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On complex-stepped Runge-Kutta methods for exact time integration of linear PDEs

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Abstract

We extend the approaches developed in the papers by Kaps and Rentrop [Numerische Mathematik **33**, (1979)] and by Schatzman [J. Sci. Comput. **17**, (2002)], and suggest a method for highly accurate time integration of linear first-order evolutionary PDEs. The main idea is to generalise the Runge-Kutta methods for complex timesteps, which reduces the problem of accurate integration to the search for an optimal temporal path on the complex plane. Theoretically the method admits *exact* integration with *finite* timesteps.

Keywords: Runge-Kutta methods, linear partial differential equations, time integration, conservative finite difference schemes, complex plane.

1 Introduction

Consider the one-dimensional Cauchy problem

$$\frac{\partial\varphi}{\partial t} = L\varphi, \quad (x,t) \in \mathbb{R} \times (0,+\infty), \tag{1}$$

$$\varphi|_{t=0} = g(x). \tag{2}$$

Here $\varphi = \varphi(x,t)$ is the function to be sought, $L = L(x, w_1, \ldots, w_L)$ is a linear operator, and $\{w_l\}_{l=1}^{L}$ is a fixed set of parameters. In order to integrate problem (1)-(2) numerically, in accordance with Euler's method we can write

$$\varphi^{n+1} = \varphi^n + \tau L\varphi^n. \tag{3}$$

It is known that Euler's method is of first order of accuracy. If the solution φ is not very smooth, then for improving the precision we can go one of the following two ways [1, 2]:

1. Using Euler's method, to fractionise the timestep τ (upon this, additionally we can employ the Richardson (or Bulirsch-Stoer-Deuflhard, as an advanced) extrapolation [2-4]. For example, in case of $\tau_2 = \frac{1}{2}\tau$ we obtain

$$\varphi^{n+1/2} = \varphi^n + \frac{\tau}{2}L\varphi^n, \tag{4}$$

$$\varphi^{n+1} = \varphi^{n+1/2} + \frac{\tau}{2}L\varphi^{n+1/2} = \varphi^n + \tau L\varphi^n + \frac{\tau^2}{4}L^2\varphi^n; \tag{5}$$

in case of $\tau_4 = \frac{1}{4}\tau$ it holds

$$\varphi^{n+1/4} = \varphi^n + \frac{\tau}{4}L\varphi^n, \tag{6}$$

$$\varphi^{n+2/4} = \varphi^{n+1/4} + \frac{\tau}{4}L\varphi^{n+1/4}, \tag{7}$$

$$\varphi^{n+3/4} = \varphi^{n+2/4} + \frac{\tau}{4}L\varphi^{n+2/4}, \tag{8}$$

$$\begin{aligned} \varphi^{n+1} &= \varphi^{n+3/4} + \frac{\tau}{4}L\varphi^{n+3/4} = \dots \\ &= \varphi^n + \tau L\varphi^n + \frac{6\tau^2}{16}L^2\varphi^n + \frac{\tau^3}{16}L^3\varphi^n + \frac{\tau^4}{256}L^4\varphi^n. \end{aligned} \tag{9}$$

2. To employ Runge-Kutta methods of higher orders [1, 2] (Euler's method is then treated as Runge-Kutta of order one). For instance, for the midpoint method (or Runge-Kutta of order two) we have

$$k_1 = \tau L\varphi^n, \tag{10}$$

$$k_2 = \tau L(\varphi^n + \frac{1}{2}k_1), \tag{11}$$

$$\varphi^{n+1} = \varphi^n + k_2 = \varphi^n + \tau L\varphi^n + \frac{\tau^2}{2}L^2\varphi^n; \tag{12}$$

for the classical fourth-order Runge-Kutta method it holds

$$k_1 = \tau L\varphi^n, \tag{13}$$

$$k_2 = \tau L(\varphi^n + \frac{1}{2}k_1), \tag{14}$$

$$k_3 = \tau L(\varphi^n + \frac{1}{2}k_2), \tag{15}$$

$$k_4 = \tau L(\varphi^n + k_3), \tag{16}$$

$$\begin{aligned} \varphi^{n+1} &= \varphi^n + \frac{k_1}{6} + \frac{k_2}{3} + \frac{k_3}{3} + \frac{k_4}{6} = \cdots \\ &= \varphi^n + \tau L\varphi^n + \frac{\tau^2}{2}L^2\varphi^n + \frac{\tau^3}{6}L^3\varphi^n + \frac{\tau^4}{24}L^4\varphi^n. \end{aligned} \tag{17}$$

Clearly that when infinitely fractionising the timestep, the numerical solution φ^{n+1} converges to the exact one. However, in practice we cannot compute φ^{n+1} with infinitely small timesteps; furthermore, beginning with some $j \in \mathbb{N}$ fractionising $\tau = j\tau_j$ will lead to the growth of round-off errors, which will disturb the solution. Therefore, in order to increase the integration accuracy one has to employ advanced solvers, such as, e.g., [5-7]. Nevertheless, even when using advanced methods, the error between the numerical and exact solutions is still distinct from zero, which leaves the possibility for improvement.

2 Runge-Kutta methods on the complex plane

It is noteworthy that the Runge-Kutta methods can also be treated as timestep fractionising methods, if we extend possible values of the timestep to complex numbers. Indeed, let us define $\tau_2^{\alpha_1} = \frac{1}{\alpha_1}\tau$, $\tau_2^{\alpha_2} = \frac{1}{\alpha_2}\tau$, and go two steps in time from the point t_n to the point t_{n+1}: The first step is from t_n to t_{n+1/α_1} with the timestep $\tau_2^{\alpha_1}$, the second one — from t_{n+1/α_1} to t_{n+1} with $\tau_2^{\alpha_2}$. It holds:

$$\varphi^{n+1/\alpha_1} = \varphi^n + \frac{\tau}{\alpha_1}L\varphi^n, \tag{18}$$

$$\begin{aligned} \varphi^{n+1} &= \varphi^{n+1/\alpha_1} + \frac{\tau}{\alpha_2}L\varphi^{n+1/\alpha_1} = \cdots \\ &= \varphi^n + \Big(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\Big)\tau L\varphi^n + \frac{1}{\alpha_2\alpha_1}\tau^2L^2\varphi^n. \end{aligned} \tag{19}$$

It can be seen that if $\alpha_1 = \alpha_2 = 2$ then expression (19) coincides with (5); however, if

$$\begin{cases} \frac{1}{\alpha_1} + \frac{1}{\alpha_2} = 1, \\ \frac{1}{\alpha_2\alpha_1} = \frac{1}{2}, \end{cases} \tag{20}$$

then (19) is equivalent to (12), from where $\alpha_1 = 1 \pm i$, $\alpha_2 = 1 \mp i$, and hence, $\tau_2^{\alpha_1} = \frac{1 \mp i}{2}\tau$, $\tau_2^{\alpha_2} = \frac{1 \pm i}{2}\tau$. In Fig. 1 we sketch the process of searching for the solution by folmulae (18)-(19); obviously, the intermediate solution φ^{n+1/α_1} has no physical meaning.

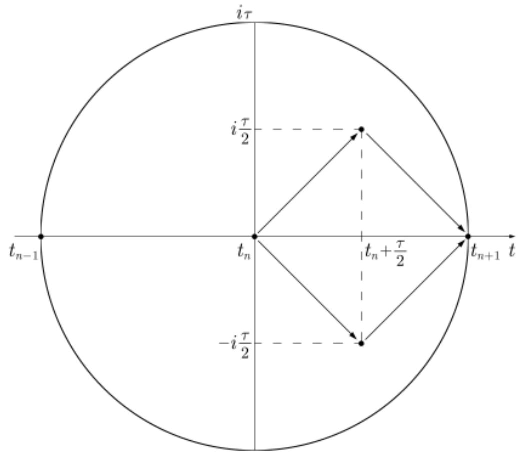

Figure 1: Integration of problem (1)-(2) by formulae (18)-(19) with complex timesteps $\tau_2^{\alpha_1}$ and $\tau_2^{\alpha_2}$ defined via (20).

Analogously, let now $\tau_4^{\alpha_r} = \frac{1}{\alpha_r}\tau$, $r = \overline{1,4}$. Then

$$\varphi^{n+1/\alpha_1} = \varphi^n + \frac{\tau}{\alpha_1} L\varphi^n, \tag{21}$$

$$\varphi^{n+1/\alpha_1+1/\alpha_2} = \varphi^{n+1/\alpha_1} + \frac{\tau}{\alpha_2} L\varphi^{n+1/\alpha_1}, \tag{22}$$

$$\varphi^{n+1/\alpha_1+1/\alpha_2+1/\alpha_3} = \varphi^{n+1/\alpha_1+1/\alpha_2} + \frac{\tau}{\alpha_3} L\varphi^{n+1/\alpha_1+1/\alpha_2}, \tag{23}$$

$$\begin{aligned}
\varphi^{n+1} &= \varphi^{n+1/\alpha_1+1/\alpha_2+1/\alpha_3} + \frac{\tau}{\alpha_4} L\varphi^{n+1/\alpha_1+1/\alpha_2+1/\alpha_3} = \cdots \\
&= \varphi^n + \Big(\frac{1}{\alpha_1} + \frac{1}{\alpha_2} + \frac{1}{\alpha_3} + \frac{1}{\alpha_4}\Big)\tau L\varphi^n \\
&+ \Big(\frac{1}{\alpha_2\alpha_1} + \frac{1}{\alpha_3\alpha_1} + \frac{1}{\alpha_3\alpha_2} + \frac{1}{\alpha_4\alpha_1} + \frac{1}{\alpha_4\alpha_2} + \frac{1}{\alpha_4\alpha_3}\Big)\tau^2 L^2\varphi^n \\
&+ \Big(\frac{1}{\alpha_3\alpha_2\alpha_1} + \frac{1}{\alpha_4\alpha_2\alpha_1} + \frac{1}{\alpha_4\alpha_3\alpha_1} + \frac{1}{\alpha_4\alpha_3\alpha_2}\Big)\tau^3 L^3\varphi^n \\
&+ \frac{1}{\alpha_4\alpha_3\alpha_2\alpha_1}\tau^4 L^4\varphi^n.
\end{aligned} \tag{24}$$

From the system

$$\left\{\begin{array}{l}\frac{1}{\alpha_1}+\frac{1}{\alpha_2}+\frac{1}{\alpha_3}+\frac{1}{\alpha_4}=1,\\ \frac{1}{\alpha_2\alpha_1}+\frac{1}{\alpha_3\alpha_1}+\frac{1}{\alpha_3\alpha_2}+\frac{1}{\alpha_4\alpha_1}+\frac{1}{\alpha_4\alpha_2}+\frac{1}{\alpha_4\alpha_3}=\frac{1}{2},\\ \frac{1}{\alpha_3\alpha_2\alpha_1}+\frac{1}{\alpha_4\alpha_2\alpha_1}+\frac{1}{\alpha_4\alpha_3\alpha_1}+\frac{1}{\alpha_4\alpha_3\alpha_2}=\frac{1}{6},\\ \frac{1}{\alpha_4\alpha_3\alpha_2\alpha_1}=\frac{1}{24}\end{array}\right. \tag{25}$$

we find $\{\alpha_r\}_{r=1}^4$. Here, similarly to (18), the intermediate solutions (21)-(23) do not have physical meaning.

