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Probability of an inequality applied in statistical theory of communications

Roman E. Goot* and Alex Messel

Center for Industrial and Applied Mathematics,
Holon Institute of Technology, 52 Golomb St., Holon 58102, Israel
* *Corresponding author: goot@hit.ac.il*
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Abstract

We evaluate the probability $\Pr(\xi_1 \leq \xi_2)$ with ξ_1 and ξ_2 are non-central chi-square random variables and find the closed form expression for that probability.

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1 Introduction

In several problems of statistical theory of communications the necessity arises to evaluate the probability $\Pr(\xi_1 \leq \xi_2)$, where ξ_1 and ξ_2 are independent random variables distributed according to "Noncentral Chi-square" law. That probability is of interest for analyzing noise immunity of communication systems [1,2], performance of synchronization systems [3] and also for other areas, so that the problem is of rather common interest for applications. This probability can be written as follows

$$\Pr(\xi_1 \leq \xi_2) = \int_0^\infty f_1(x_1)\, dx_1 \int_{x_1}^\infty f_2(x_2)\, dx_2. \quad (1)$$

with $f_i(x)$ $(i = 1,2)$ be " Noncentral Chi-square" density having the view [4]:

$$f_i(x) = f(x, \lambda_i, N_i/2) = \frac{1}{2}\left(\frac{x}{\lambda_i}\right)^{\frac{N_i-2}{4}} \exp\left(-\frac{x+\lambda_i}{2}\right) I_{\frac{N_i}{2}-1}\left(\sqrt{\lambda_i x}\right), \tag{2}$$

where $\lambda_i \geq 0$ is the noncentrality parameter, N_i is integer and referred to as the number degrees of freedom.

The closed form expression for the integral (1) is known only for some particular cases. So, if $N_1 = N_2 = 2$, then [1,2]

$$\Pr(\xi_1 \leq \xi_2) = Q\left(\sqrt{\frac{\lambda_2}{2}}, \sqrt{\frac{\lambda_1}{2}}\right) - \frac{1}{2}\exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) I_0\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right), \tag{3}$$

where

$$Q(x,y) = \int_y^{\infty} t \exp\left(-\frac{t^2+x^2}{2}\right) I_0(xt)\, dx, \tag{4}$$

.

is Marcum's $Q-$function [5].

Also, the closed form is obtained for the case $N_1 = N_2 = 2q$ (q is integer) and $\lambda_2 = 0$ (see [1,2] with reference to [6]):

$$\Pr(\xi_1 \leq \xi_2) = 2^{-q} \exp\left(-\frac{\lambda_1}{4}\right) \sum_{k=0}^{q-1} 2^{-k} L_k^{q-1}\left(-\frac{\lambda_1}{4}\right), \tag{5}$$

where $L_k^{q-1}(\cdot)$ is Lagguer's polynom.

If, however, λ_1 and λ_2 are arbitrary positive values, there is a representation in the form of infinite series,

$$\Pr(\xi_1 \leq \xi_2) = \tag{6}$$
$$= 1 - 2^{-q} \exp\left(-\frac{\lambda_1+\lambda_2}{4}\right) \sum_{m=0}^{\infty} \frac{(\lambda_1/2)^m}{m!} \sum_{k=0}^{m+q-1} 2^{-k} L_k^{q-1}\left(-\frac{\lambda_2}{4}\right).$$

In some applications the necessity arises to find probability $\Pr(\xi_1 \leq \xi_2)$ in the case of $N_1 = 2q_1,\ N_2 = 2q_2$. Below closed expression will be obtained for the both cases ($q_1 = q_2$ and $q_1 \neq q_2$). Note, the " Noncentral Chi-square" distribution is the special case of the law with density

$$f\left(x, \lambda, p\right) = \frac{1}{2}\left(\frac{x}{\lambda}\right)^{\frac{p-1}{2}} \exp\left(-\frac{x+\lambda}{2}\right) I_{p-1}\left(\sqrt{\lambda x}\right), \tag{7}$$

where $\lambda > 0$, $p > 0$ are arbitrary real numbers [4].

Below we shall obtain the expression for $\Pr\left(\xi_1 \leq \xi_2\right)$ in the case when ξ_i (i = 1,2) follow the density (7).

2 Preliminary results

For derivation the formulas we need some properties of the density (7) and the function $\Pr\left(\xi_1 \leq \xi_2\right) = P\left(\lambda_1, p_1; \lambda_2, p_2\right)$, where λ_i, p_i $(i = 1, 2)$ are the corresponding parameters of the density (7).

Property 1

Let $p > 1$, then

$$\frac{d}{dx} f\left(x, \lambda, p\right) = \frac{1}{2}\left[f\left(x, \lambda, p-1\right) - f\left(x, \lambda, p\right)\right], \tag{8}$$

where *f(x, λ, p)* is the density (7).

Proof follows immediately from the known property of the modyfied Bessel function [7]:

$$\frac{d}{dx} x^{\nu} I_{\nu}\left(x\right) = x^{\nu} I_{\nu-1}\left(x\right), \tag{9}$$

where ν is an arbitrary real number.

Property 2

Let $p_i (i = 1, 2)$ be arbitrary real numbers, such that $p_i > 1$. Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) = \frac{1}{2}\left[P\left(\lambda_1, p_1;\ \lambda_2, p_2 - 1\right) + P\left(\lambda_1, p_1 - 1;\ \lambda_2, p_2\right)\right] \tag{10}$$

Proof:

It follows from (8), that

$$f\left(x_2, \lambda_2, p_2\right) = f\left(x_2, \lambda_2, p_2 - 1\right) - 2\frac{d}{dx} f\left(x_2, \lambda_2, p_2\right). \tag{11}$$

By using this connection in (1), we obtain

$$P(\lambda_1, p_1;\ \lambda_2, p_2) \ = \ P(\lambda_1, p_1;\ \lambda_2, p_2 - 1) \ + \ 2 \int\limits_0^{\infty} f(x, \lambda_1, p_1)\, f(x, \lambda_2, p_2)\ dx \tag{12}$$

Evaluate the last integral by part:

$$2 \int\limits_0^{\infty} f(x, \lambda_1, p_1) f(x_2, \lambda_2, p_2)\ dx = -2 f(x_1, \lambda_1, p_1) \int\limits_0^{\infty} f(x_2, \lambda_2, p_2) dx. \Bigg|_{x_1}^{\infty} +$$

$$+ \int\limits_0^{\infty} \left[\frac{d}{dx_1} f(x_1, \lambda_1, p_1)\right] dx_1 \int\limits_{x_1}^{\infty} f(x_2, \lambda_2, p_2) dx_2 =$$

$$P(\lambda_1, p_1 - 1; \lambda_2, p_2) \ - \ P(\lambda_1, p_1;\ \lambda_2, p_2)\ . \tag{13}$$

The last passage was realized with use of (8). Substitution (13) into (12) proves (10).

