta \to 0+$

$$\phi(G(a,b,\alpha,\beta),\delta) \sim (\beta b^2/e)^{\beta} \ \delta^{1/b} \ |\log\delta|^{-\beta},$$

and as $\delta \to \infty$

$$\phi(G(a,b,\alpha,\beta),\delta) \sim (a^2\alpha/e)^{\alpha}\delta^{1/a} \ (\log\delta)^{-\alpha}.$$

In the case $b = \infty, \beta < 0$ we have denoting

$$\phi_3(\delta) = (\beta/e)^{\beta} \ |\log\delta|^{-|\beta|}, \ \delta \in (0, \exp(-h|\beta|)),$$

$$\phi_3(\delta) = h^{-|\beta|}\delta^{1/h}, \ \delta \ge \exp(-h|\beta|):$$

$$\phi(G(a,\infty;\alpha,-\beta),\delta) = \max(\phi_1(\delta),\phi_3(\delta)),$$

and we receive as $\delta \to 0+$ and as $\delta \to \infty$ correspondingly:

$$\phi(G(a,\infty;\alpha,-\beta),\delta) \sim (\beta)^{|\beta|}|\log\delta|^{-|\beta|},$$

$$\phi(G(a,\infty;\alpha,-\beta),\delta) \sim (a^2\alpha/e)^{\alpha} \ \delta^{1/a}(\log\delta)^{-a}.$$

2 Connection with another classical r.i. spaces

We define here the equivalence between a two Banach spaces $(Y_1, \ ||\cdot||Y_1)$ and $(Y_2, ||\cdot||Y_2)$ as the set coincidence and norm equivalent ness:

$$||f||Y_1 \le C_1||f||Y_2 \le C_2||f||Y_1.$$

Theorem 2. **A.** *Let* $\psi(\cdot) \in EX\Psi$, *such that* $\exists g : X \to R, \ \psi(p) \asymp |g(\cdot)|_p, \ p \in (a,b)$. *Denote*

$$N^{(-1)}(1/\delta) = 1/(\phi(G(\psi), \delta)), \ \delta \in (0, \infty),$$

where $N^{(-1)}$ *denotes the left inverse function to the* $N(\cdot)$ *on the set* R_+. *If*

$$\forall \epsilon > 0 \ \int_X N(\epsilon|g(x)|) \ \mu(dx) = \infty, \tag{2.1}$$

then the space $G(\psi)$ *is not equivalent to arbitrary Orlicz's space* $Or(X, \mu, \Phi)$.
B. *Denote* $T(x) = (1/\phi(x))^{(-1)}$. *If*

$$\sup_{p \in B(\psi)} \left[\left(\int_0^\infty x^{p-1} T(x) dx \right) / \psi(p) \right]^{1/p} = \infty, \tag{2.2}$$

then the space $G(\psi)$ *is not equivalent to arbitrary Marcinkiewicz's space* $M(\theta)$.
C. *Let* $\psi(\cdot) \in U\Psi, \ B(\psi) = (a,b), \ 1 \le a < b < \infty$. *Then the space* $G(\psi)$ *is not equivalent to arbitrary Lorentz's space* $L(\chi)$.
Proof. **A.** Assume converse, i.e. that $G(\psi) \sim Or(\Phi)$, where $Or(\Phi)$ is some Orlicz's space on the set (X, Σ, μ) with corresponding (convex, even, $\Phi(0) = 0$ etc.) Orlicz's function $\Phi(u), u \in R$. Since for $A \in \Sigma, \mu(A) \in (0, \infty)$

$$\phi(Or(\Phi); \mu(A)) = ||I(A)||Or(\Phi) = 1/\left[\Phi^{-1}(1/\mu(A))\right],$$

we conclude that $\Phi(u) = N(u)$. Note that, because of our condition (2.1) $g(\cdot) \in G(\psi) = Or(\Phi)$, but $g(\cdot) \notin Or(\Phi)$. This contradiction proves the assertion **A.**

As a consequence:
Lemma 2. The space $G(a, b; \alpha, \beta)$ are equivalent to the Orlicz's space *only in the case* $\alpha = 0, b = \infty, \beta < 0$.

(The case $\alpha = 0, b = \infty, \beta < 0$ was considered in [12].)

Proof B. Assume converse, i.e. that the space $G(\psi) = G(\psi, a, b)$ is equivalent to some Marcinkiewicz's space $M(\theta)$ over the our measurable space (X, μ). Recall here that in the considered case $a \geq 1; b > a$ the norm of a function $f : X \to R$ in the Marcinkiewicz's space may be calculated by the formula (up to norm equivalence)

$$||f||M(\theta) = \sup_{\delta > 0} \left[\theta(\delta) T^{(-1)}(f, \delta) \right]$$

and that the fundamental function for the $M(\theta)$ space in equal to

$$\phi(M(\theta), \delta) = 1/\theta(\delta),$$

(see, for example, [21], p. 187). Therefore, if the space $G(\psi)$ is equivalent to some Marcinkiewicz's space $M(\theta)$, then

$$\theta(\delta) = \delta/\phi(G(\psi), \delta)).$$

Let us consider the function $f : X \to R$ with the tail - function $T(f, x) \sim T(x), x \in (0, \infty)$, where as usual the tail function for the measurable function $f : X \to R$ is defined by equality

$$T(f, z) = \mu\{x; x \in X, \ |f(x)| > z\}, \ z \geq 0;$$

then $f \in M(\theta)$, but it follows from our condition (2.2) that $f \notin G(\psi)$.

For example, all the spaces $G(a, b; \alpha, \beta)$ are not equivalent to arbitrary Marcinkiewicz's space.

Proof C is very simple, again by means of the method of "reductio ad absurdum". Suppose $G(\psi) \sim L(\chi)$, where $L(\chi)$ denotes the Lorentz's space with some (quasi) - concave generating function $\chi(\cdot)$. Since

$$\phi(L(\chi), \delta) = \chi(\delta) \to 0, \delta \to 0+$$

and $\chi(\delta) \to \infty, \ \delta \to \infty$, we conclude that the space $L(\chi) = G(\psi)$ is separable ([22], p. 150.) But we will prove further (in the section 4) that the space $G(\psi)$ are non - separable.

3 Norm's absolute continuity of the function

We will say that the function $f \in G(\psi), \ \psi \in U\Psi$ has *absolute continuous norm* in the space $G(\psi)$ and write $f \in GA(\psi)$, if

$$\lim_{\delta \to 0+} \sup_{A: \mu(A) \leq \delta} ||f \ I_A||G(\psi) = 0.$$

The subspaces $GA(\psi), GB(\psi), G^0(\psi)$ are closed subspaces of the space $G(\psi)$.