Generally, for an arbitrary $m \in \mathbb{N}$ we can write

$$\varphi^{n+1} = \varphi^n + \sum_{r=1}^{m} A_r \tau^r L^r \varphi^n, \tag{26}$$

where

$$A_r = \sum_{p=r}^{m} \frac{1}{\alpha_p} \sum_{1\le q_1<\cdots<q_{r-1}<p} \frac{1}{\alpha_{q_{r-1}} \cdot \ldots \cdot \alpha_{q_1}}. \tag{27}$$

Denote by $\widetilde{\varphi}^{n+1}$ the exact solution to problem (1)-(2) at the moment t_{n+1}. Then for the error ε^{n+1} of formula (26) we have

$$\varepsilon^{n+1} \equiv \widetilde{\varphi}^{n+1} - \varphi^{n+1} = \widetilde{\varphi}^{n+1} - \varphi^n - \sum_{r=1}^{m} A_r \tau^r L^r \varphi^n. \tag{28}$$

Consequently, the increase of the integration accuracy of problem (1)-(2) is equivalent to the minimisation of the quantity ε^{n+1}, that is,

$$||\varepsilon^{n+1}|| \xrightarrow{\{\alpha_r\}_{r=1}^m} \min. \tag{29}$$

In other words, on the temporal complex plane it is required to find such an integration path determined by the parameters $\{\alpha_r\}_{r=1}^m$, for which the error ε^{n+1} is minimal. This idea is in some sense close to the stepsize adjustment algorithm originally invented by Fehlberg and further extended by Kaps and Rentrop and by Cash and Karp [2, 5, 6]; it can also be considered as a generalisation of the multistep-multivalue methods [1, 2] and preconditioned Runge-Kutta methods suggested by Schatzman [7] (see the quoted papers for details). However, the key difference of our method is that it admits *exact* time integration with finite timesteps

Proposition 1. *For any pair (m, n), where $m \in \mathbb{N}$ and $n \in \{0\} \cup \mathbb{N}$, there exist complex numbers $\{\alpha_r\}_{r=1}^m$, such that*

$$\varepsilon^{n+1} = 0. \tag{30}$$

Proof. To prove the proposition, it is sufficient to show that the equation

$$\sum_{r=1}^{m} A_r \Phi_r = \Theta, \tag{31}$$

where

$$\Phi_r = \tau^r L^r \varphi^n, \quad \Theta = \widetilde{\varphi}^{n+1} - \varphi^n, \tag{32}$$

has at least one solution. For this we observe that since all $\{\Phi_r\}_{r=1}^m$ and Θ are real (for a fixed $x \in \mathbb{R}$), there exist m numbers $c_r \in \mathbb{R}\backslash\{0\}$, such that

$$\sum_{r=1}^{m} c_r \Phi_r = \Theta. \tag{33}$$

Consequently, equation (31) can be rewritten in the form

$$A_r = c_r, \quad r = \overline{1,m}. \tag{34}$$

Denote $\beta_r = \alpha_r^{-1}$. Then (34) is equivalent to the system

$$B_r = c_r, \quad r = \overline{1,m}, \tag{35}$$

where

$$B_r = \sum_{p=r}^{m} \beta_p \sum_{1 \le q_1 < \cdots < q_{r-1} < p} \beta_{q_{r-1}} \cdot \ldots \cdot \beta_{q_1}. \tag{36}$$

It can easily be seen that (35)-(36) are relations between the coefficients $\{c_r\}_{r=1}^m$ and the elementary symmetrical functions [8] of the roots $\{\beta_r\}_{r=1}^m$ of the polynomial

$$\beta^m - c_1\beta^{m-1} + c_2\beta^{m-2} + \cdots + (-1)^{m-1}c_{m-1}\beta + (-1)^m c_m = 0. \tag{37}$$

These relations are also known as Vieta's formulae. In accordance with the fundamental theorem of algebra [8], equation (37) has exactly m roots $\{\beta_r\}_{r=1}^m$. Since $c_m \neq 0$, all $\beta_r \neq 0$, and therefore, $\exists \alpha_r = \beta_r^{-1}$, $r = \overline{1,m}$. ■

3 Choice of the parameters $\{\alpha_r\}_{r=1}^m$

The proposition 1 states that formula (26) admits the *exact* integration of problem (1)-(2). Nevertheless, the number Θ on the right-hand side of (31) is unknown, and so the proof of the statement is not constructive: The proposition 1 merely asserts that such an optimal path does exist, but it does not provide an idea how to find a set of $\{\alpha_r\}_{r=1}^m$ (for a fixed $m \in \mathbb{N}$) in order (30) hold. Hence, we can only try to solve the original problem (29).

For this, let us recall that many partial differential equations studied in scientific computing are based on some or other physical laws of conservation; a classical example is the well known advection equation which is based on the mass conservation law. Consequently, for finding a path $\{\alpha_r\}_{r=1}^m$ we can require that the difference (i.e., temporally discretised) equation satisfy the appropriate laws of conservation. Then, in this "physically adequate" sense, the quantity ε^{n+1} in (29) will be minimal, while the corresponding finite difference scheme will be *conservative* (see, e.g., [1]). The latter property is extremely important when numerically solving various practical problems of mathematical physics, since conservative finite difference schemes carefully take into account physical laws underlying the phenomena to be studied while discretising the original (continuous) equations, and thus the resulting numerical solutions possess the property of *physical fidelity* (or adequacy) [2, Chapter 19].

To give an example, one can easily verify that the finite difference scheme

$$\varphi_j^{n+1} = \varphi_j^n - \Big(\frac{1}{\alpha_1} + \frac{1}{\alpha_2}\Big)\tau w \frac{\varphi_{j+1}^n - \varphi_{j-1}^n}{2\Delta x} + \frac{1}{\alpha_2\alpha_1}\tau^2 w^2 \frac{\varphi_{j+1}^n - 2\varphi_j^n + \varphi_{j-1}^n}{\Delta x^2} \tag{38}$$

for the advection equation

$$\frac{\partial\varphi}{\partial t} = L\varphi \equiv -w\frac{\partial\varphi}{\partial x}, \quad w = const > 0 \tag{39}$$

conserves both mass ($\sum_j \varphi_j^{n+1}$) and energy ($\sum_j [\varphi_j^{n+1}]^2$) of the solution if $\alpha_1 = 1 \pm i$ and $\alpha_2 = 2 - \alpha_1$.

In conclusion we remark that all the calculations presented above can be modified for a d-dimensional case ($\mathbf{x} = (x_1, \ldots, x_d)^{\mathrm{T}}$), as well as for solving a boundary value problem instead of the Cauchy problem, with the presence of a non-zero forcing on the right-hand side of (1).

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Local geometric characterization of generalized quasiconformal mappings

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Abstract

One of characteristic properties of quasiconformal mappings is the quasiinvariance of n-module. The quasiinvariance is crucial in various applications. We provide here general conditions which are formulated in terms of the inequalities for certain set functions.

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Key words: quasiconformal mappings, mappings with distributional derivatives, inner and outer mean dilatations, p-moduli of separating sets and joining curves

1 Introduction

In this paper we continue to investigate the classes of mappings with finite mean dilatations. Our main goal is to generalize certain important classes of mappings: quasiconformal, quasiconformal in mean, etc.

The theory quasiconformal mappings in the plane has been appeared on the end of 1920-th in the works of Grötzsch and Lavrent'ev and is now a far developed area of geometric function theory with extremely reach applications.

A concept of quasiconformality in $\mathbb{R}^n$ was introduced by Lavrent'ev in 1938 as a suitable tool to construct some mathematical models of certain hydrodynamic problems. A point is that in multidimensional case the conditions of comformality are very rigid and therefore the class of conformal mappings in $\mathbb{R}^n, n > 2$, is narrow. Namely, by the fundamental Liouville theorem (1850) the only conformal mappings are the Möbius mappings, that is the finite compositions of reflections in spheres.

One of characteristic properties of quasiconformal mappings is a quasiinvariance of n-module of families of joining curves (or conformal capacity). It states that the module of a curve family is changed under a K-quasiconformal mapping only up to a factor at most K. All other properties of K-quasiconformal mappings can be derived from this inequality for K-quasiconformality (see, e.g., [1]). We shall consider more general inequalities than the quasiinvariance of n-moduli.

An essential deficiency of the moduli methods is that the exact values of moduli were found only for simple types of domains. In general case, one only can estimate these moduli, and in the most cases these estimates are not sharp. Moreover, these estimates do not exist for all values of parameter p of p-module.

Another approach to investigation of generalized quasiconformal mappings involves some local geometric characteristics which are based on appropriate change of the radii of the normal neighborhood systems. We discuss also the equivalence of the above methods. On other geometric methods, we refer to the survey paper of Srebro [2].

2 Quasiconformal dilatations

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a linear bijection. The numbers

$$H_I(A) = \frac{|detA|}{l^n(A)}, \quad H_O(A) = \frac{L^n(A)}{|detA|}, \quad H(A) = \frac{L(A)}{l(A)},$$

are called the *inner*, the *outer* and the *linear* dilatations of A, respectively. Here

$$l(A) = \min_{|h|=1} |Ah|, \quad L(A) = \max_{|h|=1} |Ah|,$$

and $detA$ is the determinant of A (see, e.g., [3]).

Obviously, all three dilatations are not less than 1. They have the following geometric interpretation. The image of the unit ball B^n under A is an ellipsoid $E(A)$. Let $B_I(A)$ and $B_O(A)$ be the inscribed and the circumscribed balls of $E(A)$, respectively.

Then

$$H_I(A) = \frac{mE(A)}{mB_I(A)}, \quad H_O(A) = \frac{mB_O(A)}{mE(A)},$$

and $H(A)$ is the ratio of the greatest and the smallest semi-axis of $E(A)$. Here $mA = m_n A$ denotes the n-dimensional Lebesgue measure of a set A.

Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be the semi-axes of $E(A)$. Then

$$L(A) = \lambda_1, \quad l(A) = \lambda_n, \quad |detA| = \lambda_1 \cdot \ldots \cdot \lambda_n,$$

and we can also write

$$H_I(A) = \frac{\lambda_1 \cdot \ldots \cdot \lambda_{n-1}}{\lambda_n^{n-1}}, \quad H_O(A) = \frac{\lambda_1^{n-1}}{\lambda_2 \cdot \ldots \cdot \lambda_n}, \quad H(A) = \frac{\lambda_1}{\lambda_n}.$$

If $n = 2$ then $H_I(A) = H_O(A) = H(A)$. In the general case, we have the relations:

$$\begin{aligned} H(A) &\leq \min(H_I(A), H_O(A)) \leq H^{n/2}(A) \\ &\leq \max(H_I(A), H_O(A)) \leq H^{n-1}(A). \end{aligned} \tag{1}$$

Let G and G^* be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and let a mapping $f : G \to G^*$ be differentiable at a point $x \in G$. This means there exists a linear mapping $f'(x) : \mathbb{R}^n \to \mathbb{R}^n$, called the (strong) derivative of the mapping f at x, such that

$$f(x+h) = f(x) + f'(x)h + \omega(x,h)|h|,$$

where $\omega(x,h) \to 0$ as $h \to 0$.