Corollary

Let us $p_2 \ = \ p_1 \ + \ n$, n is natural. Then

$$\begin{aligned} & P(\lambda_1,\ p_1; \lambda_2,\ p_2) \ = \\ = \ & P(\lambda_1, p_1;\ \lambda_2, p_1) \ + \ \sum_{i=0}^{n-1} \int\limits_0^{\infty} f(x, \lambda_1, p_1) f(x, \lambda_2, p_1 + i - 1)\, dx \end{aligned} \tag{14}$$

Proof follows from n - fold application of formula (12).

Property 3

Let us $p_i \ > \ 1 \ \ (i \ = \ 1, 2)$ are real numbers. Then

$$\begin{aligned} & P(\lambda_1, p_1; \lambda_2, p_2) \ = \ P(\lambda_1, p_1 - 1; \lambda_2, p_2 - 1) \ + \\ & + \int\limits_0^{\infty} [f(x, \lambda_1, p_1 - 1) \ f(x, \lambda_1, p_1)\, f(x, \lambda_2, p_2 - \ 1)] dx \end{aligned} \tag{15}$$

Proof:

With use of (12) we obtain

$$P(\lambda_1, p_1 - 1; \lambda_2, p_2) = \tag{16}$$

$$= \quad P(\lambda_1, p_1 - 1; \lambda_2, p_2 - 1) + 2\int\limits_0^\infty f(x, \lambda_1, p_1 - 1) f(x, \lambda_2, p_2)\, dx$$

It follows from (12), that

$$2\int\limits_0^\infty f(x, \lambda_1, p_1) f(x, \lambda_2 p_2 - 1)\, dx = \tag{17}$$

$$= \quad P(\lambda_1, p_1 - 1;\ \lambda_2, p_2 - 1) - P(\lambda_1, p_1;\ \lambda_2, p_2 - 1)$$

Validity of (15) follows from (16) and (17) in view of (12).
Corollary
If $p_1 = p_2 = q$, q is natural, then

$$P(\lambda_1, q; \lambda_2, q) = P(\lambda_1, 1;\ \lambda_2, 1) +$$
$$+\sum_{k=1}^{q-1}\int\limits_0^\infty [f(x, \lambda_1, k)\, f(x, \lambda_2,\ k+1) - f(x, \lambda_1, k+1)\, f(x, \lambda_2, k)]dx \tag{18}$$

Property 4
Let us $p_i \geq 0$ $(i = 1, 2)$ are real numbers. Then

$$P(\lambda_1, p_1; \lambda_2, p_2) = 2\sum_{k=0}^{\infty}\int\limits_0^\infty f(x, \lambda_1, p_1 + k + 1)\, f(x, \lambda_2, p_2) dx \tag{19}$$

Proof:
It is evident, that (1) can be represented as follows:

$$P(\lambda_1, p_1; \lambda_2, p_2) = \int\limits_0^\infty f(x, \lambda_2, p_2) dx \int\limits_0^x f(y, \lambda_1, p_1) dy. \tag{20}$$

Using the known series [7],

$$\sum_{k=0}^{\infty} t^k I_{k+\nu}(z) \;=\; z^{-\nu} \exp\left(\frac{tz}{2}\right) \int_0^z \tau^{\nu} \exp\left(-\frac{t\tau}{2z}\right) d\tau, \tag{21}$$

and the formula (7), it is easy to show, that

$$2\sum_{k=0}^{\infty} f\left(x, \lambda_1, p_1 + k + 1\right) \;=\; \int_0^x f\left(y, \lambda_1, p_1\right) dy \tag{22}$$

Substitution of (22) into (19) and changing order of integration and summation prove validity of (19).

3 Basic results

Statement 1

Let us ξ_1 and ξ_2 are independent random values having density (2) and parameters$(\lambda_i, N_i)_{i=1,2}$, $N_1 = N_2 = 2q$, q is natural. Then

$$\Pr\left(\xi_1 < \xi_2\right) \;=\; P\left(\lambda_1, q; \lambda_2, q\right) \;=\; Q\left(\sqrt{\tfrac{\lambda_2}{2}}, \sqrt{\tfrac{\lambda_1}{2}}\right) \;-\; \tfrac{1}{2}\exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \times$$

$$\times \left[I_0\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \;-\; \sum_{k=0}^{q-1} C_{k,q} \tfrac{\lambda_1^k - \lambda_2^k}{\left(\lambda_1\lambda_2\right)^{1/k}} I_k\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \right] , \tag{23}$$

where

$$C_{k,q} \;=\; 2^{-2k} \sum_{m=0}^{q-k-1} 2^{-2m} \frac{k}{m+k} \begin{pmatrix} 2m + 2k \\ m \end{pmatrix} \tag{24}$$

Proof

According to the corollary from the property 3 this probability can be represented as follows:

$$P\left(\lambda_1, q; \lambda_2, q\right) \;=\; P\left(\lambda_1, 1; \lambda_2, 1\right) \;+\; \Delta. \tag{25}$$

Since the closed expression for $P\left(\lambda_1, 1; \lambda_2, 1\right)$ is known (see formula (3)), to demonstrate validity (23), (24) it is sufficient to evaluate Δ:

$$\Delta = \sum_{k=1}^{q-1} \int_0^\infty [f(x,\lambda_1,k)\ f(x,\lambda_2,k+1) - f(x,\lambda_1,k+1)\, f(x,\lambda_2,k)]dx \tag{26}$$

Substitute the density (2) into (26) and change the variable for integration $z = \sqrt{x}$. Then

$$\Delta = \tfrac{1}{2}\exp\Big(-\tfrac{\lambda_1+\lambda_2}{2}\Big)\sum_{k=1}^{q-1}\Big[\ \lambda_1^{-(k-1)/2}\lambda_2^{-k/2}$$

$$\int_0^\infty z^{2k}\exp\left(-z^2\right) I_k\left(z\sqrt{\lambda_2}\right) I_{k-1}\left(z\sqrt{\lambda_1}\right)dz - \tag{27}$$

$$-\lambda_1^{-k/2}\lambda_2^{-(k-1)/2}\int_0^\infty z^{2k}\exp\left(-z^2\right)\ I_k\left(z\sqrt{\lambda_1}\right) I_{k-1}\left(z\sqrt{\lambda_2}\right)\ dz\ \Big]$$