Theorem 3. *Let* $\psi \in U\Psi$. *Then the spaces* $G^o, GB(\psi), GA(\psi)$ *are equal:*

$$G^(\psi) = GB(\psi) = GA(\psi).$$

For example, if $\min(\alpha, |\beta|) > 0, 1 \le a < b \le \infty$, then

$$G^o(a, b; \alpha, \beta) = GB(a, b; \alpha, \beta) = GA(a, b; \alpha, \beta).$$

Proof. The inclusions $GB \subset GA, GA \subset GB, \ GA \subset G^o$ are obvious.

Let now $f \in G^0$; for simplicity we will suppose $b < \infty, \mu(X) = 1$. Then $\lim_{p \to b-0} |f|_p / \psi(p) = 0$. Let $\epsilon > 0$. We have: $||f \ I(|f| \ge N)||G \le$

$$\sup_{p \in [1, b-\delta]} |f \ I(|f| \ge N|_p / \psi(p) + \sup_{p \in (b-\delta, b)} |f|_p / \psi(p) = \Sigma_1 + \Sigma_2;$$

$$\Sigma_2 \le \sup_{p \in [b-\delta, b)} |f|_p / \psi(p) \le \epsilon/2$$

for some $\delta \in (0, b)$ by virtue of condition $f \in G^o$.

Further, there exists a value $N \ge 1$ such that

$$\Sigma_1 \le C|f \ I(|f| \ge N)|_{b-\delta} \le \epsilon/2$$

as long as $f \in L_{b-\delta}$. Following, $f \in GB$; thus $G^0 \subset GB$.

Now we prove the inverse embedding. Let $f \in GB, \epsilon > 0$. Then $\exists g$, $\sup_x |g(x)| = B < \infty, \forall p \in [1, b) \ \Rightarrow |f - g|_p / \psi(p) < \epsilon/2$,

$$|f|_p \le |g|_p + 0.5\epsilon\psi(p), \ p \in [1, b);$$

$$|f|_p / \psi(p) \le |g|_p / \psi(p) + 0.5\epsilon < 0.5\epsilon + 0.5\epsilon \le \epsilon, \ |p - b| < \delta$$

for sufficiently small value δ. Theorem 3 is proved.

We investigate here the *sufficient* condition for the convergence

$$||f_n - f_\infty||G(\psi, a, b) \to 0, \ n \to \infty. \tag{3.1}$$

Assume at first that (the necessary condition)

$$\mathbf{A}. \forall p \in (a, b) \ \Rightarrow |f_n - f_\infty|_p \to 0, \ n \to \infty.$$

Theorem 4. *Let* $f_n, f_\infty \in G(\psi)$. *Assume that (in addition to the condition* **A**)

B. $\exists \psi_2(\cdot) \in U\Psi, \ \psi << \psi_2$, such that

$$\sup_{n \le \infty} ||f_n||G(\psi_2) < \infty.$$

Then the convergence (3.1) holds.
Proof. We need to use the following auxiliary well - known facts.
1. Let $1 \le a < b \in (1,\infty)$. We assert that

$$\sup_{p \in (a,b)} |f|_p < \infty \ \Leftrightarrow \max(|f|_a, |f|_b) < \infty.$$

This proposition follows from the formula

$$|f|_p^p = p \int_0^\infty z^{p-1} T(f,z) dz,$$

Tchebychev's inequality and Fatou's lemma.
2. Let $1 \le p(1) \le p \le p(2) < \infty, \max(|f|_{p(1)}, |f|_{p(2)}) < \infty$. Then $|f|_p \le$

$$|f|_{p(1)}^{(p(2)-p)/(p(2)-p(1))} \cdot |f|_{p(2)}^{(p-p(1))/(p(2)-p(1))} \stackrel{def}{=} Z(p, p(1), p(2); |f|_{p(1)}, |f|_{p(2)}).$$

Proposition **2** follows from the Hölder's inequality.

It is sufficient to investigate the case $b < \infty$; another cases may be proved analogously. Consider the norm

$$\Sigma \stackrel{def}{=} ||f_n - f_\infty||G(\psi) = \sup_{p \in (a,b)} |f_n - f_\infty|_p / \psi(p).$$

Let $\epsilon = const > 0$. We have: $\Sigma \le \Sigma_1 + \Sigma_2 + \Sigma_3$, where $\Sigma_1 =$

$$\sup_{p \in (a,a+\delta)} |f_n - f_\infty|_p / \psi(p) \le$$

$$\sup_p [|f_n - f_\infty|_p / \psi_2(p)] \cdot \sup_{p \in (a,a+\delta)} \psi(p)/\psi_2(p) \le C(a,\delta) < \epsilon/3,$$

if $\delta = \delta(\epsilon)$ is sufficiently small. Further, $\Sigma_3 =$

$$\sup_{p \in (b-\delta,\delta)} [|f_n - f_\infty|_p / \psi_2(p)] \cdot \sup_{p \in (b-\delta,b)} [\psi(p)/\psi_2(\delta)] \le C(b,\delta) < \epsilon/3.$$

Finally, $\Sigma_2 \le$

$$\sup_{p \in (a+\delta,b-\delta)} |f|_p/\psi(p) \le CZ\left(p, a+\delta, b-\delta, |f_n - f_\infty|_{a+\delta}, |f_n - f_\infty|_{b-\delta}\right)$$

$< \epsilon/3$ for sufficiently large values n.

Analogously may be proved the following assertion about the $G(\psi, a, b)$ convergence.

Lemma 3. *If the sequence of a functions* $\{f_n(\cdot)\}$ *convergens in all the* L_p *norms:*

$$\forall p \in (a,b) \ \Rightarrow \lim_{n\to\infty} |f_n - f_\infty|_p = 0$$

and has a uniform absolute continuous norms in the $G(\psi,a,b)$ *space:*

$$\lim_{\delta\to 0+} \sup_{n\le\infty} \sup_{A:\mu(A)\le\delta} ||f_n \ I(A)||G(\psi,a,b) = 0,$$

then $||f_n - f_\infty||G(\psi,a,b) \to 0, \ n\to\infty.$

In the case $\mu(X) < \infty$ the condition of lemma 3 may be replaced on the measure convergence: $\forall \epsilon > 0 \ \Rightarrow$

$$\lim_{n\to\infty} \mu\{x : |f_n(x) - f(x)| > \epsilon\} = 0.$$

Note that the first condition of the lemma 3 is not sufficient for $G(\psi)$ convergence. Let us consider the following example. Ler X be the interval $X = [0,1]$ with the classical Lebesque's measure and let ξ be a measurable function (random variable) with standard Gaussian distribution. Let also $\xi(n) = \xi \cdot I(|\xi| < n)$. Then $\xi(n) \in G^o(\psi_{0.5})), \xi \in G \setminus G^o(\psi_{0.5})$, where $\psi_{0.5}(p) \stackrel{def}{=} |\xi|_p \sim \sqrt{p}, \ p \in [1,\infty)$.