We denote

$$H_I(x,f) = H_I(f'(x)), \quad H_O(x,f) = H_O(f'(x)),$$

and

$$L(x,f) = L(f'(x)), \quad l(x,f) = l(f'(x)), \quad J(x,f) = det(f'(x)).$$

Proposition 1. *Let $f : G \to G^*$ be a K-quasiconformal homeomorphism, $1 \leq K < \infty$. Then*

(i) f is ACL (absolutely continuous on lines);

(ii) $f \in W^1_{n,loc}(G)$ (Sobolev class);

(iii) for almost every $x \in G$,

$$H_I(x,f) \leq K, \quad H_O(x,f) \leq K.$$

3 Global characteristics of quasiconformality

Now we define the quasiconformality of homeomorphisms in other terms (geometric or modular). Let $\mathcal{S}_k$ be a family of k-dimensional surfaces $\mathcal{S}$ in $\mathbb{R}^n$, $1 \le k \le n-1$ (curves for $k = 1$). $\mathcal{S}$ is a k-dimensional surface if $\mathcal{S} :\ D_s \to \mathbb{R}^n$ is a homeomorphic image of the closed domain $D_s \subset \mathbb{R}^k$.

The p-module of $\mathcal{S}_k$ is defined by

$$M_p(\mathcal{S}_k) = \inf \int\limits_{\mathbb{R}^n} \rho^p \, dx, \quad p \ge 1,$$

where the infimum is taken over all Borel measurable functions $\rho \ge 0$ and such that

$$\int S \rho^k \, d\sigma_k \ge 1$$

for every $\mathcal{S} \in \mathcal{S}_k$. We call each such ρ to be an *admissible function* for $\mathcal{S}_k$.

The following proposition characterizes K-quasiconformality in the terms of n-moduli of $\mathcal{S}_k$ (see, e.g., [4], cf. [5]).

Proposition 2. *A homeomorphism f of a domain $G \subset \overline{\mathbb{R}^n}$ is K-quasiconformal, $1 \le K < \infty$, if for any family $\mathcal{S}_k$, $1 \le k \le n-1$, of k-dimensional surfaces in G the double inequality*

$$K^{\frac{k-n}{n-1}} M_n(\mathcal{S}_k) \le M_n(f(\mathcal{S}_k)) \le K^{\frac{n-k}{n-1}} M_n(\mathcal{S}_k),$$

holds.

For more details about geometric definitions of quasiconformality see also [2].

A ring domain $D \subset \mathbb{R}^n$ is a finite domain whose complement consists of two components C_0 and C_1. Setting $F_0 = \partial C_0$ and $F_1 = \partial C_1$, we obtain two boundary components of D. For definiteness, let us assume that $\infty \in C_1$.

We say that a curve γ which *joins the boundary components* in D, if γ lies in D excluding its endpoints, one of which lies on F_0 and the second on F_1. A compact set Σ is said to *separate the boundary components* of D if $\Sigma \subset D$ and if C_0 and C_1 are located in different components of the complement $C\Sigma$ of Σ. Denote by Γ_D the family of all locally rectifiable curves γ, which join the boundary components of D, and by Σ_D the family of all compact piecewise smooth $(n-1)$-dimensional surfaces Σ, which separate the boundary components of D. For each quantity V associated with D such as subset of D or a family of sets contained in D, we let V^* denote its image under f.

Theorem 1. *A homeomorphism $f: G \to G^*$ is K-quasiconformal ($1 \le K < \infty$) if and only if there exists a constant K, $1 \le K < \infty$, such that for any ring domain $D \subset G$ either the inequalities*

$$M_p^n(\Sigma_D^*) \le K^{\frac{p}{n-1}} \big(mD^*\big)^{n-p} M_n^p(\Sigma_D),$$

$$M_q^n(\Sigma_D) \le K^{\frac{q}{n-1}} \big(mD\big)^{n-q} M_n^q(\Sigma_D^*),$$

hold for $n-1 < p < q \le n$ or inequalities

$$M_n^p(\Sigma_D^*) \le K^{\frac{p}{n-1}} \big(mD\big)^{p-n} M_p^n(\Sigma_D),$$

$$M_n^q(\Sigma_D) \le K^{\frac{q}{n-1}} \big(mD^*\big)^{q-n} M_q^n(\Sigma_D^*),$$

hold for $n \le p < q < (n-1)^2/(n-2)$.

The proof of this theorem is given in [6].

4 Gehring's characterization of quasiconformality

In 1973 Gehring proved the following important result

Gehring's Lemma. *Let G be an open subset of $\mathbb{R}^n$ and let $1 < p < \infty$. Suppose a nonnegative function h on G satisfies*

$$\fint_Q h^p \le A_p \Big(\fint_Q h\Big)^p$$

for all cubes $Q \subset G$ with a constant A_p independent of the cube. Then there exist a new exponent $s > p$ and a constant A_s depending only on p, n and A_p such that

$$\fint_Q h^s \le A_s \Big(\fint_Q h\Big)^s.$$

Here the symbol $\fint_Q h$ stands for the L^1-mean of h over the cube Q,

$$\fint_Q h = \frac{1}{mQ} \int_Q h.$$

In the same paper [7] Gehring derived the inequalities related to L^1 and L^n-means of the differential of K-quasiconformal mappings. He proved the following result (see, also [8]).

Integrability Theorem. *Every K-quasiconformal mapping $f: G \to \mathbb{R}^n$ belongs to the Sobolev space $W^1_{p,loc}$ with an exponent $p = p(n, K)$ greater than the dimension.*

This theorem not only extends an earlier result of Bojarski [9] for the planar case but also gives a new characteristic property for K-quasiconformal mappings.

An essential complement to Gehring's result is given by Reshetnyak [10].

5 Generalized quasiconformal mappings

The classes of mappings quasiconformal in the mean are studied more than 40 years (see, e.g., [11]). One of recent developments in this field is provided in [12].

Consider the quantities

$$H_{I,\alpha}(A) = \frac{|J(A)|}{l^\alpha(A)}, \quad H_{O,\alpha}(A) = \frac{L^\alpha(A)}{|J(A)|}, \quad \alpha \geq 1.$$

Such dilatations were applied in [6, 13-17]. For $\alpha = n$, the values of $H_{I,\alpha}(A)$ and $H_{O,\alpha}(A)$ coincide with $H_I(A)$ and $H_O(A)$, respectively.

We consider the homeomorphisms f which are differentiable almost everywhere in G and fix the real numbers α, β satisfying $1 \leq \alpha < \beta < \infty$. Put

$$HI_{\alpha,\beta}(f) = \int\limits_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x, f)\, dx, \quad HO_{\alpha,\beta}(f) = \int\limits_G H_{O,\beta}^{\frac{\alpha}{\beta-\alpha}}(x, f)\, dx,$$

where $H_{I,\alpha}(x, f) = H_{I,\alpha}(f'(x))$, $H_{O,\beta}(x, f) = H_{O,\beta}(f'(x))$. We call these values the *inner* and the *outer* mean dilatations of a mapping f of a given domain G.

The main purpose to introduce the inner and outer mean dilatations relies on the following theorem.

Theorem 2 ([18]). *Let $f: G \to G^*$ be a homeomorphism satisfying:*

(iv) f and f^{-1} are ACL;

(v) f and f^{-1} are differentiable a.e. in G and G^*, respectively;

(vi) the Jacobians $J(x, f)$ and $J(y, f^{-1})$ do not vanish a.e. in G and G^*, respectively.

Then for every fixed values $\alpha, \beta, \gamma, \delta$ such that $1 \leq \alpha < \beta < \infty$, $1 \leq \gamma < \delta < \infty$, and for any ring domain $D \subset G$ the inequalities

$$M_\alpha^\beta(\mathcal{S}_k^*) \leq HI_{\alpha,\beta}^{\beta-\alpha}(f) M_\beta^\alpha(\mathcal{S}_k),$$

$$M_\gamma^\delta(\mathcal{S}_k) \le HO_{\gamma,\delta}^{\delta-\gamma}(f) M_\delta^\gamma(\mathcal{S}_k^*),$$

hold, where $\mathcal{S}_k^* = f(\mathcal{S}_k)$.

Define for the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$, $1 \le \gamma < \delta < \infty$, the *class* $\mathcal{B}(G)$ of such homeomorphisms $f:\ G \to G^*$ which satisfy:
(vii) f and f^{-1} are ACL-homeomorphisms,
(viii) f and f^{-1} are differentiable, with the Jacobians $J(x,f) \ne 0$ and $J(y, f^{-1}) \ne 0$ a.e. in G and G^*, respectively,
(ix) the inner and the outer mean dilatations $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.

In other words, the class $\mathcal{B}(G)$ consists of *mappings with finite mean dilatations.* The case $\beta = \delta = n$ was considered in [19].

It follows that for any $f \in \mathcal{B}(G)$, we have the equalities

$$HI_{\alpha,\beta}(f^{-1}) = HO_{\alpha,\beta}(f), \quad HO_{\alpha,\beta}(f^{-1}) = HI_{\alpha,\beta}(f).$$

The relations (1) show that in the classical case of quasiconformal mappings, their dilatations are finite or infinity simultaneously. However, this is not true for the mean dilatations. The following example shows that the unboundedness of one of dilatations does not depend on the value of another mean dilatation.

Example 1. Consider the unit cube

$$G = \{x = (x_1, \dots, x_n) : 0 < x_k < 1, k = 1, \dots, n\},$$

and let

$$f(x) = \left(x_1, \dots, x_{n-1}, \frac{x_n^{1-c}}{1-c}\right), \quad 0 < c < 1.$$

The calculation gives

$$HI_{\alpha,\beta}(f) = \int\limits_G H_{I,\alpha}^{\frac{\beta}{\beta-\alpha}}(x,f)\, dx = \int\limits_0^1 x_n^{-\frac{c\beta}{\beta-\alpha}} dx_n,$$

$$HO_{\gamma,\delta}(f) = \int\limits_G H_{O,\delta}^{\frac{\gamma}{\delta-\gamma}}(x,f)\, dx = \int\limits_0^1 x_n^{-\frac{c(\delta-1)\gamma}{\delta-\gamma}} dx_n,$$

which yields

$$HI_{\alpha,\beta}(f) < \infty \iff 0 < c < 1 - \alpha/\beta,$$

$$HI_{\alpha,\beta}(f) = \infty \iff 1 - \alpha/\beta \le c < 1,$$

$$HO_{\gamma,\delta}(f) < \infty \iff 0 < c < 1 - (\gamma - 1)\delta/(\delta - 1)\gamma,$$

$$HO_{\gamma,\delta}(f) = \infty \iff 1 - (\gamma - 1)\delta/(\delta - 1)\gamma \le c < 1.$$

This allows us to choose the parameters $c, \alpha, \beta, \gamma, \delta$ so that one obtains the desired relations between $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$.