The integrals from (27) are evaluated in appendix 3. After substitution the results of these evaluations into (27), we obtain:

$$\Delta \;=\; \frac{1}{2}\exp\left(-\frac{\lambda_1-\lambda_2}{4}\right)\sum_{k=1}^{q-1} 2^{-2k}\times \tag{28}$$

$$\times \sum_{m=-k-1}^{k.} \binom{2k-1}{k-1-m}\left[\left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} - \left(\frac{\lambda_2}{\lambda_1}\right)^{m/2}\right] I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right)$$

Now we can evaluate formulas (23) and (24) from (3), (25) and (28) if to take into account that the modified Bessel functions are even for integer index and to use the following evidential identities:

$$\binom{a}{b} = \frac{b+1}{a-b}\binom{a}{b+1};\binom{a-1}{b} = \frac{a-b}{a}\binom{a}{b}.$$

Some identities for sums with binomial coefficients are proved in appendices 1 and 2. With help of these it can obtain the following representations for values $C_{k,q}$determined by (24):

$$C_{k,q} = 2^{-(q+k-1)}\sum_{m=0}^{q-k+1} 2^{-m}\binom{m+q+k-1}{m}, \tag{29}$$

$$C_{k,q} = 2^{-2(q-1)} \sum_{m=0}^{q-k+1} \left(\begin{array}{c} 2q-1 \\ m \end{array} \right), \tag{30}$$

$$C_{k.q} = 2\frac{B_{1/2}\,(q+k, q-k)}{B\,(q+k, q-k)}, \tag{31}$$

where$B_x\,(a,b)$, $B\,(a,b)$ are incomplete and complete beta functions correspondingly.

Note that at $\lambda_2 \to 0$, the expression (20) reduces to formula (4). For demonstration of that it is sufficient to use the representation (26), Lagguer's polynoms determination

$$L_n^{\alpha}\,(y) = \sum_{m=0}^{n} \left(\begin{array}{c} n+\alpha \\ n-m \end{array} \right) \frac{(\,-y)^m}{m!}, \tag{32}$$

Marcum's Q- function limit value [2],

$$\begin{array}{l} \lim Q\,(x,y) = \exp\left(-\frac{y^2}{2} \right) \\ x \to 0 \end{array}, \tag{33}$$

and modified Bessel functions asymptotic values [7],

$$\begin{array}{l} I_k\,(y) \approx \frac{(y/2)^k}{k\,!} \\ y \to 0 \end{array}. \tag{34}$$

Statement 2

Let ξ_1 and ξ_2 are independent random values with density (2) and parameters $(\lambda_i, N_i)_{i=1,2}$, $N_1 = 2q_1 < N_2 = 2q_2$, q_i $(i = 1,2)$ are natural. Then

$$\begin{array}{l} \Pr\,(\xi_1 < \xi_2) = P\,(\lambda_1, q_1;\, \lambda_2, q_2) = \\ = P\,(\lambda_1, q_1; \lambda_2, q_1) + \exp\left(-\frac{\lambda_1+\lambda_2}{4} \right) \sum\limits_{m=-q_1+1}^{q_2-1} \left(\frac{\lambda_1}{\lambda_2} \right)^{m/2} \times \\ \times I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2} \right) \sum\limits_{i=\max(q_1,m)}^{q_2-1} 2^{-(q_1+i)} \left(\begin{array}{c} q_1+i-1 \\ q_1+m-1 \end{array} \right) \end{array} \tag{35}$$

Proof

Using the corollary from property 2, we shall represent the considered probability in the following form:

$$P(\lambda_1, q_1;\ \lambda_2, q_2) \ = \ P(\lambda_1, q_1;\ \lambda_2, q_1) \ + \ \Delta, \tag{36}$$

where

$$\Delta \ = \ 2 \sum_{i=0}^{q_2-q_1-1} \int\limits_0^{\infty} f(x, \lambda_1, q)\ f(x, \lambda_2, q_1 + i + 1)\ dx. \tag{37}$$

Substitute the expression for density (2) into (36). Then after some transformations value Δ may be reduced to sum of the integrals, evaluated in Appendix 3. It will lead us to the following expression:

$$\begin{aligned} \Delta \ &= \ \exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \sum_{i=0}^{q_2-q-1} 2^{-(2q_1+i)} \times \\ &\times \sum_{m=-q_1+1}^{q_1+i} \left(\begin{array}{c} 2q_1 + i \ - \ 1 \\ m \ + \ q_1 \ - \ 1 \end{array} \right) \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \ = \\ &= \exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \sum_{m=-q+1_1}^{q_2-1} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \times \\ &\times \sum_{i=\max(q,m_1)}^{q-1_2} 2^{-(q+i_1)} \left(\begin{array}{c} i + q_1 - 1 \\ m + q_1 - 1 \end{array} \right) \end{aligned} \tag{38}$$

The validity of (35) follows from (38) and (36).

Statement 3

Let ξ_1 and ξ_2 are independent random values with density (2) and parameters $(\lambda_i, N_i)_{i=1,2}$, $N_1 = 2q_1$, $N_2 = 2q_2$, q_i $(i \ = \ 1, 2)$ are natural. Then

$$\begin{aligned} &\Pr(\xi_1 \ < \ \xi_2) \ = \ P(\lambda_1, q_1; \lambda_2, q_2) \ = \\ &= Q\left(\sqrt{\tfrac{\lambda_1}{2}}, \sqrt{\tfrac{\lambda_2}{2}}\right) \ + \ \tfrac{1}{2} \exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \times \\ &\times \left\{ \sum_{m=1}^{q_2-1} C_{m-k,q} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \right. \\ &\left. - \sum_{m=0}^{q_1-1} C_{m+k,q} \left(\tfrac{\lambda_2}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \right\}, \end{aligned} \tag{39}$$

where

$$k = (q_1 - q_2)/2, \; q = (q_1 + q_2)/2, \tag{40}$$

and $C_{m-k,q}$, $C_{m+k,q}$ are the coefficients, determined by (29)-(31).

Proof

At first we will prove validity of (39) for the case $q_1 < q_2$. For that we use the representation of the probability in form (36) and (38). The value Δ may be written as the sum:

$$\Delta = A + B \tag{41}$$

where

$$A = \exp\left(-\frac{\lambda_1 + \lambda_2}{4} \right) \times \tag{42}$$

$$\times \sum_{m=-q_1+1}^{q-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m \left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \sum_{i=q_1}^{q_2-1} 2^{-(q+i)} \binom{i + q_1 - 1}{m + q_1 - 1},$$

$$B = \exp\left(-\frac{\lambda_1 + \lambda_2}{4} \right) \times \tag{43}$$

$$\times \sum_{m=q_1}^{q_2-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m \left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \sum_{i=m}^{q_2-1} 2^{-(q_1-i)} \binom{i + q_1 - 1}{m + q_1 - 1}.$$

Using (29), from (42) and (43) we obtain

$$A = \tfrac{1}{2} \exp\left(-\tfrac{\lambda_1+\lambda_2}{4} \right) \times$$

$$\times \left\{ \sum_{m=0}^{q_1-1} \left(\tfrac{\lambda_2}{\lambda_1}\right)^{m/2} I_{-m} \left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \left(C_{-(k+m),q} - C_{-m,q_1}\right) + \right. \tag{44}$$

$$\left. + \sum_{m=1}^{q_1-1} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m \left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \left(C_{m-k,q} - C_{m,q_1}\right) \right\},$$

$$B = \frac{1}{2} \exp\left(-\frac{\lambda_1 + \lambda_2}{4} \right) \sum_{m=q_1}^{q_2-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m \left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right). \tag{45}$$

The values k and q from (44) and (45) are determined by (40).