It is easy to verify that $\forall p \in [1,\infty) \ |\xi - \xi(n)|_p \to 0, \ n \to \infty$, but $||\xi - \xi(n)||G(\psi_{0.5})$ does not convergent to 0 as $n\to\infty$ since $\xi \notin G^o(\psi_{0.5})$.

Theorem 5. *Let* $\psi \in U\Psi$. *We assert that* $||f||G/G^o = ||f||G/GA =$

$$||f||G/GB = \inf_{g\in GB} ||f-g||G = \overline{\lim}_{\delta\to 0+} \sup_{A:\mu(A)\le\delta} ||fI(A)||G.$$

Here the notation G/G^o *denotes the factor - space.*

Proof. Suppose for simplicity $b \in (1,\infty), \ \mu(X) = 1, G = G(\psi), \psi(a) < \infty, \psi(b) = \infty; \ f \in G \setminus G^o$. Put

$$\gamma = \overline{\lim}_{\delta\to 0} \sup_{A:\mu(A)\le\delta} ||f \ I(A)||G > 0.$$

Let also $g = g(x)$ be a measurable bounded function: $\sup_x |g(x)| = B \in (0,\infty); k = const \ge 2$. We conclude using the elementary inequality: $X \ge kY > 0, k > 2, Y \le B = const \ \Rightarrow$

$$\frac{(X-Y)^p}{X^p - B^p} \ge \frac{(k-1)^p}{k^p - 1} :$$

$$||f-g||G \ge \sup_{p\in[1,b)} \left[\int_{\{x:|f(x)|>k|g(x)|\}} |f(x)-g(x)|^p \ \mu(dx) \right]^{1/p} / \psi(p) \ge$$

$$\overline{\lim}_{p\to b-0} \left[\int_{\{|f(x)|\ge kB\}} (k-1)^p (k^p-1)^{-1} (|f|^p - B^p)\ \mu(dx) \right]^{1/p} / \psi(p) \ge$$

$$(k-1)(k^b-1)^{-1/b}\ \overline{\lim}_{\delta\to 0} ||f\ I(A)||G = (k-1)(k^b-1)^{-1/b}\ \gamma.$$

Since the value of k is arbitrary, it follows from the last inequality that $||f-g||G \ge \gamma$; this proves that $\inf_{g\in GB} ||f-g||G \ge \gamma$; the inverse inequality is evident.

4 Non-separability

Recall that $\min(\psi(a), \psi(b)) = \infty$.

Theorem 6. *The spaces $G(\psi)$, $\psi \in U\Psi$ are non-separable.*

Proof. The assertion of theorem 6 is trivial if the metric space $(\Sigma, \rho(A,B))$, $\rho(A,B) = \arctan(\mu(A\Delta B))$ is non-separable. Therefore by virtue of Rocklin's theorem we can suppose that the space X is equipped by the distance $d = d(x_1, x_2)$ such that the space (X, d) is complete and separable, the measure μ is Borelian and diffuse.

Conversely, assume that the space $G(\psi)$ is separable. Let $\{u_n(x)\}$ be a enumerable dense subset of $G(\psi)$. By virtue of Lusin's and Prokhorov's theorems we conclude that there exists a compact subset Y of X with $\mu(Y) > 0$ such that on the subspace Y all the functions $u_n(x)$ are continuous. We consider now the space $G(Y, \psi)$. The functions $\{u_n(x)\}, x \in Y$ belong to the space $G^o_Y(\psi)$. Let $w(x), x \in Y$, be some function from the space $G_Y(\psi) \setminus G^o_Y(\psi)$ and define $w(x) = 0,\ x \in X \setminus Y$. We get:

$$\inf_n ||w - u_n||G_X \ge \inf_n ||w - u_n||G_Y \ge \inf_{g\in GB_Y} ||w - g||G_Y > 0,$$

in contradiction. This completes the proof of theorem 3.

Our proof of theorem 3 is the same as proof of non-separability of Orlicz's spaces ([1], p. 103; [2], p. 127).

5 Adjoint spaces

The complete description of the spaces conjugated to the spaces $\cap_p L_p$, see in [3], [4]. The spaces which are conjugate to the Orlicz's spaces are described in [2], p. 128 - 132. The structure of spaces $G^*(\psi)$ is analogous.

It is easy to verify using the classical theorem of Radon and Nicodim that the structure of linear continuous functionals over the space $G^0(\psi) = GA = GB$ is follows: $\forall l \in G^{0*}(\psi) \Rightarrow \exists g : X \to R$,

$$l(f) = \int_X f(x)g(x)\ \mu(dx) \stackrel{def}{=} l_g(f).$$

We investigate here only some necessary conditions for the inclusion $g \in G^*(\psi)$. Note at first that if $\psi \in U\Psi(a,b),\ q \in (b/(b-1), a/(a-1))$ and $g \in L_q$, then $g \in G^*(\psi)$.

Theorem 7. *If* $g \in G^*$, *then* $\exists K = K(g) < \infty\ \Rightarrow$

$$\forall z > 0\ \Rightarrow \int_z^\infty T(g,u)du \le K\phi(G, T(g,z)).$$

Recall that $\phi(G,\delta)$ denotes the fundamental function of the space G.

Proof. Let $l_g \in G^*$. It follows from the uniform boundedness principle that $\forall f \in G\ \Rightarrow$

$$|l_g(f)| = \left| \int_X f(x)\ g(x)\mu(dx) \right| \le K||f||_G.$$

Put $f = I_A(x), A \in \Sigma, A = \{x : |g(x)| > z\},\ z > 0$; then

$$\int_z^\infty T(g,u)du = \int_X |g(x)| I(|g(x)| > z)\ \mu(dx) \le K\phi(G, T(g,z)).$$

Let now $\psi \in U\Psi,\ B(\psi) = (a,b), b < \infty$. Introduce the following $N-$ Orlicz's function

$$N_\psi(u) = \sup_{p \in (a,b)} \left[|u|^p \psi^{-p}(p) \right],$$

then the following implication holds:

$$\exists \epsilon > 0 \int_X N_\psi(\epsilon f)\mu(dx) < \infty\ \Rightarrow f \in G(\psi).$$

Therefore, the Orlicz's space $Or(N, X, \mu)$ is subspace of $G(\psi)$. Following,

$$(G(\psi))^* \subset (L(N_\psi))^*.$$

Since the function $N_\psi(u)$ satisfies the Δ_2 condition, the adjoint space $(L(H_\psi))^*$ may be described as a new Orlicz's space, namely

$$(L(N_\psi))^* = L(\Phi_\psi),\ \Phi_\psi(u) = \sup_{z \in R}(uz - N_\psi(z)).$$