6 Relationship between classes with finite mean dilatations

The following theorems describe the relationship between the classes $\mathcal{B}(G)$.

Theorem 3. *Suppose that $\alpha, \beta, \gamma, \delta$ are fixed real numbers such that $n-1 \le \alpha < \beta < \infty$, $n-1 \le \gamma < \delta < \infty$. Then the mappings of $\mathcal{B}(G)$ belong to the Sobolev classes $W^1_{p,loc}(G)$ and $W^1_{q,loc}(G^*)$ with $p = \max\bigl(\gamma, \beta/(\beta - n + 1)\bigr)$ and $q = \max\bigl(\alpha, \delta/(\delta - n + 1)\bigr)$.*

The proof of this theorem is given in [6].

Theorem 4. *Let $\alpha, \beta, \gamma, \delta, r, s, t, u$ be fixed real numbers such that $1 \le r < \alpha < \beta < s < \infty$ and $1 \le t < \gamma < \delta < u < \infty$. Then*
(a) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, r, \beta, \gamma, \delta)$,
(b) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, s, \gamma, \delta)$,
(c) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, \beta, t, \delta)$,
(d) $\mathcal{B}(G, G^*, \alpha, \beta, \gamma, \delta) \subset \mathcal{B}(G, G^*, \alpha, \beta, \gamma, u)$.

Example 2. Let $n \le \alpha < \beta < \infty$ and $n \le \gamma < \delta < \infty$. Fix two numbers $0 < a < 1$ and $p > 0$ and consider two spherical systems of coordinates (r, φ_i) and (ρ, ψ_i) on the n-dimensional balls $B(0, a)$ and $B\bigl(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p} \frac{1}{a}\bigr)$ respectively. Here $B(x, h) = \{y \in \mathbb{R}^n : |y - x| < h\}$ denotes n-dimensional ball of radius h centered at x, $\Omega_n = mB(0, 1)$, $\omega_{n-1} = m_{n-1}B(0, 1)$

It is easy to verify that the mapping

$$g = \Big\{ \rho = r^{\frac{\gamma-n}{\gamma}} \ln^{-p}\frac{1}{r}, 0 < r < a < 1, \psi_i = \varphi_i, 0 \le \varphi_i < \pi, i = 1, \dots, n-2,$$
$$0 \le \varphi_{n-1} < 2\pi, \rho(0) = 0 \Big\}$$

moves the ball $B(0,a)$ to $B\big(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p}\frac{1}{a}\big)$. Therefore, g and g^{-1} are differentiable a.e. with nonzero Jacobians in $B(0,a)$ and $B\big(0, a^{\frac{\gamma-n}{\gamma}} \ln^{-p}\frac{1}{a}\big)$, respectively, which yields that the value

$$\begin{aligned} HO_{\gamma,\delta}(g) &= \int\limits_{B(0,a)} \left(\frac{L^\delta(x,g)}{|J(x,g)|}\right)^{\frac{\gamma}{\delta-\gamma}} dx \\ &= \omega_{n-1} \int\limits_0^a r^{\frac{n(n-\gamma)}{\delta-\gamma}-1} \ln^{-\frac{p\gamma(\delta-n)}{\delta-\gamma}} \frac{1}{r} \left(\frac{\gamma-n}{\gamma} + \ln^{-1}\frac{1}{r}\right)^{\frac{\gamma}{\delta-\gamma}} dr \end{aligned}$$

is finite only when

$$n = \gamma \quad \text{and} \quad 0 < p \le \frac{\delta}{n(\delta-n)},$$

and $HO_{\gamma,\delta}(g) = \infty$ for $n < \gamma$. Moreover,

$$\int\limits_{B(0,a)} L^\gamma(x,g)dx = \omega_{n-1} \int\limits_0^a r^{-1} \ln^{-p\gamma}\frac{1}{r} \left(\frac{\gamma-n}{\gamma} + \ln^{-1}\frac{1}{r}\right)^\gamma dr < \infty$$

under $p > 1/\gamma$, which shows that this mapping, being in the Sobolev space $W^1_{\gamma,loc}$ does not belong to our class $\mathcal{B}(G)$.

Hence, for the indicated restrictions to parameters $\alpha, \beta, \gamma, \delta$ the class $\mathcal{B}(G)$ is a proper subset of $W^1_{\gamma,loc}(G)$.

On the other hand,

$$\int\limits_{B(0,a)} L^{\gamma+\varepsilon}(x,g)dx = \omega_{n-1} \int\limits_0^a r^{-1-\frac{n\varepsilon}{\gamma}} \ln^{-p(\gamma+\varepsilon)}\frac{1}{r} \left(\frac{\gamma-n}{\gamma} + \ln^{-1}\frac{1}{r}\right)^{\gamma+\varepsilon} dr = \infty$$

for any positive ε and p. Thus the mapping g belongs to the class $\mathcal{B}(G)$ for

$$\frac{1}{n} \le p \le \frac{\delta}{n(\delta-n)}$$

and $W^1_{n,loc}(G)$, but g does not belong for $W^1_{n+\varepsilon,loc}(G)$. We conclude that the Integrability Theorem does not hold for the class of mappings with finite mean dilatations.

7 Global characterization of generalized quasiconformal mappings

We now establish the inequalities which generalize the quasiinvariance of module in the case of homeomorphisms with finite mean dilatations and derive some differential and geometric properties of such mappings.

To this end, we consider also certain set functions.

Let Φ be a finite nonnegative function in domain G defined for open subsets E of G so that

$$\sum_{k=1}^{m} \Phi(E_k) \leqslant \Phi(E)$$

for any finite collection $\{E_k\}_{k=1}^m$ of nonintersecting open sets $E_k \subset E$. We denote the class of such set functions Φ by $\mathcal{F}$.

Now we introduce some new classes of homeomorphisms depending on the values of parameters $\alpha, \beta, \gamma, \delta$ and on the set functions.

Fix the numbers $\alpha, \beta, \gamma, \delta$ which satisfy

$$n-1 \le \alpha < \beta < \infty, \quad n-1 \le \gamma < \delta < \infty,$$

and assume that there exists a nonempty family of homeomorphisms $f : G \to G^*$ such that there are two set functions $\Phi, \Psi \in \mathcal{F}$ not depending from f so that for each ring domain $D \subset G$ the inequalities

$$M_\alpha^\beta(\Sigma_D^*) \le \Phi^{\beta-\alpha}(D) M_\beta^\alpha(\Sigma_D), \tag{2}$$

$$M_\gamma^\delta(\Sigma_D) \le \Psi^{\delta-\gamma}(D) M_\delta^\gamma(\Sigma_D^*), \tag{3}$$

hold.

The class of such homeomorphisms will be denoted by $\mathcal{MS}(G)$. (In fact, it depends also on $\alpha, \beta, \gamma, \delta$.) Their main properties of this class are given by following

Theorem 5 ([18]). *Let* $n-1 < \alpha < \beta \le n$ *and* $n-1 < \gamma < \delta \le n$ *or*

$$n \le \alpha < \beta < \frac{(n-1)^2}{n-2} \quad \textit{and} \quad n \le \gamma < \delta < \frac{(n-1)^2}{n-2}.$$

Then every mapping $f \in \mathcal{MS}(G)$ admits the following properties:

(a') f is ACL in G;

(b') f^{-1} is ACL in G^*;

(c') $f \in W^1_{a,loc}(G)$ with $a = \max\Big(\gamma, \beta/(\beta - n + 1)\Big)$;

(d') $f^{-1} \in W^1_{b,loc}(G^*)$ with $b = \max\Big(\alpha, \delta/(\delta - n + 1)\Big)$.

Remarks.

1. It is enough only one inequality ((2) or (3)) to provide (a') and (b').
2. f and f^{-1} both possess N-property (see, e.g., [20]).
3. To characterize a quasiconformal mapping again enough only one inequality ((2) or (3)) with $\alpha = \beta = n$ or $\gamma = \delta = n$.
4. Another characterization of the class $\mathcal{MS}(G)$ can be given in the terms of the moduli of Γ_D instead of moduli of Σ_D.

8 Local characterization of generalized quasiconformal mappings

Let x be an arbitrary point in $\mathbb{R}^n$. Assume that some closed neighborhood $\mathcal{G}_t(x)$ of x is defined for any $t \in (0,1]$. We say that a set of the neighborhoods $\mathcal{G}_t(x)$ of the point x constitutes a *normal system*, if there exists a continuous function $v : R^n \to \mathbb{R}$ such that $v(x) = 0$, $v(y) > 0$ for any $y \neq x$. Here $\mathcal{G}_t(x) = \{y \in \mathbb{R}^n : v(y) \leq t\}$ for any $t \in (0,1]$. Let $\Gamma_t(x) = \{y \in \mathbb{R}^n : v(y) = t\}$ denote the boundary of $\mathcal{G}_t(x)$.

The function v is called the *generating function* for a given normal system $\{\mathcal{G}_t(x)\}$.

Denote

$$r(x,t) = \inf_{y \in \Gamma_t(x)} |y - x|, \quad \mathcal{R}(x,t) = \sup_{y \in \Gamma_t(x)} |y - x|.$$

These values $r(x,t)$ and $\mathcal{R}(x,t)$ are equal, respectively, to the minimal and the maximal radii of the neighborhood $\mathcal{G}_t(x)$. The limit

$$\Delta(x) = \limsup_{t \to 0} \frac{\mathcal{R}(x,t)}{r(x,t)}$$

is called the *regularity parameter* of the family $\{\mathcal{G}_t(x), 0 < t \leq 1\}$. Any such system $\{\mathcal{G}_t(x)\}$ is called the *regular normal system*, provided $\Delta(x) < \infty$.

Let $f: \ G \to G^*$ be a homeomorphism and let $\{\mathcal{G}_t(x)\}$ be a normal system of neighborhoods of $x \in G$. One can introduce similar to above the minimal and the maximal radii for the image of $\mathcal{G}_t(x)$ by

$$r^*(x,t) = \inf_{y \in \Gamma_t(x)} |f(y) - f(x)|, \quad \mathcal{R}^*(x,t) = \sup_{y \in \Gamma_t(x)} |f(y) - f(x)|$$

and

$$\Delta^*(x) = \limsup_{t \to 0} \frac{\mathcal{R}^*(x,t)}{r^*(x,t)}.$$

Yu.G. Reshetnyak [10] has investigated quasiconformal mappings of the space domains using the radii of the normal regular system of neighborhoods. He called a mapping f quasiconformal at a point $x \in G$ if there exists a normal regular system $\{\mathcal{G}_t(x)\}$ of neighborhoods of x such that $\Delta(x)\Delta^*(x) < \infty$.