It is demonstrated in Appendix 2, that

$$C_{-(k+m),q} = 2 - C_{k+m,q};\ C_{-m,k} = 2 - C_{m,k} \tag{46}$$

It is easy to obtain from (30) or (31), that

$$C_{0,q} = 1 \tag{47}$$

Since $I_{-m}(x) = I_m(x)$, substitution of (45) and (47) into (44) brings the following expression:

$$A = \frac{1}{2}\exp\left(-\frac{\lambda_1+\lambda_2}{4}\right)\left\{I_0\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \sum_{m=1}^{q_1-1} C_{m,q_1}\frac{\lambda_1^m-\lambda_2^m}{(\lambda_1\lambda_2)^{m/2}} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \right.$$

$$- \sum_{m=0}^{q_1-1} C_{m-k,q}\left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \sum_{m=0}^{q_1-1} C_{m+k,q}\left(\frac{\lambda_2}{\lambda_1}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \tag{48}$$

Substitutions of (45) and (48) into (41) and also (23) and (41) into (38) prove the validity of (39) for the event $q_1 < q_2$.

Now, let $q_1 > q_2$. We use the evidential equality:

$$P(\lambda_1, q_1; \lambda_2, q_2) = 1 - P(\lambda_2, q_2; \lambda_1, q_1)\,. \tag{49}$$

The validity of (39) for $P(\lambda_2, q_2; \lambda_1, q_1)$ has been proved. Now, if to substitute the known expression for $P(\lambda_2, q_2; \lambda_1, q_1)$ to (45), to take into consideration (47) and, also, the known property of Marcum's Q- function [2]:

$$Q(x,y) + Q(y,x) = 1 + \exp\left(-\frac{x^2+y^2}{2}\right) I_0(x,y)\,, \tag{50}$$

the validity of (38) may be easy proved for the event $q_1 < q_2$.

It is evident that for $q_1 = q_2$, (23) and (38) coincide. The equivalence of (28), (31) for that case if$k$ and $q$are integer or half-integer coincidentally $(q \geq 1, |k| \leq q-1)$is proved in Appendix 2.

Statement 4

Let ξ_1 and ξ_2 are independent random values with density (7) and parameters $(\lambda_i, p_i)_{i=1,2}$, λ_i, p_i are nonnegative numbers. Then probability $P(\xi_1 < \xi_2)$ is

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) = \exp\left(-\frac{\lambda_1 + \lambda_2}{2}\right) \times \times \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} . \frac{\cdot\left(\frac{\lambda_1}{2}\right)^m \left(\frac{\lambda_2}{2}\right)^n}{m!\, n!} S\left(p_1 + m,\, p_2 + n\right)^{\cdot} \tag{51}$$

with

$$S\left(x, y\right) = \frac{B_{1/2}\left(x, y\right)}{B\left(x, y\right)}. \tag{52}$$

Proof

Substitute the density (7) into (19) and fulfill change of the variable $y = \sqrt{x}$. Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) = \lambda_1^{-p_1/2} . \lambda_2^{-\ (p_2-1)/2} \times$$

$$\times \exp\left(-\tfrac{\lambda_1+\lambda_2}{2}\right) \sum_{k=0}^{\infty} \int_0^{\infty} y^{p_1+p_2+k} \exp\left(-y^2\right) \times \tag{53}$$

$$\times I_{p_1+k}\left(y\sqrt{\lambda_1}\right)\ I_{p_2-1}\left(y\sqrt{\lambda_2}\right) dy$$

The integral from (53) is known [7] :

$$\int_0^{\infty} x^{\alpha-1} \exp\left(-px^2\right) I_\mu\left(bx\right) I_\nu\left(cx\right) dx =$$

$$= \frac{b^\mu c^\nu p^{-.(\mu+\nu+\alpha)/2}}{2^{\mu+\nu+1}\Gamma\left(\nu+1\right)} \sum_{k=0}^{\infty} \frac{\Gamma\left(k + \left(\alpha+\mu+\nu\right)/2\right)}{\Gamma\left(\mu+k+1\right)} \left(\frac{b^2}{4p}\right)^k \times \tag{54}$$

$$\times {}_2F_1\left(-k, -\left(\mu+k\right), \nu+1, \tfrac{c^2}{b^2}\right)$$

Substitute integral (54) into (44) with following meanings of the parameters:

$b = \sqrt{\lambda_1},\ c = \sqrt{\lambda_2},\ \mu = p_1 + k,\ \nu = p_2 - 1,\ \alpha = p_1 + p_2 + k + 1, p = 1$.Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) =$$

$$= \frac{2^{-(p_1+p_2)}}{\Gamma\left(p_2\right)} \exp\left(-\tfrac{\lambda_1+\lambda_2}{2}\right) \sum_{k=0}^{\infty} \sum_{k_1=0}^{\infty} 2^{-(k+k_1)} \frac{\Gamma\left(k+k_1+p_1+p_2\right)}{k_1!\,\Gamma\left(k+k_1+p_1+1\right)} \times$$

$$\times \left(\tfrac{\lambda_2}{2}\right)^{k_1} {}_2F_1\left(-k_1, -p_1, -k-k_1, p_2, \tfrac{\lambda_2}{\lambda_1}\right) \tag{55}$$

Now we use the polynomial form of the hypergeometrical function and take into account absolute convergence of the series. Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) =$$

$$= 2^{-(p_1+p_2)} \exp\left(-\tfrac{\lambda_1+\lambda_2}{2}\right) \sum_{k_1=0}^{\infty} \sum_{n=0}^{k_1} 2^{-k_1} \tfrac{\Gamma(\ k_1+p_1+p_2)}{\Gamma(p_2+n)\,\Gamma(k_1+p_1-n+1)} \times \tag{56}$$

$$\times \tfrac{(\lambda_1/2)^{k_1-n}(\lambda_2/2)^n}{n!\,(k_1-n)!}\ {}_2F_1\left(1, p_1+p_2+k_1, p_1+k_1-n+1, \tfrac{1}{2}\right)$$

Substitute the known representation of Hypergeometrical Gauss function [7],

$${}_2F_1\left(1, b, c, z\right) = z^{1-c}\left(1-z\right)^{1-c}\left(1-z\right)^{c-1}\left(c-1\right) B_z\left(c-1, b-c+1\right), \tag{57}$$

into (56), where $B_z\left(x, y\right)$ be uncomplete beta-function Then, after simple transformations, (56) takes the following form:

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) = \tag{58}$$

$$= \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) \sum_{k_1=0}^{\infty} \sum_{n=0}^{k_1} \frac{(\lambda_1/2)^{k_1-n}\,(\lambda_2/2)^n}{n!\,(k_1-n)!} S\left(p_1+k+n,\, p_2+n\right),$$

where function$S(x, y)$ is determined by (52).