Thus, we obtained: $\psi \in U\Psi(a,b),\ 1 \le a < b < \infty\ \Rightarrow$

$$(G(\psi))^* \subset L(\Phi_\psi).$$

6 Tail behavior

Let $f \in G(\psi),\ \psi \in U\Psi(a,b), b \le \infty$. It follows from Tchebychev's inequality that

$$T(f,u) \le \inf_{p \in (a,b)} \left[||f||^p \psi^p(p)/u^p \right],\ u > 0.$$

Conversely,

$$|f|_p^p = p \int_0^\infty u^{p-1} T(f,u) du,\ p \ge 1;$$

therefore

$$||f||G(\psi) = \sup_{p \in B(\psi)} \left[p \ \left[\int_0^\infty u^{p-1} T(f,u)\ du \right]^{1/p} \ /\psi(p) \right].$$

In the particular case the spaces $G(a,b;\alpha,\beta)$ we obtain after simple calculations:

Theorem 8. **A**. *Let* $f \in G(a,b;\alpha,\beta), 1 \le a < b < \infty$. *Then*

$$u \in (0,1/2) \ \Rightarrow T(f,u) \le C_1(a,b,\alpha,\beta) |\log u|^{a\alpha} u^{-a}; \tag{5.1}$$

$$u \ge 2 \ \Rightarrow T(f,u) \le C_2(a,b,\alpha,\beta) (\log u)^{b\beta} u^{-b}. \tag{5.2}$$

B. *Conversely, suppose* $\exists a,b, 1 \le a < b < \infty, \gamma, \tau \ge 0, C_j > 0$ such that

$$T(f,u) \le C_1 |\log u|^\gamma u^{-a},\ u \in (0,1/2);\ T(f,u) \le C_2 (\log u)^\tau u^{-b},\ u \ge 2.$$

Then $f \in G(a,b;\gamma+1,\tau+1)$.

C. *Let now* $f \in G(a,\infty;\alpha,-\beta),\ \beta > 0$. *We propose that*

$$T(f,u) \le C_1 |\log u|^{a\alpha}\ u^{-a},\ u \in (0,1/2],$$

$$T(f,u) \le C_2 \exp\left(-C_3 u^{1/\beta}\right), u \ge 1/2;$$

D. *Conversely, if* $\exists a \ge 1, \beta > 0, \gamma \ge 0$,

$$T(f,u) \le C_1 |\log u|^\gamma\ u^{-a}, u \in (0,1/2), a = const > 0, \gamma \ge 0,$$

$$T(f,u) \le C_2 \exp\left(-C_3 u^{1/\beta}\right), \beta > 0,$$

then $f \in G(a,\infty;\gamma+1,-\beta)$.

Note in addition that at $\min(\alpha,\beta) > 0, b < \infty$

$$T(f,u) \sim C_1 |\log u|^{a\alpha} u^{-a}, u \to 0+ \Leftrightarrow |f|_p \sim C_2 (p-a)^{-\alpha}, p \to a+0;$$

$$T(f,u) \sim C_3 |\log u|^{b\beta} u^{-b}, u \to \infty \Leftrightarrow |f|_p \sim C_4 (b-p)^{-\beta}, p \to b-0$$

(Richter's theorem).

We can show despite the well - known Richter's theorem that both the inequalities (5.1) and (5.2) are exact. Let us consider the correspondent examples.

EXAMPLE 5.1. Let $\mu(X) = 1$, i.e. let (X, Σ, μ) be the probability space and let μ be diffuse. Consider the (measurable) discrete - valued function $f : X \to R$ such that

$$\mu\{x : f(x) = \exp(\exp(k))\} = C \exp(\beta b k - b \exp k), k = 1, 2, \ldots;$$

$$1/C = \sum_{k=1}^{\infty} \exp(\beta b k - b \exp(k)),$$

and denote $\gamma = \beta b,\ a(k) = a(k, \gamma, \epsilon) = \exp(k\gamma - \epsilon \exp(k)),$

$$\epsilon = b - p \to 0+,\ k(0) \stackrel{def}{=} [\log(\gamma/\epsilon)],\ x(k) = \exp(\exp(k)),$$

here $[z]$ denotes the integer part of z. We get:

$$W(\epsilon) \stackrel{def}{=} C^{-1} |f|_p^p = \sum_{k=1}^{\infty} a(k, \gamma, \epsilon) \geq$$

$$C_2 a(k(0), \gamma, \epsilon) \geq C_3 (b-p)^{-b\beta},$$

therefore $|f|_p \geq C_4 (b-p)^{-\beta}$.

Further, we have at $k > k(0)$ and $k < k(0)$ correspondently

$$a(k+1)/a(k) < \exp(\gamma(e-2)) < 1,\ a(k-1)/a(k) < \exp(-\gamma/e) < 1,$$

hence

$$W(\epsilon) \leq C_3 a(k(0), \gamma, \epsilon) \leq C \epsilon^{-p\beta},$$

following $|f|_p \leq C_5 (b-p)^{-\beta}, p \in (1, b)$. Thus $f \in G(1, b; 0, \beta)$. However,

$$T(f, x(k)) > C \exp(b\beta k - b \exp k) = C (\log x(k))^{b\beta} x(k)^{-b}.$$

(we used the discrete analog of saddle - point method).

EXAMPLE 5.2. Let $X = R^1_+, \mu(dx) = dx, Q(k) = \exp(a\alpha k + a \exp(k)), a = const \geq 1, S(k) = \sum_{l=1}^{k} Q(l),\ b \in (a, \infty),$

$$g(x) = \sum_{k=1}^{\infty} \exp(-\exp(k))\ I(x \in (S(k-1), S(k)]),$$

$u(k) = \exp(-\exp(k))$. We obtain analogously to the example 5.1:

$$p \in (a,b) \Rightarrow |g|_p \asymp (p-a)^{-\alpha},$$

but

$$T(g, u(k)) \geq C(a,b,\alpha) \ |\log u(k)|^{a\alpha} u(k)^{-a}.$$

7 Fourier's series and transform

In this section we investigate the boundedness of certain Fourier's operators, convergence and divergence Fourier's series and transforms in $G(\psi)$ spaces. Let $X = [-\pi, \pi]$ or $X = R = (-\infty, \infty)$, $\mu(dx) = dx$, $X = R$, $\mu(dx) = dx/(2\pi)$ in the case $X = [-\pi, \pi]; c(n) = c(n, f) =$

$$\int_{-\pi}^{\pi} \exp(inx) f(x) dx, n = 0, \pm 1, \pm 2 \ldots; \ 2\pi s_M[f](x) =$$

$$\sum_{\{n:|n| \leq M\}} c(n) \exp(-inx), \ s^*[f] = \sup_{M \geq 1} |s_M[f]|,$$

$$F[f](x) = \lim_{M \to \infty} \int_{-M}^{M} \exp(itx) f(t) dt,$$

$$F^*[f](x) = \sup_{M > 0} \int_{-M}^{M} \exp(itx) f(t) dt,$$

$$S_M[f](x) = (2\pi)^{-1} \int_{-M}^{M} \exp(-itx) F[f](t) dt,$$

$$S^*[f](x) = \sup_{M > 0} |S_M[f](x)|.$$

Recall that if $f \in L_p(R), p \in [1,2]$, then operators F, F^* are well defined; for the values $p > 1$, $f \in L_p$ are well defined the operators s_M, s^*, S_M, S^*.