The *upper* and *lower* derivatives of a set function $\Phi \in \mathcal{F}$ at a point $x \in G$ are defined by

$$\overline{\Phi'}(x) = \lim_{h \to 0} \sup_{d(Q)<h} \frac{\Phi(Q)}{mQ}, \qquad \underline{\Phi'}(x) = \lim_{h \to 0} \inf_{d(Q)<h} \frac{\Phi(Q)}{mQ},$$

where Q ranges over all open cubes and open balls such that $x \in Q \subset G$ and $d(Q) = diamQ$. Due to [21], these derivatives have the following properties:

(x) $\overline{\Phi'}(x)$ and $\underline{\Phi'}(x)$ are Borel's functions;

(xi) $\overline{\Phi'}(x) = \underline{\Phi'}(x) < \infty$ a.e. in G;

(xii) for each open set $V \subset G$,

$$\int_V \overline{\Phi'}(x)\, dx \le \Phi(V).$$

Using these set functions, we define for the fixed real numbers $\alpha, \beta, \gamma, \delta$ such that $1 \le \alpha < \beta < \infty$ and $1 \le \gamma < \delta < \infty$, the class $\mathcal{H}(G)$ of homeomorphisms $f : G \to G^*$ which satisfy:

(xiii) there exist $\Phi, \Psi \in \mathcal{F}$ in G,

(xiv) for any point $x \in G$ there exists $\{\mathcal{G}_t(x)\} \subset G$,

(xv) the inequalities

$$\limsup_{t \to 0} \frac{mf(B(x, \mathcal{R}(x,t)))\mathcal{R}^{\alpha-n}(x,t)}{\Omega_n r^{*\alpha}(x,t)} \le \left[\Phi'(x)\right]^{\frac{\beta-\alpha}{\beta}}, \tag{4}$$

$$\limsup_{t \to 0} \frac{\Omega_n \mathcal{R}^{*\delta}(x,t)}{mf(B(x, r(x,t)))r^{\delta-n}(x,t)} \le \left[\Psi'(x)\right]^{\frac{\delta-\gamma}{\gamma}}. \tag{5}$$

hold for all points $x \in G$ at which the derivatives $\Phi'(x)$ and $\Psi'(x)$ exist.

9 Equivalence of analytic and geometric descriptions

We now are enable to establish that classes $\mathcal{B}(G)$ and $\mathcal{H}(G)$ are coincide. The proof of this fact is accomplished in several steps and relies on an idea of the classical Menshoff paper [22]. (See also [10]).

Lemma 1. *Let $f \in \mathcal{H}(G)$, then f is ACL-mapping and is differentiable a.e. in G.*

Sketch of the proof. First, we show that f is ACL-mapping in G. Fix for each $x \in G$ a normal regular system $\{\mathcal{G}_t(x)\}$ of neighborhoods such that $\mathcal{G}_t(x) \subset G$ for any $t \in (0,1]$. Consider an arbitrary point $a = (a_1, \ldots, a_n) \in G$ and $h > 0$ so that the cube $\bar{Q} = \bar{Q}(a,h)$ belongs to G, where $Q(a,h) = \{x : |x_i - a_i| < h, i = 1, \ldots, n\}$. Denote by C_k, $k = 1, \ldots, n$, the intersection of $\bar{Q}$ with the plane $P_k(a) = \{x \in \mathbb{R}^n : x_k = a_k\}$, and let $p(x)$ be the segment $|x_k - a_k| \leq h/2$ of the line passing through x ($x \in C_k$) parallel to the kth coordinate axis x_k. Similarly, let $p(A)$ denote the union of all segments $p(x)$, when $A \subset C_k$ and $x \in A$.

We show that for almost all $x \in C_k$ (with respect to $(n-1)$-dimensional Lebesgue measure) the restriction of f to $p(x)$ admits the Lusin N-property.

Further, for $x, \tilde{x} \in G, \tilde{x} \neq x$, we define

$$k(x) = \limsup_{\tilde{x} \to x} \frac{|f(\tilde{x}) - f(x)|}{|\tilde{x} - x|}.$$

Then $k(x) < \infty$ for almost all $x \in G$, and for any open set $E \subset G$ we have

$$\int\limits_E k^{\frac{\beta}{\beta-n+1}}(x)\, dx \leq \Big[\Phi(E)\Big]^{\frac{\beta-\alpha}{\beta-n+1}} \Big[mf(E)\Big]^{\frac{\alpha-n+1}{\beta-n+1}} < \infty,$$

$$\int\limits_E k^{\gamma}(x)\, dx \leq \Big[\Phi(E)\Big]^{\frac{\delta-\gamma}{\delta}} \Big[mf(E)\Big]^{\frac{\gamma}{\delta}} < \infty.$$

These inequalities allow us to the classical Stepanov theorem [23] on differentiability and obtain that f is differentiable almost everywhere in G.

Lemma 2. *Let $f \in \mathcal{H}(G)$, then $HI_{\alpha,\beta}(f)$ and $HO_{\gamma,\delta}(f)$ are finite.*

Lemma 3. *If $f \in \mathcal{B}(G)$, then $f \in \mathcal{H}(G)$.*

The proofs of Lemmas 2 and 3 are similar to the corresponding lemmas from [16]. Now we can now formulate the following theorem which is the main result of this section.

Theorem 6. *The classes $\mathcal{B}(G)$ and $\mathcal{H}(G)$ coincide.*

We now mention about some special cases. The first of them is the standard quasiconformality in $\mathbb{R}^n$. One can obtain the class of usual quasiconformal mappings not only when $\alpha = \beta = \gamma = \delta = n$, but also if the following conditions hold:
(xvi) only one of pairs α, γ or β, δ is equal to n;
(xvii) the set functions of D are reduced to the product $CmesE$, where $mesE$ denote the Lebesgue n-measure on D or D^*.

We set

$$K_I(x, \{\mathcal{G}_t(x)\}) = \limsup_{t \to 0} \frac{mf(B(x, \mathcal{R}(x,t)))}{\Omega_n r^{*n}(x,t)},$$

and

$$K_O(x, \{\mathcal{G}_t(x)\}) = \limsup_{t \to 0} \frac{\Omega_n \mathcal{R}^{*n}(x,t)}{mf(B(x, r(x,t)))}.$$

Theorem 7. *A homeomorphism $f : G \to G^*$ is quasiconformal in the domain G if and only if for almost all $x \in G$ there exist the normal regular systems $\{\mathcal{G}_t(x)\} \subset G$ of neighborhoods of x which satisfy*

$$K_I(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty, \quad K_O(x, \{\mathcal{G}_t(x)\}) \leq K(\{\mathcal{G}_t\}) < \infty.$$

Put now

$$K = \inf K(\{\mathcal{G}_t\}),$$

where the infinum is taken over all such systems of neighborhoods. This value is the *quasiconformality coefficient* of f in G.

In the planar case, $n = 2$, we obtain instead of (4),(5) the following bounds

$$\limsup_{t \to 0} \frac{\mathcal{R}^*(x,t)}{r(x,t)} \left(\frac{\mathcal{R}(x,t)}{r^*(x,t)} \right)^{\alpha - 1} \leq \left[\Phi'(x) \right]^{\frac{\beta - \alpha}{\beta}},$$

$$\limsup_{t \to 0} \frac{\mathcal{R}(x,t)}{r^*(x,t)} \left(\frac{\mathcal{R}^*(x,t)}{r(x,t)} \right)^{\delta - 1} \leq \left[\Psi'(x) \right]^{\frac{\delta - \gamma}{\gamma}}.$$

In other words, we can characterize geometrically in terms of radii of normal neighborhood systems many well-known classes of mappings (quasiconformal, quasiconformal in the mean, enc).

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Probability of an inequality applied in statistical theory of communications

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Abstract

We evaluate the probability $\Pr(\xi_1 \leq \xi_2)$ with ξ_1 and ξ_2 are noncentral chi-square random variables and find the closed form expression for that probability.

Keywords: Noncentral chi-square distribution, probability of inequality, closed form representation

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1 Introduction

In several problems of statistical theory of communications the necessity arises to evaluate the probability $\Pr(\xi_1 \leq \xi_2)$, where ξ_1 and ξ_2 are independent random variables distributed according to "Noncentral Chi-square" law. That probability is of interest for analyzing noise immunity of communication systems [1,2], performance of synchronization systems [3] and also for other areas, so that the problem is of rather common interest for applications. This probability can be written as follows

$$\Pr(\xi_1 \leq \xi_2) = \int_0^\infty f_1(x_1)\,dx_1 \int_{x_1}^\infty f_2(x_2)\,dx_2. \tag{1}$$

with $f_i(x)$ $(i = 1,2)$ be " Noncentral Chi-square" density having the view [4]:

$$f_i(x) = f(x, \lambda_i, N_i/2) = \frac{1}{2}\left(\frac{x}{\lambda_i}\right)^{\frac{N_i-2}{4}} \exp\left(-\frac{x+\lambda_i}{2}\right) I_{\frac{N_i}{2}-1}\left(\sqrt{\lambda_i x}\right), \tag{2}$$

where $\lambda_i \geq 0$ is the noncentrality parameter, N_i is integer and referred to as the number degrees of freedom.

The closed form expression for the integral (1) is known only for some particular cases. So, if $N_1 = N_2 = 2$, then [1,2]

$$\Pr(\xi_1 \leq \xi_2) = Q\left(\sqrt{\frac{\lambda_2}{2}}, \sqrt{\frac{\lambda_1}{2}}\right) - \frac{1}{2}\exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) I_0\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right), \tag{3}$$

where

$$Q(x,y) = \int\limits_y^\infty t \exp\left(-\frac{t^2+x^2}{2}\right) I_0(xt)\, dx, \tag{4}$$

.

is Marcum's $Q-$function [5].

Also, the closed form is obtained for the case $N_1 = N_2 = 2q$ (q is integer) and $\lambda_2 = 0$ (see [1,2] with reference to [6]):

$$\Pr(\xi_1 \leq \xi_2) = 2^{-q} \exp\left(-\frac{\lambda_1}{4}\right) \sum_{k=0}^{q-1} 2^{-k} L_k^{q-1}\left(-\frac{\lambda_1}{4}\right), \tag{5}$$

where $L_k^{q-1}(\cdot)$ is Lagguer's polynom.

If, however, λ_1 and λ_2 are arbitrary positive values, there is a representation in the form of infinite series,

$$\Pr(\xi_1 \leq \xi_2) = \tag{6}$$

$$= 1 - 2^{-q} \exp\left(-\frac{\lambda_1+\lambda_2}{4}\right) \sum_{m=0}^{\infty} \frac{(\lambda_1/2)^m}{m!} \sum_{k=0}^{m+q-1} 2^{-k} L_k^{q-1}\left(-\frac{\lambda_2}{4}\right).$$

In some applications the necessity arises to find probability $\Pr(\xi_1 \leq \xi_2)$ in the case of $N_1 = 2q_1,\ N_2 = 2q_2$. Below closed expression will be obtained for the both cases ($q_1 = q_2$ and $q_1 \neq q_2$). Note, the " Noncentral Chi-square" distribution is the special case of the law with density

$$f(x, \lambda, p) = \frac{1}{2} \left(\frac{x}{\lambda} \right)^{\frac{p-1}{2}} \exp\left(-\frac{x+\lambda}{2} \right) I_{p-1}\left(\sqrt{\lambda x} \right), \tag{7}$$

where $\lambda > 0$, $p > 0$ are arbitrary real numbers [4].