By changing $m = k_1 - n$, (58) is reduced to (51). Note, that the recursive formulas and expansion to continued fraction exist for $S(x, y)$ [7].

4 Additional results

Integrals

In the course of demonstrating of the results above we obtained closed form for the following integral:

$$\int\limits_0^\infty x^{k+l+1} \exp\left(-px^2\right) I_k\left(ax\right) I_l\left(bx\right) dx \;=\; \left(2p\right)^{-(p+l+1)} a^k b^l \exp\left(\tfrac{a^2+b^2}{4p}\right) \times$$

$$\times \sum_{i=-l}^{k} \left(\begin{array}{c} k+l \\ i+l \end{array} \right) \left(\tfrac{b}{a}\right)^i I_i\left(\tfrac{ab}{2p}\right) \tag{59}$$

where k, l are integer, such that $k \geq 0, |l| \leq k$.

The deduction of (59) is brought in Appendix 3 as having no direct relation to the treated problem. With usage of (59) the following integral was evaluated:

$$\int\limits_0^\infty x^M \exp\left(-\tfrac{px^2}{2}\right) I_{M-1}\left(cx\right) Q_M\left(b, ax\right)\, dx \;= \tfrac{1}{c}\left(\tfrac{c}{p^2}\right)^M \exp\left(\tfrac{c^2}{2p^2}\right) \times$$

$$\times \left\{ Q_M\left(U,V\right) \,-\, \exp\left(\, -\tfrac{U^2+V^2}{2}\right) \sum_{k=-(M-1)}^{M-1} \tfrac{B_Z(M-k,M+k)}{B(M-k,M+k)} \left(\tfrac{V}{U}\right)^k I_k\left(UV\right) \right\} \tag{60}$$

where $Q_M\left(U,V\right)$ is so-called Q_M- function [2],

$$U = \frac{bp}{\sqrt{p^2+a^2}},\; V = \frac{ac}{p\sqrt{p^2+a^2}}, Z = \frac{a^2}{p^2+a^2}.$$

Derivation of (60) is brought in Appendix 4. Formula (59) can complete the table of integrals with Q_M-function [1].

Transformation of a known formula

In order to illustrate the usage of our finding, transform the formula for probability of errors when autocorrelation demodulation of binary DPSK signals is performed under the carrier detuning ([2], formula (8.18)):

$$P_b \;=\; \frac{1}{2^{\cdot q}} \exp\left(-2h_2^2 - \, h_1^2 \right) \sum_{n=0}^{\infty} \frac{\left(2h_2^2\right)^n}{n!} \sum_{k=0}^{q+n-1} \frac{1}{2^k} \sum_{m=0}^{k} \frac{\left(h_1^2\right)^m}{m\,!} \left(\begin{array}{c} k+q-1 \\ k-m \end{array} \right) \tag{61}$$

(61) is obtained as the For the details of the evaluation of that formula the authors of [2] refer to [8], where result of transformation for the following starting formula:

$$P_b = \frac{1}{2^{2q}} \exp\left[-2\left(h_1^2 + h_2^2\right)\right] h_1^{1-q} h_2^{1-q} \int\limits_0^\infty x_2^{(q-1)/2} e^{-x_2/2} I_{q-1}\left(2h_2\sqrt{x_2}\right) dx_2 \times$$

$$\times \int\limits_0^{x_2} x_1^{(q-1)/2} e^{-x_1/2} I_{q-1}\left(2h_1\sqrt{x_1}\right) dx_1 \tag{62}$$

Denote $\lambda_1 = (4h_1)^2$, $\lambda_2 = (2h_2)^2$. Then it may be rewritten as follows:

$$\begin{aligned} P_b = & \int\limits_0^\infty \frac{1}{2}\left(\frac{x_2}{\lambda_2}\right)^{(q-1)/2} \exp\left(\frac{x_2+\lambda_2}{2}\right) I_{q-1}\left(\sqrt{\lambda_2 x_2}\right) dx_2 \times \\ & \times \int\limits_0^{x_2} \frac{1}{2}\left(\frac{x_1}{\lambda_2}\right)^{(q-1)/2} \exp\left(\frac{x_1+\lambda_1}{2}\right) I_{q-1}\left(\sqrt{\lambda_1 x_1}\right) dx_1. \end{aligned} \tag{63}$$

.

The latter is probability of the inequality $x_1 < x_2$, where x_1 and x_2 are "Noncentral χ^2" random variables with parameters of noncentrality $\lambda_1$and λ_2 and the numbers of degrees of freedom are equal to $2q$. Hence the conditions of the ***Statement 1*** are valid and instead of the infinite series (60) we can use formula (23).

5 Conclusion

In the paper we obtained the closed form expression for probability $\Pr(\xi_1 \leq \xi_2\ )$ with ξ_1 and ξ_2 are noncentral chi-square random variables with even number degrees of freedom, which is of considerable important under evaluation reliability of networks. For the case add number of degrees the chi-square density is expressed by elementary functions and corresponding computations do not represent any difficulties.

Appendix 1

Statement 5

Let k and q are natural, such that $k \leq q-1$. Then it is valid

$$C_{k,q} = 2^{-2k} \sum_{m=0}^{q-k-1} 2^{-2m} \frac{k}{m+k} \binom{2m+2k}{m} = \qquad (64)$$

$$= 2^{-(q+k-1)} \sum_{m=0}^{q-k-1} 2^{-m} \binom{m+q+k-1}{m}.$$

Proof

First we shall demonstrate that (5) is reduced to the expression, coinciding with (20), within coefficients.