We introduce also for arbitrary function ψ, such that $\psi(\cdot) \in U\Psi$, $B(\psi) \supset (1,2]$, $\psi_1(p) = \psi(p/(p-1))$; for $s = const \in (1, \infty)$ and for $\psi(\cdot) \in U\psi$, $B(\psi) \supset (1, s)$

$$\psi_{(s)}(p) = \psi(sp/(s-p)); \ p = \infty \ \Rightarrow p/(p-1) = +\infty;$$

for $\psi \in U\Psi$, $B(\psi) \supset [1, s/(s-1))$,

$$\psi^{(s)}(p) = \psi[ps/(s-1)/(p + s/(s-1))].$$

Let $\lambda, \gamma = const \geq 0$; we denote for $\psi \in U\Psi(1,\infty)$

$$\psi_{\lambda,\gamma}(p) = p^{\lambda+\gamma}\psi(p)\ (p-1)^{-\gamma}.$$

It is easy to verify that if $\psi \in EX\Psi$, then $\psi_{\lambda,\gamma} \in EX\Psi$.

Let Y_1, Y_2 be a two Banach spaces and let $Q : Y_1 \to Y_2$ be an operator (not necessary linear or sublinear) defined on the space Y_1 with values in Y_2. The operator Q is said to be bounded from the space Y_1 into the space Y_2, using the following notation:

$$||Q||[Y_1 \to Y_2] < \infty,$$

if for arbitrary $f \in Y_1 \ \Rightarrow \ ||Q[f]||Y_2 \leq C \cdot ||f||Y_1$.

Theorem 9. *Let* $\psi \in U\Psi, (1,2] \subset B(\psi)$. *The FourierŠs operator* F *is bounded from the space* $G(\psi)$ *into the space* $G(\psi_1)$:

$$||F||[G(\psi) \to G(\psi_1)] < \infty.$$

Proof. We will use the classical result of Hardy - Littlewood - Young:

$$|F[f]|_{p/(p-1)} \leq C|f|_p,\ p \in (1,2].$$

Here C is an absolute constant.

If $f \in G(\psi)$, then $|f|_p \leq ||f||G \cdot \psi(p)$, therefore

$$|F[f]|_p \leq \psi(p/(p-1))\ ||f||G(\psi) = \psi_1(p)\ ||f||G(\psi).$$

Theorem 10. *Let* $X = [-\pi,\pi], \psi \in U\Psi,\ B(\psi) \supset (1,\infty)$. *We assert that*

$$\sup_{M \geq 1} ||s_M||[G(\psi) \to G(\psi_{1,1})] < \infty.$$

Proof. Now we use the well - known result of M.Riesz:

$$||s_M[f]||[L_p \to L_p] \leq Cp^2/(p-1),\ p \in (1,\infty).$$

with an absolute constant C. If $f \in G(\psi)$, then $|f|_p \leq$

$$\psi(p)||f||G(\psi),\ |s_M\ f|_p \leq Cp^2||f||G(\psi)/(p-1) = C||f||G(\psi) \cdot \psi_{1,1}(p).$$

Corollary 3. Assume in addition to the conditions of theorem 10 that $B(\psi) \subset (a,b)$ for some $a = const > 1, a < b = const < \infty$. Then

$$\psi_{1,1}(p) \asymp \psi(p),\ p \in (a,b).$$

Therefore, in this case

$$\sup_{M \geq 1} ||s_M||[G(\psi) \to G(\psi)] < \infty.$$

However, this assertion does not means that $\forall f \in G(\psi) \ \Rightarrow$

$$\lim_{M \to \infty} ||s_M[f] - f||G(\psi) = 0;$$

see counterexamples further, in the lemma 4. If $\nu(\cdot) \in U\Psi, \ \nu << \psi_{1,1}, f \in G(\psi)$, then

$$\lim_{M \to \infty} ||s_M[f] - f||G(\nu) = 0,$$

i.e. the sequence $s_M[f]$ convergent to the function f in the $G(\nu)$ sense.

At the same assertion is true if $f \in G^0(\psi)$.

The assertion analogous to the assertion of theorem 10 is true for the maximal Fourier's operator s^*, Fourier's transform S_M and maximal Fourier's transform S^* etc.

Namely, it is proved in [13], p. 163 that $\forall f \in L_p, p \in (1,2]$ $|F^*[f]|_p \leq Cp^4(p-1)^{-2}|f|_p$. Following,

$$||F^*||[G(\psi) \to G\left(\psi_{2,2}\right)] < \infty.$$

Let us show the exactness of theorem 9. Let $f(x) = f_{a,b}(x) = |x|^{-1/b}, |x| \in (0,1); f(x) = |x|^{-1/a}, |x| \geq 1; G = G(a,b;1/a,1/b), G^{/} = G(b/(b-1), a/(a-1), (b-1)/b, (a-1)/a)$; then $f \in G$. It is easy to calculate that $F[f_{a,b}](t) \asymp f_{b/(b-1),a/(a-1)}(t), \ t \in R$, so

$$F[f_{a,b}] \in G^{/} \setminus G^{/0}.$$

This example is true even in the case $a = 1$; then $a/(a-1) + \infty$.

It is well known for the Fourier series $\sum_n c(n) \exp(inx)$ on the basis of Riesz's theorem that

$$f \in L_p[-\pi,\pi], \exists p > 1 \ \Rightarrow \lim_{M \to \infty} |s_M[f] - f|_p = 0.$$

This fact is true also in the Orlicz's spaces (instead the $L(p)$ spaces) with $N-$ function satisfying the so-called $\Delta_2 \cap \nabla_2$ conditions ([6], p. 196 - 197). Conversely, in the exponential Orlicz's spaces there exist a functions f, belonging to this spaces but such that Fourier's series (or integrals) does not convergent to f in the Orlicz's norm sense [5]. Analogously, this effect is true also in $G(\psi)$ spaces.

Lemma 4. *Let* $\psi \in EX\Psi, X = [-\pi, \pi]$. *There exists a function* $f \in G(\psi)$ *for which the Fourier's series does not convergent in the* $G(\psi)$ *norm to the function* f.

Proof. Since $\psi \in EX\Psi$, there exists a function $f : X \to R$ for which $|f|_p \asymp \psi(p),\ p \in (a, b)$; then $f \in G \setminus G^0(\psi)$. Assume conversely, i.e.

$$\lim_{M\to\infty} ||s_M[f] - f||G(\psi) = 0.$$

Since the trigonometrical system is bounded, this means that $f \in G^0$, in contradiction.