Below we shall obtain the expression for $\Pr(\xi_1 \leq \xi_2)$ in the case when ξ_i ($i = 1,2$) follow the density (7).

2 Preliminary results

For derivation the formulas we need some properties of the density (7) and the function $\Pr(\xi_1 \leq \xi_2) = P(\lambda_1, p_1; \lambda_2, p_2)$, where λ_i, p_i $(i = 1, 2)$ are the corresponding parameters of the density (7).

Property 1

Let $p > 1$, then

$$\frac{d}{dx} f(x, \lambda, p) = \frac{1}{2} \left[f(x, \lambda, p-1) - f(x, \lambda, p) \right], \tag{8}$$

where *f(x, λ, p)* is the density (7).

Proof follows immediately from the known property of the modyfied Bessel function [7]:

$$\frac{d}{dx} x^{\nu} I_{\nu}(x) = x^{\nu} I_{\nu-1}(x), \tag{9}$$

where ν is an arbitrary real number.

Property 2

Let $p_i (i = 1, 2)$ be arbitrary real numbers, such that $p_i > 1$. Then

$$P(\lambda_1, p_1; \lambda_2, p_2) = \frac{1}{2} \left[P(\lambda_1, p_1; \lambda_2, p_2 - 1) + P(\lambda_1, p_1 - 1; \lambda_2, p_2) \right] \tag{10}$$

Proof:

It follows from (8), that

$$f(x_2, \lambda_2, p_2) = f(x_2, \lambda_2, p_2 - 1) - 2 \frac{d}{dx} f(x_2, \lambda_2, p_2). \tag{11}$$

By using this connection in (1), we obtain

$$P(\lambda_1, p_1;\ \lambda_2, p_2) \ = \ P(\lambda_1, p_1;\ \lambda_2, p_2 - \ 1) \ + \ 2 \int\limits_0^{\infty} f(x, \lambda_1, p_1)\, f(x, \lambda_2, p_2)\ dx \tag{12}$$

Evaluate the last integral by part:

$$2 \int\limits_0^{\infty} f(x, \lambda_1, p_1) f(x_2, \lambda_2, p_2)\ dx = -2 f(x_1, \lambda_1, p_1) \int\limits_0^{\infty} f(x_2, \lambda_2, p_2) dx. \Bigg|_{x_1}^{\infty} \ +$$

$$+ \int\limits_0^{\infty} \left[\frac{d}{dx_1} f(x_1, \lambda_1, p_1)\right] dx_1 \int\limits_{x_1}^{\infty} f(x_2, \lambda_2, p_2) dx_2 =$$

$$P(\lambda_1, p_1 - \ 1; \lambda_2, p_2) \ - \ P(\lambda_1, p_1;\ \lambda_2, p_2)\ . \tag{13}$$

The last passage was realized with use of (8). Substitution (13) into (12) proves (10).

Corollary

Let us $p_2 \ = \ p_1 \ + \ n$, n is natural. Then

$$\begin{aligned} & P(\lambda_1,\ p_1; \lambda_2,\ p_2) \ = \\ = \quad & P(\lambda_1, p_1;\ \lambda_2, p_1) \ + \sum_{i=0}^{n-1} \int\limits_0^{\infty} f(x, \lambda_1, p_1) f(x, \lambda_2, p_1 + i - 1)\, dx \end{aligned} \tag{14}$$

Proof follows from n - fold application of formula (12).

Property 3

Let us $p_i \ > \ 1 \ (i \ = \ 1, 2)$ are real numbers. Then

$$\begin{aligned} & P(\lambda_1, p_1; \lambda_2, p_2) \ = \ P(\lambda_1, p_1 - 1; \lambda_2, p_2 - 1) \ + \\ & + \int\limits_0^{\infty} [f(x, \lambda_1, p_1 - 1)\ f(x, \lambda_1, p_1)\, f(x, \lambda_2, p_2 - \ 1)] dx \end{aligned} \tag{15}$$

Proof:

With use of (12) we obtain

$$P\,(\lambda_1, p_1 - 1; \lambda_2, p_2) \; = \tag{16}$$

$$= \quad P\,(\lambda_1, p_1 - 1; \lambda_2, p_2 - 1) \; + \; 2\int\limits_0^{\infty} f\,(x, \lambda_1, p_1 - 1) f\,(x,\; \lambda_2, p_2)\,dx$$

It follows from (12), that

$$2\int\limits_0^{\infty} f\,(x, \lambda_1, p_1) f\,(x, \lambda_2 p_2 - 1)\,dx \; = \tag{17}$$

$$= \quad P\,(\lambda_1, p_1 - 1;\; \lambda_2, p_2 - 1\,) \; - \; P\,(\lambda_1, p_1;\; \lambda_2, p_2 - 1)$$

Validity of (15) follows from (16) and (17) in view of (12).
Corollary
If $p_1 = p_2 = q$, q is natural, then

$$P\,(\lambda_1, q; \lambda_2, q) \; = \; P\,(\lambda_1, 1;\; \lambda_2, 1) \; +$$
$$+ \sum_{k=1}^{q-1} \int\limits_0^{\infty} [f\,(x, \lambda_1, k)\, f\,(x, \lambda_2,\; k+1) \; - f\,(x, \lambda_1, k+1)\, f\,(x, \lambda_2, k)]dx \tag{18}$$

Property 4
Let us $p_i \geq 0$ $(i \; = \; 1, 2)$ are real numbers. Then

$$P\,(\lambda_1, p_1; \lambda_2, p_2) \; = \; 2\sum_{k=0}^{\infty} \int\limits_0^{\infty} f\,(x, \lambda_1, p_1 + k + 1)\, f\,(x, \lambda_2, p_2) dx \tag{19}$$

Proof:
It is evident, that (1) can be represented as follows:

$$P\,(\lambda_1, p_1; \lambda_2, p_2) \; = \int\limits_0^{\infty} f\,(x, \lambda_2, p_2) dx \int\limits_0^{x} f\,(y, \lambda_1, p_1) dy. \tag{20}$$

Using the known series [7],

$$\sum_{k=0}^{\infty} t^k I_{k+\nu}(z) = z^{-\nu} \exp\left(\frac{tz}{2}\right) \int_0^z \tau^{\nu} \exp\left(-\frac{t\tau}{2z}\right) d\tau, \tag{21}$$

and the formula (7), it is easy to show, that

$$2\sum_{k=0}^{\infty} f(x, \lambda_1, p_1 + k + 1) = \int_0^x f(y, \lambda_1, p_1) dy \tag{22}$$

Substitution of (22) into (19) and changing order of integration and summation prove validity of (19).

3 Basic results

Statement 1

Let us ξ_1 and ξ_2 are independent random values having density (2) and parameters$(\lambda_i, N_i)_{i=1,2}$, $N_1 = N_2 = 2q$, q is natural. Then

$$\Pr(\xi_1 < \xi_2) = P(\lambda_1, q; \lambda_2, q) = Q\left(\sqrt{\tfrac{\lambda_2}{2}}, \sqrt{\tfrac{\lambda_1}{2}}\right) - \tfrac{1}{2}\exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \times$$

$$\times \left[I_0\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \sum_{k=0}^{q-1} C_{k,q} \tfrac{\lambda_1^k - \lambda_2^k}{(\lambda_1\lambda_2)^{1/k}} I_k\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\right], \tag{23}$$

where

$$C_{k,q} = 2^{-2k} \sum_{m=0}^{q-k-1} 2^{-2m} \frac{k}{m+k} \begin{pmatrix} 2m+2k \\ m \end{pmatrix} \tag{24}$$

Proof

According to the corollary from the property 3 this probability can be represented as follows:

$$P(\lambda_1, q; \lambda_2, q) = P(\lambda_1, 1; \lambda_2, 1) + \Delta. \tag{25}$$

Since the closed expression for $P(\lambda_1, 1; \lambda_2, 1)$ is known (see formula (3)), to demonstrate validity (23), (24) it is sufficient to evaluate Δ:

$$\Delta = \sum_{k=1}^{q-1} \int\limits_0^\infty [f(x,\lambda_1,k)\ f(x,\lambda_2,k+1) - f(x,\lambda_1,k+1)\, f(x,\lambda_2,k)]dx \tag{26}$$

Substitute the density (2) into (26) and change the variable for integration $z = \sqrt{x}$. Then

$$\Delta = \tfrac{1}{2} \exp\Big(-\tfrac{\lambda_1+\lambda_2}{2}\Big) \sum_{k=1}^{q-1} [\ \lambda_1^{-(k-1)/2} \lambda_2^{-k/2}$$

$$\int\limits_0^\infty z^{2k} \exp\big(-z^2\big)\, I_k\big(z\sqrt{\lambda_2}\big) I_{k-1}\big(z\sqrt{\lambda_1}\big)\, dz - \tag{27}$$

$$-\lambda_1^{-k/2}\lambda_2^{-(k-1)/2} \int\limits_0^\infty z^{2k} \exp\big(-z^2\big)\ I_k\big(z\sqrt{\lambda_1}\big)\, I_{k-1}\big(z\sqrt{\lambda_2}\big)\ dz\ ]$$

The integrals from (27) are evaluated in appendix 3. After substitution the results of these evaluations into (27), we obtain:

$$\Delta \ = \ \frac{1}{2} \exp\left(-\frac{\lambda_1 - \lambda_2}{4}\right) \sum_{k=1}^{q-1} 2^{-2k} \times \tag{28}$$

$$\times \sum_{m=-k-1}^{k.} \left(\begin{array}{c} 2k-1 \\ k-1-m \end{array} \right) \left[\left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} - \left(\frac{\lambda_2}{\lambda_1}\right)^{m/2} \right] I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right)$$

Now we can evaluate formulas (23) and (24) from (3), (25) and (28) if to take into account that the modified Bessel functions are even for integer index and to use the following evidential identities:

$$\left(\begin{array}{c} a \\ b \end{array} \right) = \frac{b+1}{a-b} \left(\begin{array}{c} a \\ b+1 \end{array} \right); \left(\begin{array}{c} a-1 \\ b \end{array} \right) = \frac{a-b}{a} \left(\begin{array}{c} a \\ b \end{array} \right).$$

Some identities for sums with binomial coefficients are proved in appendices 1 and 2. With help of these it can obtain the following representations for values $C_{k,q}$determined by (24):

$$C_{k,q} = 2^{-(q+k-1)} \sum_{m=0}^{q-k+1} 2^{-m} \left(\begin{array}{c} m+q+k-1 \\ m \end{array} \right), \tag{29}$$

$$C_{k,q} = 2^{-2(q-1)} \sum_{m=0}^{q-k+1} \binom{2q-1}{m}, \tag{30}$$

$$C_{k.q} = 2\frac{B_{1/2}(q+k, q-k)}{B(q+k, q-k)}, \tag{31}$$

where$B_x(a,b)$, $B(a,b)$ are incomplete and complete beta functions correspondingly.