Introduce the notations:

$$z = \frac{\lambda_1}{2},\ x = \frac{\lambda_2}{4},\ M = \frac{N-2}{2} = q-1 \qquad (65)$$

Represent the internal sum in (6) in the following form:

$$\sum_{k=0}^{m+M} 2^{-(k+M+1)} L_k^M(-x) = \qquad (66)$$

$$= \sum_{k=0}^{M-1} 2^{-(k+M+1)} L_k^M(-x) \qquad + 2^{-(2M+1)} \sum_{k=0}^{m} L_{k+M}^M(-x).$$

Substitute (66) into (6). Then

$$\Pr(\xi_1 \lhd \xi_2) = 1 - 2^{-(M+1)} \exp(-x) \times \qquad (67)$$

$$\times \sum_{k=0}^{M-1} 2^{-k} L_k^M(-x) - 2^{-(2M+1)} \exp[-(x+z)]\, G(z,x),$$

where

$$G(z,x) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k=0}^{m} 2^{-k} L_{k+M}^M(-x). \qquad (68)$$

It is obvious, that function $G(x, z)$ satisfies the following integral equation:

$$G(z,x) = H(z,x) + \int_0^z G(t,x)\,dt, \tag{69}$$

where

$$H(z,x) = \sum_{m=0}^{\infty} \frac{(z/2)^m}{m!} L_{m+M}^{M}(-x). \tag{70}$$

By solving equation (68) we obtain the following representation

$$G(z,x) = H(z,x) + \exp(z) \int_0^z \exp(-t)\, H(t,x)\, dt. \tag{71}$$

It follows from (70), that

$$H(z,x) = \frac{d^{2M}}{dy^{2M}} \left\{ y^M \sum_{m=0}^{\infty} \frac{y^m}{(m+M)\,!} L_m^M(-x) \right\}_{y=z/2}. \tag{72}$$

The internal series in (72) is the generating function for Lagguer's polynoms [7]:

$$\sum_{m=0}^{\infty} \frac{y^m}{(m+M)!} L_m^M(-x) = (xy)^{-M/2} \exp(y)\; I_m(2\sqrt{xy}). \tag{73}$$

By substitution of (73) into (72) we obtain

$$H(z,x) = \frac{d^{2M}}{dy^{2M}} \left\{ \left(\frac{y}{x}\right)^{M/2} \exp(-y)\, I_m(2\sqrt{xy}) \right\}_{y=z/2}. \tag{74}$$

After substitution (74) into (71) and $2M$-fold integration by part, expression (71) takes the following form:

$$G(z,x) = 2^M \left\{ \sum_{i=0}^{2M} 2^{-i} \frac{d^i}{dy^i} \left[\left(\frac{y}{x}\right)^{M/2} \exp(y)\, I_M\left(2\sqrt{xy}\right) \right]_{y=z/2} - \right.$$

$$- \exp(z) \sum_{i=0}^{2M-1} 2^{-i} \frac{d^i}{dy^i} \left[\left(\frac{y}{x}\right)^{M/2} \exp\ (y)\, I_M\left(2\sqrt{xy}\right) \right]_{y=0} + \tag{75}$$

$$\left. + 2\ \exp\ (z) \int_0^{z/2} \left(\frac{t}{2}\right)^{M/2} \exp\ (\ -z)\ I_M\left(2\sqrt{xt}\right)\ dt \right\}.$$

Transform (75) by $M-$fold integration by part with usage formulas (9),(32) and (34) and taking into account the evenness of modified Bessel function respect to integer index. Then substitute the obtained expression into (67). Then

$$\Pr(\xi_1 < \xi_2) = 1 - \exp(\ -x) \int_0^{z/2} \exp(\ -t)\ I_0\left(2\sqrt{xt}\right) dt +$$

$$+\frac{1}{2} \exp(\ -x - z/2) \sum_{k=1}^{M} \left[\left(\frac{z}{2x}\right)^{k/2} (2 - C_{-k,M+1}) - \left(\frac{z}{2x}\right)^{-z/2} C_{k,M+1} \right] I_k\left(2\sqrt{\frac{xz}{2}}\right) -$$

$$+\frac{1}{2} \exp(\ -x - z/2)\, I_0\left(2\sqrt{\frac{xz}{2}}\right), \tag{76}$$

where the coefficients $C_{k,M+1}$ are given by expression (64).

It follows from determination of Q-function [1,2] that

$$1\ -\ \exp(\ -x) \int_0^{z/2} \exp(\ -t)\, I_0\left(z\sqrt{xt}\right)\ dt\ =\ Q\left(\sqrt{2z},\ \sqrt{z}\right). \tag{77}$$

Also it is shown in Appendix 2, that

$$C_{-k,M+1} + C_{k,\,M+1} = 2 \tag{78}$$

Taking into account notation (65), we obtain from (76) - (78):

$$\Pr\ (\xi_1 \lhd \xi_2)\ =\ Q\left(\sqrt{\frac{\lambda_2}{2}},\ \sqrt{\frac{\lambda_1}{2}}\right)\ +\ \frac{1}{2}\ \exp\left(\ -\frac{\lambda_1+\lambda_2}{4}\right)\ \times$$

$$\times \left[\ -I_0\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\ +\ \sum_{k=0}^{q-1} C_{k,q} \frac{\lambda_1^k - \lambda_2^k}{(\lambda_1\lambda_2)^{k/2}}\, I_0\left(\frac{1}{2}\sqrt{\lambda_1\,\lambda_2}\right)\right], \tag{79}$$

where

$$C_{k,q} = 2^{-(q+k-1)} \sum_{m=0}^{q-k-1} 2^{-m} \left(\begin{array}{c} m+q+k-1 \\ \\ m \end{array} \right). \qquad (80)$$

As (79) identically equals to (23) for each permissible values of parameters λ_1 andλ_2, the validity of (64) is proved.

Appendix 2

Statement 6

Let q and k are integer or half-integer coincidentally such, that $q \geq 1$, ?k? $\leq q$- 1. Then

$$\begin{aligned} C_{k,q} &= 2\frac{B_{1/2}\,(q+k, q-k)}{B\,(q+k, q-k)} = \qquad (81) \\ &= 2^{-(q+k-1)} \sum_{m=0}^{q+k-1} 2^{-m} \left(\begin{array}{c} m+q+k-1 \\ m \end{array} \right) \end{aligned}$$

Proof

Denote

$$a = q-k, \quad b = q+k, \quad 1-z = 1/2. \qquad (82)$$

According to the known property of incomplete beta-function [7], rewrite the left-side of (81) in the following form

$$C_{k,q} = 2\left(1 - \frac{B_z\,(a,b)}{B\,(a,b)}\right). \qquad (83)$$

Use now the expression (57). Then

$$B_z\,(a,b) = \frac{1}{a} z^a\,(1-z)^b \;\, {}_2F_1\,(1,\, a+b,\, a+1,\, z)\,. \qquad (84)$$