8 Martingales

Let (f_n, F_n) be a martingale, i.e. a monotonically non - decreasing sequence of F_n − sigmaŰsubalgebras of the sigma-algebra Σ and F_n measurable functions f_n such that $\mathbf{E}f_{n+1}/F_n = f_n$.

In this section we will use the probabilistic notations

$$\mathbf{E}f = \int_X f(x)\mu(dx),\ |f|_p = \mathbf{E}^{1/p}|f|^p$$

and notation $\mathbf{E}f/F$ for the conditional expectation.

For the L_p − theory of conditional expectations- and theory of martingales in the case $\mu(X) = \infty$ and some applications see, for example, in the book [7], pp. 330 - 347.

The Orlicz's norm estimates for martingales are used in the modern non-parametrical statistics, for example, in the so-called regression problem ([10], [11], [12]) etc. Namely, let us consider the following problem. Given is the observation $\{\xi(i)\},\ i = 1, 2, 3, \ldots, n;\ n \to \infty$ of a view

$$\xi(i) = g(z(i)) + \epsilon(i),\ i = 1, 2, \ldots,$$

where $g(\cdot)$ is an unknown estimated function, $\{\epsilon(i)\}$ are the errors of measurements and may be an independent random variables or martingale differences, $\{z(i)\}$ is some dense set in a metric space (Z, ρ) with Borel measure $\nu :\ \ z(i) \in Z$.

Let $\{\phi_k(z)\}$ be some complete orthonormal sequence of a functions, for example, the classical trigonometrical sequence, Legengre's or Hermitae's polynomials etc. Put

$$c_k(n) = n^{-1}\sum_{i=1}^{n}\phi_k(z(i)),\ \tau(N) = \tau(N, n) = \sum_{k=N+1}^{2N}(c_k(n))^2,$$

$$M = argmin_{N \in [1,n/3]} \tau(N), \ f_n(z) = \sum_{k=1}^{M} c_k(n) \phi_k(z).$$

Via an investigation of confidence region for estimating function f in the $L(p)$ norm $|f_n - f|_p$ are used the exponential bounds for the tail of distribution of polynomial martingales.

The next facts about martingales in the unbounded case $\mu(X) = \infty$ either there are in [7], p. 347 - 351, or are simple generalization of the classical results in the case $\mu(X) = 1$ ([8], [9]).

1. Let the martingale (f_n, F_n) be non - negative, $c, d = const, 0 < c < d < \infty$ and assume that for some $p \geq 1 \ \sup_n |f_n|_p < \infty$. Denote by $\nu = \nu(c, d)$ the number of upcrossing of interval (c, d) by the (random) sequence $\{f_n\}$. Then

$$\mathbf{E}\nu \leq (d - c)^{-p} \left[2^{p-1} \sup_n |f_n|_p^p + 2^{p-1} c^p + (d - c)^p \right].$$

2. Almost everywhere convergence. If for some $p \geq 1 \ \sup_n |f_n|_p < \infty$, then $\exists f_\infty(x) = \lim_{n \to \infty} f_n(x) \ (mod \ \mu), \ |f_\infty|_p < \infty$.

3. Convergence in L_p norms. If $\exists p > 1 \ \Rightarrow \sup_n |f_n|_p < \infty$, then

$$\lim_{n \to \infty} |f_n - f_\infty|_p = 0.$$

4. Doob's inequality: $p > 1 \ \Rightarrow$

$$\left| \sup_n f_n \right|_p \leq \sup_n \left[|f_n|_p \right] \ p/(p - 1).$$

In the bounded case $\mu(X) = 1$ the convergence of martingale $\{f_n\}$ to some function f_∞ $(mod \ \mu)$ is true under the sufficient condition $\sup_n |f_n|_1 < \infty$; let us show here that in unbounded case $(\mu(X) = \infty)$ our condition is unimproved. Namely, we consider the sequence of independent identically distributed functions $h_j = h_j(x)$ such that for some $p \geq 1$

$$|h_j|_p < \infty; \ \forall s, s \neq p, s \geq 1 \ \Rightarrow |h_j|_s = \infty.$$

Put

$$f_n(x) = \sum_{j=1}^{n} 2^{-j} h_j(x), \ F_n = \sigma\{h_j, j \leq n\};$$

then the convergence $f_n(\cdot)$ $(mod \ \mu)$ is true, despite $\forall \ s \neq p \ |f_n|_s = \infty$.

It is proved in the book [10], p. 252, see also [11] that if in some Orlicz's space $Or(X, \Sigma, \mu; N) = Or(N)$, with $\mu(X) = 1$ and with the $N -$ Orlicz's function satisfying the so-called $\Delta_2 \cap \nabla_2$ condition the martingale $\{f_n\}$ is bounded:

$$\sup_n ||f_n||Or(N) < \infty,$$

then the martingale $\{f_n\}$ convergent in the correspondent Orlicz's norm:

$$\lim_{n \to \infty} ||f_n - f_\infty||Or(N) = 0.$$

It is showed in the article [12] that in the *exponential* Orlicz's spaces $Or(N)$ the $Or(N)$ bounded martingale may divergent in the $Or(N)$ norm sense. Let us prove that in the $Or(N)$ spaces is the same case.

Lemma 5. *Let* $\psi \in EX\Psi$, *so that* $\psi(p) \asymp |f|_p$, *and let the* $\sigma -$ *algebra* $\sigma(f)$ *be an union of finite* $\sigma -$ *algebras:*

$$\sigma(f) = \cup_{n=1}^{\infty} \sigma_n, \ card(\sigma_n) = n < \infty,$$

$$\sigma_n = \sigma\{A_1^{(n)}, A_2^{(n)}, \ldots, A_n^{(n)}\},$$

with finite subsets:

$$\forall i \le n-1 \ \Rightarrow \mu(A_i^{(n-1)}) < \infty.$$

Then there exists a bounded but divergent in the $G(\psi) -$ *sense martingale*

$$(f_n, F_n): \ \sup_n ||f_n||G(\psi) < \infty, \ \overline{\lim}_{n \to \infty} ||f_n - f_\infty||G(\psi) > 0.$$

Proof. Let us consider some function $f \in G(\psi) \setminus G^0(\psi)$. Put $F_n = \sigma_n, \ f_n = \mathbf{E} f/F_n$; then (f_n, F_n) is a (regular) bounded martingale:

$$\sup_n ||f_n||G = \sup_{p \in (a,b)} |f_n|_p/\psi(p) \le \sup_{p \in (a,b)} |f|_p/\psi(p) = ||f||G < \infty;$$

we used the Iensen inequality $|f_n|_p \le |f|_p$.