Note that at $\lambda_2 \to 0$, the expression (20) reduces to formula (4). For demonstration of that it is sufficient to use the representation (26), Lagguer's polynoms determination

$$L_n^\alpha(y) = \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-y)^m}{m!}, \tag{32}$$

Marcum's Q- function limit value [2],

$$\begin{array}{l} \lim Q(x,y) = \exp\left(-\frac{y^2}{2}\right) \\ x \to 0 \end{array}, \tag{33}$$

and modified Bessel functions asymptotic values [7],

$$\begin{array}{l} I_k(y) \approx \frac{(y/2)^k}{k!} \\ y \to 0 \end{array}. \tag{34}$$

Statement 2

Let ξ_1 and ξ_2 are independent random values with density (2) and parameters $(\lambda_i, N_i)_{i=1,2}$, $N_1 = 2q_1 < N_2 = 2q_2$, q_i $(i = 1,2)$ are natural. Then

$$\begin{aligned} &\Pr(\xi_1 < \xi_2) = P(\lambda_1, q_1; \lambda_2, q_2) = \\ &= P(\lambda_1, q_1; \lambda_2, q_1) + \exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \sum_{m=-q_1+1}^{q_2-1} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} \times \\ &\times I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \sum_{i=\max(q_1,m)}^{q_2-1} 2^{-(q_1+i)} \binom{q_1+i-1}{q_1+m-1} \end{aligned} \tag{35}$$

Proof

Using the corollary from property 2, we shall represent the considered probability in the following form:

$$P\left(\lambda_1, q_1;\ \lambda_2, q_2\right)\ =\ P\left(\lambda_1, q_1;\ \lambda_2, q_1\right)\ +\ \Delta, \tag{36}$$

where

$$\Delta\ =\ 2\sum_{i=0}^{q_2-q_1-1}\int_0^\infty f\left(x, \lambda_1, q\right)\, f\left(x, \lambda_2, q_1+i+\ 1\right)\, dx. \tag{37}$$

Substitute the expression for density (2) into (36). Then after some transformations value Δ may be reduced to sum of the integrals, evaluated in Appendix 3. It will lead us to the following expression:

$$\begin{aligned}
\Delta\ &=\ \exp\left(\ -\tfrac{\lambda_1+\lambda_2}{4}\right)\sum_{i=0}^{q_2-q-1} 2^{-(2q_1+i)}\times\\
&\times\sum_{m=-q_1+1}^{q_1+i}\begin{pmatrix} 2q_1+i\ -\ 1\\ m\ +\ q_1\ -\ 1\end{pmatrix}\left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\ =\\
&= \exp\left(\ -\tfrac{\lambda_1+\lambda_2}{4}\right)\sum_{m=-q+1_1}^{q_2-1}\left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\times\\
&\times\sum_{i=\max(q,m_1)}^{q-1_2} 2^{-(q+i_1)}\begin{pmatrix} i+q_1-1\\ m+q_1-1\end{pmatrix}
\end{aligned} \tag{38}$$

The validity of (35) follows from (38) and (36).

Statement 3

Let ξ_1 and ξ_2 are independent random values with density (2) and parameters $(\lambda_i, N_i)_{i=1,2}$, $N_1 = 2q_1$, $N_2 = 2q_2$, q_i $(i\ =\ 1,2)$ are natural. Then

$$\begin{aligned}
&\Pr\left(\xi_1<\ \xi_2\right)\ =\ P\left(\lambda_1, q_1; \lambda_2, q_2\right)\ =\\
&=\ Q\left(\sqrt{\tfrac{\lambda_1}{2}}, \sqrt{\tfrac{\lambda_2}{2}}\right)\ +\ \tfrac{1}{2}\exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right)\times\\
&\times\left\{\sum_{m=1}^{q_2-1} C_{m-k,q}\left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right)-\right.\\
&\left.-\sum_{m=0}^{q_1-1} C_{m+k,q}\left(\tfrac{\lambda_2}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\right\},
\end{aligned} \tag{39}$$

where

$$k = (q_1 - q_2)/2, \ q = (q_1 + q_2)/2, \tag{40}$$

and $C_{m-k,q}$, $C_{m+k,q}$ are the coefficients, determined by (29)-(31).

Proof

At first we will prove validity of (39) for the case $q_1 < q_2$. For that we use the representation of the probability in form (36) and (38). The value Δ may be written as the sum:

$$\Delta = A + B \tag{41}$$

where

$$A = \exp\left(-\frac{\lambda_1 + \lambda_2}{4}\right) \times \tag{42}$$

$$\times \sum_{m=-q_1+1}^{q-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \sum_{i=q_1}^{q_2-1} 2^{-(q+i)} \binom{i+q_1-1}{m+q_1-1},$$

$$B = \exp\left(-\frac{\lambda_1 + \lambda_2}{4}\right) \times \tag{43}$$

$$\times \sum_{m=q_1}^{q_2-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \sum_{i=m}^{q_2-1} 2^{-(q_1-i)} \binom{i+q_1-1}{m+q_1-1}.$$

Using (29), from (42) and (43) we obtain

$$A = \tfrac{1}{2} \exp\left(-\tfrac{\lambda_1+\lambda_2}{4}\right) \times$$

$$\times \left\{ \sum_{m=0}^{q_1-1} \left(\tfrac{\lambda_2}{\lambda_1}\right)^{m/2} I_{-m}\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \left(C_{-(k+m),q} - C_{-m,q_1}\right) + \right. \tag{44}$$

$$\left. + \sum_{m=1}^{q_1-1} \left(\tfrac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\tfrac{1}{2}\sqrt{\lambda_1\lambda_2}\right) \left(C_{m-k,q} - C_{m,q_1}\right) \right\},$$

$$B = \frac{1}{2} \exp\left(-\frac{\lambda_1 + \lambda_2}{4}\right) \sum_{m=q_1}^{q_2-1} \left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right). \tag{45}$$

The values k and q from (44) and (45) are determined by (40).

It is demonstrated in Appendix 2, that

$$C_{-(k+m),q} = 2 - C_{k+m,q};\ C_{-m,k} = 2 - C_{m,k} \tag{46}$$

It is easy to obtain from (30) or (31), that

$$C_{0,q} = 1 \tag{47}$$

Since $I_{-m}(x) = I_m(x)$, substitution of (45) and (47) into (44) brings the following expression:

$$A = \frac{1}{2}\exp\left(-\frac{\lambda_1+\lambda_2}{4}\right)\left\{I_0\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \sum_{m=1}^{q_1-1} C_{m,q_1}\frac{\lambda_1^m-\lambda_2^m}{(\lambda_1\lambda_2)^{m/2}} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \right.$$

$$\left. - \sum_{m=0}^{q_1-1} C_{m-k,q}\left(\frac{\lambda_1}{\lambda_2}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right) - \sum_{m=0}^{q_1-1} C_{m+k,q}\left(\frac{\lambda_2}{\lambda_1}\right)^{m/2} I_m\left(\frac{1}{2}\sqrt{\lambda_1\lambda_2}\right)\right. \tag{48}$$

Substitutions of (45) and (48) into (41) and also (23) and (41) into (38) prove the validity of (39) for the event $q_1 < q_2$.

Now, let $q_1 > q_2$. We use the evidential equality:

$$P(\lambda_1, q_1; \lambda_2, q_2) = 1 - P(\lambda_2, q_2; \lambda_1, q_1). \tag{49}$$

The validity of (39) for $P(\lambda_2, q_2; \lambda_1, q_1)$ has been proved. Now, if to substitute the known expression for $P(\lambda_2, q_2; \lambda_1, q_1)$ to (45), to take into consideration (47) and, also, the known property of Marcum's Q- function [2]:

$$Q(x, y) + Q(y, x) = 1 + \exp\left(-\frac{x^2 + y^2}{2}\right) I_0(x, y), \tag{50}$$

the validity of (38) may be easy proved for the event $q_1 < q_2$.

It is evident that for $q_1 = q_2$, (23) and (38) coincide. The equivalence of (28), (31) for that case if$k$ and $q$are integer or half-integer coincidentally $(q \geq 1, |k| \leq q-1)$is proved in Appendix 2.

Statement 4

Let ξ_1 and ξ_2 are independent random values with density (7) and parameters $(\lambda_i, p_i)_{i=1,2}$, λ_i, p_i are nonnegative numbers. Then probability $P(\xi_1 < \xi_2)$ is

$$P(\lambda_1, p_1; \lambda_2, p_2) = \exp\left(-\frac{\lambda_1 + \lambda_2}{2} \right) \times \tag{51}$$
$$\times \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} . \frac{\cdot \left(\frac{\lambda_1}{2}\right)^m \left(\frac{\lambda_2}{2}\right)^n}{m!\, n!} S(p_1 + m,\, p_2 + n)^{\cdot}$$

with

$$S(x, y) = \frac{B_{1/2}(x, y)}{B(x, y)}. \tag{52}$$

Proof

Substitute the density (7) into (19) and fulfill change of the variable $y = \sqrt{x}$. Then

$$P(\lambda_1, p_1; \lambda_2, p_2) = \lambda_1^{-p_1/2} . \lambda_2^{-\ (p_2-1)/2} \times$$
$$\times \exp\left(-\tfrac{\lambda_1 + \lambda_2}{2} \right) \sum_{k=0}^{\infty} \int_0^{\infty} y^{p_1+p_2+k} \exp\left(-y^2 \right) \times \tag{53}$$
$$\times I_{p_1+k}\left(y\sqrt{\lambda_1} \right) I_{p_2-1}\left(y\sqrt{\lambda_2} \right) dy$$

The integral from (53) is known [7] :

$$\int_0^{\infty} x^{\alpha-1} \exp\left(-px^2 \right) I_\mu(bx)\, I_\nu(cx)\, dx =$$
$$= \frac{b^\mu c^\nu p^{-.(\mu+\nu+\alpha)/2}}{2^{\mu+\nu+1} \Gamma(\nu+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k + (\alpha+\mu+\nu)/2)}{\Gamma(\mu+k+1)} \left(\frac{b^2}{4p} \right)^k \times \tag{54}$$
$$\times {}_2F_1\left(-k, -(\mu+k), \nu+1, \tfrac{c^2}{b^2} \right)$$

Substitute integral (54) into (44) with following meanings of the parameters:

$b = \sqrt{\lambda_1},\ c = \sqrt{\lambda_2},\ \mu = p_1 + k,\ \nu = p_2 - 1,\ \alpha = p_1 + p_2 + k + 1, p = 1$.Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) =$$

$$= \frac{2^{-(p_1+p_2)}}{\Gamma\left(p_2\right)} \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) \sum_{k=0}^{\infty} \sum_{k_1=0}^{\infty} 2^{-(k+k_1)} \frac{\Gamma\left(k+k_1+p_1+p_2\right)}{k_1!\,\Gamma\left(k+k_1+p_1+1\right)} \times$$

$$\times \left(\frac{\lambda_2}{2}\right)^{k_1} {}_2F_1\left(-k_1, -p_1, -k-k_1, p_2, \frac{\lambda_2}{\lambda_1}\right) \tag{55}$$