Since [7]

$$\begin{aligned} {}_2F_1\,(1,\, l,\, n,\, z) &= (n-1)!\,(-z)^{1-n}\,(-1)^{1-\hat{\imath}}\,\frac{1}{(1-l)_n} \times \qquad (85) \\ &\times \left[(1-z)^{n-l-1} - \sum_{m=0}^{n-2} \frac{(l-n+1)_m}{m!}\, z^m \right], \end{aligned}$$

where $l = a + b$, $n = a + 1$ are natural ($l \geq n$), after some transformations, (82) takes the following form:

$$B_z(a,b) = B(a,b)\left[1 - (1-z)^b \sum_{m=0}^{a-1} \binom{b+m-1}{m} z^m\right]. \qquad (86)$$

Now, if to take into consideration notation (82), then (81) follows from (83) and (86). Also the following property of coefficients $C_{k,q}$ can be readily extracted from (81)- (83):

$$C_{-k,q} + C_{k,q} = 2. \qquad (87)$$

Statement 7

Coefficients $C_{k,q}$ have the following equivavelent to (81) representation:

$$C_{k,q} = 2^{-2(q-1)} \sum_{m=0}^{q-k-1} \binom{2q-1}{m}. \qquad (88)$$

Proof

Consider the following function:

$$C_{k,q} = f(N_1, N_2) = 2^{-N_2} \sum_{m=0}^{N_1} 2^{-m} \binom{m+N_2}{m}, \qquad (89)$$

with

$$N_1 = q - k - 1, N_2 = q + k - 1. \qquad (90)$$

Use the following known connection:

$$\binom{m+N_2}{m} = \binom{m+N_2+1}{m} - \binom{m+N_2}{m-1}. \qquad (91)$$

After some transformations we obtain

$$f(N_1, N_2) - f(N_1+1, N_2+1) = 2^{-(N_1+N_2)} \binom{N_1+N_2+1}{N_2}. \qquad (92)$$

It is evident, (76) can be represented in following form:

$$f\,(N_1,\,N_2)\;=\;\sum_{i=0}^{N_1} f\,(N_1-\,i,\,N_2+\,i)\;-\;\sum_{i=0}^{N-1_1} f\,(N_1-i\,-\,1,\,N_2+i+\,1). \tag{93}$$

According to (89),

$$f\,(\,-1,\,N_1+\,N_2)\;=\;0, \tag{94}$$

and therefore (93) takes the form

$$f\,(N_1,\,N_2)\;=\;\sum_{i=0}^{N_1}(f\,(N_1-\,i,\,N_2+\,i)-\,f\,(N_1-i\,-\,1,\,N_2+i\,+\,1)). \tag{95}$$

After substitution (87) into (89) and changing the summation index, we obtain

$$f\,(N_1,N_2)\;=\;2^{-(N_1+N_2)}\sum_{m=0}^{N_1}2^{-m}\left(\begin{array}{c}N_1+N_2+\,1\\ m\end{array}\right). \tag{96}$$

The equivalence (81) and (88) follows from (96).

Appendix 3

Evaluation of the integral

$$B\;=\;\int_0^\infty x^{k+l+1}\exp\left(\,-px^2\right)I_k\,(ax)\,I_l\,(bx)\,dx, \tag{97}$$

where $k,\ l$ are integers ($k\,\geq\,0, |l|\,\leq\,k$) and $a,\ b,\ p$ a are real positive numbers.

Introduce the following parameters:

$$g\;=\;a^2/4,\;\;h\;=\;b^2/4 \tag{98}$$

and change the integration variable $y\;=\;x^2$,

$$B\;=\;\frac{1}{2}\,g^{k/2}h^{l/2}\int_0^\infty\exp\,(\,-py)\,\varphi\,(y,k,g)\,\varphi\,(y,l,h)\,dy, \tag{99}$$

where

$$\varphi(z,n,c) \;=\; (z/c)^{n/2}\, I_n\left(2\sqrt{cz}\right). \qquad (100)$$

Fulfill $(k+l)$ -fold integration by parts. Then

$$B \;=\; A + \frac{1}{2}\, g^{k/2} h^{l/2} p^{-(k+l)} \int\limits_0^\infty \exp\left(\,-py\right) \frac{d^{k+l}}{dy^{k+l}}\left[\varphi(y,k,g)\;\varphi(y,\,l,\,h)\right]\,dy, \qquad (101)$$

with

$$A \;=\; -\frac{1}{2}\, g^{k/2} h^{l/2} \sum_{m=1}^{k+l} p^{-m} \left\{ \exp\left(\,-py\right) \frac{d^{k+l}}{dy^{k+l}}\left[\varphi(y,k,g)\;\varphi(y,\,l,\,h)\right] \right\}_{y=0}^{\infty}. \qquad (102)$$

Using (9), it is simple to show, that

$$\frac{d^m}{dz^m}\varphi(z,n,c) \;=\; \varphi(z,n-m,c)\,. \qquad (103)$$

With usage of (34) and the known asymptotic representation [7]

$$\begin{array}{l} I_n(z) \;\to\; \frac{1}{\sqrt{2\pi z}}\exp\;(z) \\ z \to \infty \end{array}\;, \qquad (104)$$

we obtain from (101)-(103), that $A = 0$ and

$$\begin{aligned} B \;&=\; \frac{1}{2}\, g^{k/2} h^{l/2} p^{-(k+l)} \sum_{m=0} \binom{k+l}{m} \times \qquad (105) \\ &\times \int\limits_0^\infty \exp\left(-py\right) \varphi(y,k-m,g)\,\varphi(y,m-k,h)\;dy. \end{aligned}$$

Now, after substitution (100) into (105) and inverse changing $x \;=\; \sqrt{y}$

$$\begin{aligned} B \;&=\; p^{-(k+l)} \sum_{m=0} \binom{k+l}{m} g^{m/2} h^{(l+k-m)/2} \times \qquad (106) \\ &\times \int\limits_0^\infty x \exp\left(\,-px^2\right)\; I_{k-m}\left(2x\sqrt{g}\right)\; I_{m-k}\left(2x\sqrt{h}\right)\;dx. \end{aligned}$$

Since for integer $n, I_n(z) = I_{-n}(z)$, we can use the known formula for the integral in (106) [7]:

$$\int\limits_0^\infty x \exp\left(-px^2\right) I_{k-m}\left(2x\sqrt{g}\right) I_{m-k}\left(2x\sqrt{h}\right) dx = \tag{107}$$
$$= \frac{1}{2p} \exp\left(\frac{g+h}{p}\right) I_{m-k}\left(\frac{2}{p}\sqrt{gh}\right).$$

It follows from (106) and (107) with accounting denote (92), that

$$B = (2p)^{-(k+l+1)} \exp\left(\frac{a^2+b^2}{4p}\right) \times \tag{108}$$
$$\times \sum_{m=0}^{k+l} \binom{k+l}{m} a^m b^{l+k-m} I_{m-k}\left(\frac{ab}{2p}\right).$$

Changing the summation index $i = k - m$ and using (107), we obtain

$$B = (2p)^{-(k+l+1)} \exp\left(\frac{a^2+b^2}{4p}\right) a^k b^l \times \tag{109}$$
$$\times \sum_{i=-l}^{k} \binom{k+l}{i+l} \left(\frac{b}{a}\right)^i I_i\left(\frac{ab}{2p}\right).$$

The correctness of (59) follows from (97) and (109).