Since the sigma - algebras σ_n are finite, $f_n \in G^0(\psi)$. Suppose $||f_n - f||G \to 0, \ n \to \infty$, then $f \in G^0$, in contradiction with choosing f.

Theorem 11. *Let* (f_n, F_n) *be a martingale,* $\psi \in U\Psi$,

$$\sup_n ||f_n||G(\psi) < \infty.$$

Then

$$\mathbf{A}. \ ||\sup_n f_n||G\left(\psi_{0,1}\right) < \infty.$$

Assume in addition that $\mathrm{supp}\ \psi = (a,b), 1 < a < b \le \infty$. *Then* $\forall \nu \in U(\psi),\ \nu << \psi_{0,1}$

$$\mathbf{B}.\ \lim_{n\to\infty} ||f_n - f_\infty||G(\nu) = 0.$$

Proof use the Doob's inequality and is the same as in theorem 8 and may be omitted.

For example, let (f_n, F_n) be a martingale, $1 \le a < b \le \infty$, $\sup_n ||f_n||G(a,b;\alpha,\beta) < \infty$. Then in the case $a > 1$ are true the following implications

$$|| \sup_n |f_n|\ ||G(a,b;\alpha,\beta) < \infty;\ \forall \Delta > 0\ \Rightarrow$$

$$\lim_{n\to\infty} ||f_n - f_\infty||G(a,b;\alpha+\Delta,\beta+\Delta[I(b<\infty) - I(b=\infty)]) = 0;$$

if $a = 1$, then

$$|| \sup_n |f_n|\ ||G(1,b;\alpha+1,\beta) < \infty;\ \forall \Delta > 0\ \Rightarrow$$

$$\lim_{n\to\infty} ||f_n - f_\infty||G(1,b;\alpha+1+\Delta,\beta+\Delta[I(b<\infty) - I(b=\infty)]) = 0.$$

It is clear that the convergence $f_n \to f_\infty$ in the norm $G(a,b;\alpha,\beta)$ is true also in the case when $f_\infty \in G^o(a,b;\alpha,\beta)$.

9 Operators

In this section we assume that there is a measurable space (X,Σ,μ) and Q is an operator not necessary linear or sublinear defined on the set $\cap_{p\in(a,b)} L_p(X,\mu), 1 \le a = const < b = const \le \infty$ and taking values in the set $\cap_{p\in(c,d)} L_p(X,\mu)$. We will investigate the problem of boundedness of operator Q from some space $G(X,\psi)$ into some *another* space $G(X,\nu)$.

The case of Orlicz's spaces and certain singular operators was consider in many publications; see, for example, [18], [19], [20].

At first we consider the regular operators.

1. Define a multiplicative operator

$$Q_f[g](x) = f(x) \cdot g(x).$$

Assume that $f \in L_s$ for some $s = const > 1$ and denote $t = t(s) = s/(s-1)$. As long as

$$|Q_f[g]|_r \le |f|_s \cdot |g|_{rt/(r+t)},\ r < s,$$

we conclude: if $B(\psi) \supset (t(s), \infty)$, then

$$||Q_f||[G(\psi) \to G(\psi_{(s)}] < |f|_s, \ \psi_{(s)}(p) = \psi(ps/(s-p)).$$

2. We consider now the convolution operator (again regular)

$$Con_f[g](x) = f * g(x) = \int_X g\left(xy^{-1}\right) f(y)\ \mu(dy),$$

where X is an unimodular Lie's group, μ is its Haar's measure. Assume that $f \in L_s(X, \mu)$ for some $s = const > 1$. Using the classical Young inequality

$$|f * g|_r \le C(r,s)|f|_s \cdot |g|_{rt(s)/(r+t(s))}, \ r > s, C(r,s) < 1,$$

we observe that

$$||Con_f|| \left[G(\psi) \to G\left(\psi^{(s)}\right)\right] \le |f|_s.$$

For example, if $\min(\alpha, \beta) > 0$, then

$$||Con_f||[G(1,\infty;\alpha,-\beta) \to G(s,\infty;\alpha,0)] \le C(\alpha,\beta,s)|f|_s, \ s > 1.$$

3. Finally we consider some classical singular operators. Assume that the operator Q satisfies the following condition: for some $\lambda, \gamma = const \ge 0$ and $\forall\ p \in (1, \infty)$

$$|Q[f]|_p \le C\ |f|_p\ p^{\lambda+\gamma}(p-1)^{-\gamma}. \tag{8.1}$$

There are many singular operators satisfying this condition, for instance, Hilbert's operator: $X = (-\pi, \pi)$ (or, analogously, $X = R$),

$$H[f](x) = \lim_{\epsilon \to 0+} H_\epsilon[f](x),$$

$$H_\epsilon[f](x) = (2\pi)^{-1} \int_{\epsilon \le |y| \le \pi} [f(x-y)/\tan(y/2)]dy, \ \lambda = \gamma = 1;$$

maximal Hilbert's operator

$$H^*[f](x) = \sup_{\epsilon \in (0,1)} |H_\epsilon[f](x)|, \ \lambda = 1, \gamma = 2;$$

operators of Caldron - Zygmund: $\lambda = \gamma = 1$, of Karlesson - Hunt: $s^*, S^*; \lambda = 1, \gamma = 3$; maximal, in particular, maximal Fourier's, operators, for example,

$$Q[f](x) \stackrel{def}{=} \sup_{M>0} \left| \int_R f(t)[\sin(M(x-t))/(x-t)]\ dt \right| : \ \lambda = \gamma = 2;$$

pseudodifferential operators ([15], p. 143): $\lambda = 1 = \gamma$, oscillating operators ([14], p. 379 - 381) etc.

The following result is obvious.

Theorem 12. *Let* $\psi \in U\Psi, B(\psi) = (1, \infty)$. *Assume that the operator* Q *satisfies the condition (8.1). Then*

$$||Q|| \left[G(\psi) \to G\left(\psi_{\lambda,\gamma}\right)\right] < \infty.$$

Let us consider some examples. Assume again that the operator Q satisfies the condition (8.1). Then Q is bounded as operator from the space $G(a, b; \alpha, \beta)$ into the space $G(a, b; \alpha_1, \beta_1)$, where at $1 < a < b < \infty \;\Rightarrow \alpha_1 = \alpha, \beta_1 = \beta$; in the case $a = 1, b < \infty \;\Rightarrow \alpha_1 = \alpha + \gamma, \beta_1 = \beta$; if $a > 1, b = \infty$ then $\alpha_1 = \alpha, \beta_1 = \beta + \lambda$; ultimately, for $a = 1, b = \infty$ we obtain: $\alpha_1 = \alpha + \gamma, \beta_1 = \beta + \lambda$.

We show now the exactness of estimations of theorem 12. Let us consider at first the singular Hilbert's operator for the functions defined on the set $(-\pi, \pi)$.