Appendix 4

Evaluation of the integral

$$A = \int\limits_0^\infty x^M \exp\left(-\frac{p^2x^2}{2}\right) I_{M-1}(cx)\, Q_M(b, ax)\, dx \tag{110}$$

where M is arbitrary natural, a, b, c, p, are real positive numbers,

$$Q_M(x, y) = \int\limits_y^\infty t\left(\frac{t}{x}\right)^{M-1} \exp\left(-\frac{t^2+x^2}{2}\right) I_{M-1}(xt)\, dt \tag{111}$$

is so-called Q_M -function (Q_1 is Marcum's Q-function).

For Q_M -function the representation exists, following from (111):

$$Q_M(x,y) = Q(x,y) + \exp\left(\frac{x^2+y^2}{2}\right)\sum_{k=1}^{M-1}\left(\frac{y}{x}\right)^k I_k(xy)\,. \qquad (112)$$

Also the following property of Q_M -function is known, which simply deduced from (112) and (50):

$$Q_M(x,y) + Q_M(y,x) = 1 + \exp\left(\frac{x^2+y^2}{2}\right)\sum_{k=-(M-1)}^{M-1}\left(\frac{x}{y}\right)^k I_k(xy)\,. \qquad (113)$$

Convey $Q_M\ (b,ax)$ with use $Q_M\ (ax,b)$ on the base of (113) and substitute into (110). Then for integral A we can write:

$$A = B - D + E \qquad (114)$$

where [5]

$$B = \int_0^\infty x^M \exp\left(-\frac{p^2x^2}{2}\right) I_{M-1}(cx)\,dx = \frac{1}{c}\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c^2}{2p^2}\right), \qquad (115)$$

and

$$\begin{aligned} D &= \int_0^\infty x^M \exp\left(-\tfrac{p^2x^2}{2}\right) I_{M-1}(cx)\, Q_M(ax,b)\,dx = \\ &= \tfrac{1}{c}\left(\tfrac{c}{p^2}\right)^M \exp\left(\tfrac{c^2}{2p^2}\right) Q_M\left(\tfrac{ac}{p\sqrt{p^2+a^2}}, \tfrac{bp}{\sqrt{p^2+a^2}}\right) \end{aligned}, \qquad (116)$$

$$\begin{aligned} E &= \exp\left(-\frac{b^2}{2}\right)\sum_{k=-(M-1)}^{M-1}\left(\frac{a}{b}\right)^k \int_0^\infty x^{M+k}\times \qquad (117) \\ &\times \exp\left(-\frac{x^2\left(p^2+a^2\right)}{2}\right) I_{M-1}(cx)\, I_k(abx)\,dx, \end{aligned}$$

The integral from (117) was evaluated in Appendix 3.Thus, by substituting (59) into (117), changing the order of summation and integration and using (107), we obtain

$$E = \frac{1}{c}\left(\frac{c}{p^2+a^2}\right)^M \exp\left(\frac{c^2-p^2b^2}{2(p^2+a^2)}\right) \sum_{k=-(M-1)}^{M-1} \left(\frac{ac}{b(p^2+a^2)}\right)^k \times$$
$$I_k\left(\frac{abc}{p^2+a^2}\right) \sum_{i=0}^{M-k-1} \begin{pmatrix} M+k+i-1 \\ i \end{pmatrix} \left(\frac{a^2}{p^2+a^2}\right)^i . \tag{118}$$

With usage of (86), (118) can be rewritten as follows:

$$E = \frac{1}{c}\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c^2-p^2b^2}{2(p^2+a^2)}\right) \sum_{k=-(M-1)}^{M-1} \left(\frac{ac}{bp^2}\right)^k I_k\left(\frac{abc}{p^2+a^2}\right) \times$$
$$\times \left[1 - \frac{B_z(M-k, M+k)}{B(M-k, M+k)}\right], \tag{119}$$

where $z = \frac{a^2}{p^2+a^2}$.

It follows from (115) and (116) with accounting (113), that

$$B - D = \frac{1}{c}\left(\frac{c}{p^2}\right)^M \left\{\exp\left(\frac{c}{p^2}\right) Q_M\left(\frac{bp}{\sqrt{p^2+a^2}}, \frac{ac}{p\sqrt{p^2+a^2}}\right) -\right.$$
$$\left. - \exp\left(\frac{c^2-p^2b^2}{2(p^2+a^2)}\right) \sum_{k=-(M-1)}^{M-1} \left(\frac{ac}{bp^2}\right)^k I_k\left(\frac{abc}{p^2+a^2}\right)\right\}. \tag{120}$$

The substitution of (191) and (120) into (114) brings:

$$A = \frac{1}{c}\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c}{p^2}\right) \left\{Q_M(U, V) - \exp\left(-\frac{U^2+V^2}{2}\right) \times\right.$$
$$\left. \times \sum_{k=-(M-1)}^{M-1} \left[1 - \frac{B_z(M-k, M+k)}{B(M-k, M+k)}\right] \left(\frac{V}{U}\right)^k I_k(VU)\right\}, \tag{121}$$

where

$$U = \frac{bp}{\sqrt{p^2+a^2}}, V = \frac{ac}{p\sqrt{p^2+a^2}}, z = \frac{a^2}{p^2+a^2}. \tag{122}$$

The validity of (60) follows from (110), (121) and (122).

Note, that with using of the following property of incomplete beta-function [7], $B_z(x, y) = 1 - B_{(1-z)}(y, x)$, and also (107) and (112), integral (110) can be represented in the following form:

$$A = \frac{1}{c}\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c}{p^2}\right) \times \tag{123}$$
$$\times \left[-\beta_0 I_0(UV) + \sum_{k=1}^{M-1} \frac{\alpha_k V^{2k} - U^{2k}}{(UV)^k} I_k(UV) \right] \Bigg\}$$

where $\alpha_k = \dfrac{B_{(1-z)}(M-k, M+k)}{B(M-k, M+k)}$, $\beta_k = \dfrac{B_z(M-k, M+k)}{B(M-k, M+k)}$ and U, V, z are determined by expression (122).

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