Put

$$f(x) = f_d(x) = \sum_{n=2}^{\infty} n^{-1} \log^d n \; \sin(nx), \; d = const \ge 0;$$

then (see [16], p. 184; [17], p. 116]) $|f(x)| \asymp (2 + |\log(|x|)|)^d, \; |f|_p \asymp p^d, p \in [1, \infty), x \in [-\pi, \pi] \setminus \{0\}$;

$$CH[f](x) = \sum_{n=2}^{\infty} n^{-1} \log^d n \; \cos(nx),$$

$$H[f](x) \asymp (2 + |\log(|x|)|)^{d+1}, \; |H[f]|_p \asymp p^{d+1}.$$

Considering the examples $d \in (0, 1), g = g_d(x) =$

$$\sum_{n=1}^{\infty} n^{d-1} \sin(nx), \; CH[g] = \sum_{n=1}^{\infty} n^{d-1} \cos(nx),$$

we can see that $|g(x)| \asymp |H[g]|(x), x \in R \setminus \{0\}$, and following $|g|_p \asymp |H[g]|_p, \; p \in (1, \infty)$.

We can built more general examples considering the functions of a view

$$f(x) = \sum_{n=2}^{\infty} n^{d-1} \; L(n) \; \sin(nx),$$

where $L(n)$ is some slowly varying as $n \to \infty$ function. See [17], p. 187 - 188.

The case of Hilbert's transform on the real axis is investigated analogously. Namely, consider the functions

$$f(x) = \int_3^\infty t^{d-1} \sin(tx)\ dt,\ d \in (0,1),$$

then (see [17], p.117) $CH[f](x) =$

$$\int_3^\infty t^{d-1} \cos(tx)\ dt,\ |H[f](x)| \asymp |f(x)| \asymp f_{1/d,1}(x),$$

following,

$$H[f](\cdot),\ f(\cdot) \in G \setminus G^o(1, 1/d; 1, d).$$

Analogously, considering the example

$$f(x) = \int_3^\infty t^{-1}\ \sin(tx)dx,\ |f(x)| \asymp f_{\infty,1}(x),$$

$x \in R \setminus \{0\}$, we observe that $|H[f](x)| \asymp |\log|x||,\ |x| \le 1/2;$

$$f(\cdot) \in G \setminus G^o(1, \infty; 1, 0),\ |CH[f](x)| \sim |\log|x||,\ x \to 0;$$

$$|H[f](x)| \asymp |x|^{-1},\ |x| \ge 1/2,$$

We can see that $H[f](\cdot) \in G \setminus G^o(1, \infty; 1, 1),$

We are very grateful for support and attention to Prof. V. Fonf, L. Beresansky, and M. Lin (Beer - Sheva, Israel). This investigation was partially supported by ISF (Israel Science Foundation), grant N^o 139/03.

References

[1] M.A. Krasnoselsky and Ya.B. Rutisky, *Convex functions and Orlicz's Spaces*, (P. Noordhoff Ltd, Groningen, The Netherland, 1961).

[2] M.M. Rao and Z.D. Ren, *Theory of Orlicz Spaces*, (Marcel Decker, New York-Basel, 1991).

[3] H.W. Davis, F.J. Murray, and J.K. Weber, *Families of L_p- spaces with inductive and projective topologies*, Pacific J. Math. **34**, 619-638 (1970).

[4] M.S. Steigenwalt and A.J. While, *Some function spaces related to L_p.*, Proc. London Math. Soc. **22**, 137-163 (1971).

[5] E. Ostrovsky and L. Sirota, *Fourier Transforms in Exponential Rearrangement Invariant Spaces*, Electronic Publ., arXiv:Math., FA/040639, v.**1**, 20.6.2004.

[6] M.M. Rao and Z.D. Ren, *Application of Orlicz Spaces*, (Marcel Decker, New York-Basel, 2002).

[7] M.M. Rao, *Measure Theory and Integration*, (Marcel Decker, New York-Basel, 2-nd ed., 2004).

[8] P. Hall and C.C. Heyde, *Martingale Limit Theorems and its Applications*, (Academic Press Inc., New York, 1980).

[9] J.L. Doob, *Stochastic Processes*, (John Wiley, New York, 1953).

[10] J, Neveu, *Discrete - Parameter Martingales*, (North-Holland Publ. Comp., Amsterdam-Oxford-New York, 1975).

[11] G. Peshkir, *Maximal Inequalities of Kahane - Khintchin type in Orlicz Spaces*, Preprint Series **33**, (Inst. of Math., University of Aarhus, Denmark, 1992).

[12] E. Ostrovsky, *Exponential Orlicz Spaces: new Norms and Applications*, Electronic Publ., arXiv/FA/0406534, v. **1**, 25.06.2004.

[13] J.A. de Reyna, *Pointwiese Convergence Fourier Series*, (Lect. Notes in Math., New York, 2004).

[14] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, (Princeton University Press, New York. 1970).

[15] M. Taylor, *Pseudodifferential Operators*, (Princeton University Press, New Jersey, 1981).

[16] A. Zygmund, *Trigonometrical Series*, Vol. **1**, (Cambridge University Press, 1959).

[17] E. Seneta, *Regularly Varying Functions*, (Nauka, Moscow, 1985) [in Russian].

[18] B. Bongioanni, L. Forzani, and E. Harboure, *Weak type and restricted weak type (p,p) operators in Orlicz spaces*, Real Analysis Exchange **28**, 381-394 (2002/2003).

[19] E. Harboure, O. Salinas, and B. Viviani, *Orlicz roundedness for certain classical operators*, Colloquium Matematicum **91**, 263-282 (2002).

[20] A. Cianchi, *Hardy inequalities in Orlicz Spaces*, Trans. AMS **351**, 2456-2478 (1999).

[21] S.G. Krein, Yu. Petunin, and E.M. Semenov, *Interpolation of linear operators*, Math. Monographs **54**, (Amer. Math. Soc., Providence, RI, 1982).

[22] A. Fiorenza, *Duality and reflexivity in grand Lebesgue spaces.* Collectanea Mathematica (electronic version) **51**, 131-148 (2000).

[23] A. Fiorenza and G.E. Karadzhov, *Grand and small Lebesgue spaces and their analogs*, Consiglio Nationale Delle Ricerche, Instituto per le Applicazioni del Calcoto Mauro Picine, Sezione di Napoli, Rapporto tecnico n. 272/03 (2005).

[24] T. Iwaniec and C. Sbordone, *On the integrability of the Jacobian under minimal hypotheses*, Archive for Rational Mechanics and Analysis **119**, 129-143 (1992).

[25] T. Iwaniec, P. Koskela, and J. Onninen, *Mapping of finite distortion: Monotonicity and continuity*, Invent. Math. **144**, 507-531 (2001).