Now we use the polynomial form of the hypergeometrical function and take into account absolute convergence of the series. Then

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) =$$

$$= 2^{-(p_1+p_2)} \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) \sum_{k_1=0}^{\infty} \sum_{n=0}^{k_1} 2^{-k_1} \frac{\Gamma(\ k_1+p_1+p_2)}{\Gamma(p_2+n)\,\Gamma(k_1+p_1-n+1)} \times \tag{56}$$

$$\times \frac{(\lambda_1/2)^{k_1-n}(\lambda_2/2)^n}{n!\,(k_1-n)!}\ {}_2F_1\left(1, p_1+p_2+k_1, p_1+k_1-n+1, \tfrac{1}{2}\right)$$

Substitute the known representation of Hypergeometrical Gauss function [7],

$${}_2F_1\left(1, b, c, z\right) = z^{1-c}\left(1-z\right)^{1-c}\left(1-z\right)^{c-1}\left(c-1\right) B_z\left(c-1, b-c+1\right), \tag{57}$$

into (56), where $B_z\left(x, y\right)$ be uncomplete beta-function Then, after simple transformations, (56) takes the following form:

$$P\left(\lambda_1, p_1; \lambda_2, p_2\right) = \tag{58}$$
$$= \quad \exp\left(-\frac{\lambda_1+\lambda_2}{2}\right) \sum_{k_1=0}^{\infty} \sum_{n=0}^{k_1} \frac{(\lambda_1/2)^{k_1-n}\,(\lambda_2/2)^n}{n!\ (k_1-n)!} S\left(p_1+k+n,\ p_2+n\right),$$

where function$S(x,y)$ is determined by (52).

By changing $m = k_1 - n$, (58) is reduced to (51). Note, that the recursive formulas and expansion to continued fraction exist for $S(x,y)$ [7].

4 Additional results

Integrals

In the course of demonstrating of the results above we obtained closed form for the following integral:

$$\int\limits_0^\infty x^{k+l+1} \exp\left(-px^2 \right) I_k\left(ax \right) I_l\left(bx \right) dx \; = \; (2p)^{-(p+l+1)}\, a^k b^l \exp\left(\tfrac{a^2+b^2}{4p} \right) \times$$

$$\times \sum_{i=-l}^{k} \left(\begin{array}{c} k+l \\ i+l \end{array} \right) \left(\tfrac{b}{a} \right)^i I_i\left(\tfrac{ab}{2p} \right) \tag{59}$$

where k, l are integer, such that $k \geq 0, |l| \leq k$.

The deduction of (59) is brought in Appendix 3 as having no direct relation to the treated problem. With usage of (59) the following integral was evaluated:

$$\int\limits_0^\infty x^M \exp\left(-\tfrac{px^2}{2} \right) I_{M-1}\left(cx \right) Q_M\left(b, ax \right)\, dx \; = \tfrac{1}{c} \left(\tfrac{c}{p^2} \right)^M \exp\left(\tfrac{c^2}{2p^2} \right) \times$$

$$\times \left\{ Q_M\left(U, V \right) \, - \, \exp\left(-\tfrac{U^2+V^2}{2} \right) \sum_{k=-(M-1)}^{M-1} \tfrac{B_Z(M-k,M+k)}{B(M-k,M+k)} \left(\tfrac{V}{U} \right)^k I_k\left(UV \right) \right\} \tag{60}$$

where $Q_M\left(U, V \right)$ is so-called Q_M- function [2],

$$U = \frac{bp}{\sqrt{p^2+a^2}},\; V = \frac{ac}{p\sqrt{p^2+a^2}}, Z = \frac{a^2}{p^2+a^2}.$$

Derivation of (60) is brought in Appendix 4. Formula (59) can complete the table of integrals with Q_M-function [1].

Transformation of a known formula

In order to illustrate the usage of our finding, transform the formula for probability of errors when autocorrelation demodulation of binary DPSK signals is performed under the carrier detuning ([2], formula (8.18)):

$$P_b \; = \; \frac{1}{2^{\cdot q}} \exp\left(-2h_2^2 - \, h_1^2 \right) \sum_{n=0}^{\infty} \frac{\left(2h_2^2 \right)^n}{n!} \sum_{k=0}^{q+n-1} \frac{1}{2^k} \sum_{m=0}^{k} \frac{\left(h_1^2 \right)^m}{m\,!} \left(\begin{array}{c} k+q-1 \\ k-m \end{array} \right) \tag{61}$$

(61) is obtained as the For the details of the evaluation of that formula the authors of [2] refer to [8], where result of transformation for the following starting formula:

$$P_b = \frac{1}{2^{2q}} \exp\left[-2\left(h_1^2 + h_2^2\right)\right] h_1^{1-q} h_2^{1-q} \int_0^{\infty} x_2^{(q-1)/2} e^{-x_2/2} I_{q-1}\left(2h_2\sqrt{x_2}\right) dx_2 \times$$
$$\times \int_0^{x_2} x_1^{(q-1)/2} e^{-x_1/2} I_{q-1}\left(2h_1\sqrt{x_1}\right) dx_1 \tag{62}$$

Denote $\lambda_1 = (4h_1)^2$, $\lambda_2 = (2h_2)^2$. Then it may be rewritten as follows:

$$\begin{aligned} P_b &= \int_0^{\infty} \frac{1}{2}\left(\frac{x_2}{\lambda_2}\right)^{(q-1)/2} \exp\left(\frac{x_2+\lambda_2}{2}\right) I_{q-1}\left(\sqrt{\lambda_2 x_2}\right) dx_2 \times \\ &\times \int_0^{x_2} \frac{1}{2}\left(\frac{x_1}{\lambda_2}\right)^{(q-1)/2} \exp\left(\frac{x_1+\lambda_1}{2}\right) I_{q-1}\left(\sqrt{\lambda_1 x_1}\right) dx_1. \end{aligned} \tag{63}$$

.

The latter is probability of the inequality $x_1 < x_2$, where x_1 and x_2 are "Noncentral χ^2" random variables with parameters of noncentrality $\lambda_1$and λ_2 and the numbers of degrees of freedom are equal to $2q$. Hence the conditions of the ***Statement 1*** are valid and instead of the infinite series (60) we can use formula (23).

5 Conclusion

In the paper we obtained the closed form expression for probability $\Pr(\xi_1 \leq \xi_2\ )$ with ξ_1 and ξ_2 are noncentral chi-square random variables with even number degrees of freedom, which is of considerable important under evaluation reliability of networks. For the case add number of degrees the chi-square density is expressed by elementary functions and corresponding computations do not represent any difficulties.

Appendix 1

Statement 5

Let k and q are natural, such that $k \leq q-1$. Then it is valid

$$\begin{aligned} C_{k,q} &= 2^{-2k} \sum_{m=0}^{q-k-1} 2^{-2m} \frac{k}{m+k} \begin{pmatrix} 2m+2k \\ \\ m \end{pmatrix} = \\ &= 2^{-(q+k-1)} \sum_{m=0}^{q-k-1} 2^{-m} \begin{pmatrix} m+q+k-1 \\ \\ m \end{pmatrix}. \end{aligned} \tag{64}$$

Proof

First we shall demonstrate that (5) is reduced to the expression, coinciding with (20), within coefficients.

Introduce the notations:

$$z = \frac{\lambda_1}{2},\ x = \frac{\lambda_2}{4},\ M = \frac{N-2}{2} = q-1 \tag{65}$$

Represent the internal sum in (6) in the following form:

$$\begin{aligned} &\sum_{k=0}^{m+M} 2^{-(k+M+1)} L_k^M(-x) = \\ &= \sum_{k=0}^{M-1} 2^{-(k+M+1)} L_k^M(-x) + 2^{-(2M+1)} \sum_{k=0}^{m} L_{k+M}^M(-x). \end{aligned} \tag{66}$$

Substitute (66) into (6). Then

$$\begin{aligned} \Pr(\xi_1 \triangleleft \xi_2) &= 1 - 2^{-(M+1)} \exp(-x) \times \\ &\times \sum_{k=0}^{M-1} 2^{-k} L_k^M(-x) - 2^{-(2M+1)} \exp[-(x+z)]\, G(z,x), \end{aligned} \tag{67}$$

where

$$G(z,x) = \sum_{m=0}^{\infty} \frac{z^m}{m!} \sum_{k=0}^{m} 2^{-k} L_{k+M}^M(-x). \tag{68}$$

It is obvious, that function $G(x, z)$ satisfies the following integral equation:

$$G(z, x) = H(z, x) + \int_0^z G(t, x)\, dt, \tag{69}$$

where

$$H(z, x) = \sum_{m=0}^{\infty} \frac{(z/2)^m}{m!} L_{m+M}^{M}(-x). \tag{70}$$

By solving equation (68) we obtain the following representation

$$G(z, x) = H(z, x) + \exp(z) \int_0^z \exp(-t)\, H(t, x)\, dt. \tag{71}$$

It follows from (70), that

$$H(z, x) = \frac{d^{2M}}{dy^{2M}} \left\{ y^M \sum_{m=0}^{\infty} \frac{y^m}{(m+M)!} L_m^M(-x) \right\}_{y=z/2}. \tag{72}$$

The internal series in (72) is the generating function for Lagguer's polynoms [7]:

$$\sum_{m=0}^{\infty} \frac{y^m}{(m+M)!} L_m^M(-x) = (xy)^{-M/2} \exp(y)\, I_m(2\sqrt{xy}). \tag{73}$$

By substitution of (73) into (72) we obtain

$$H(z, x) = \frac{d^{2M}}{dy^{2M}} \left\{ \left(\frac{y}{x}\right)^{M/2} \exp(-y)\, I_m(2\sqrt{xy}) \right\}_{y=z/2}. \tag{74}$$

After substitution (74) into (71) and $2M$-fold integration by part, expression (71) takes the following form:

$$G(z,x) = 2^M \left\{ \sum_{i=0}^{2M} 2^{-i} \frac{d^i}{dy^i} \left[\left(\frac{y}{x}\right)^{M/2} \exp(y)\, I_M\left(2\sqrt{xy}\right) \right]_{y=z/2} - \right.$$

$$- \exp(z) \sum_{i=0}^{2M-1} 2^{-i} \frac{d^i}{dy^i} \left[\left(\frac{y}{x}\right)^{M/2} \exp\ (y)\, I_M\left(2\sqrt{xy}\right) \right]_{y=0} + \tag{75}$$

$$\left. + 2 \exp\ (z) \int_0^{z/2} \left(\frac{t}{2}\right)^{M/2} \exp\ (-z)\, I_M\left(2\sqrt{xt}\right)\ dt \right\}.$$

Transform (75) by $M-$fold integration by part with usage formulas (9),(32) and (34) and taking into account the evenness of modified Bessel function respect to integer index. Then substitute the obtained expression into (67). Then

$$\Pr(\xi_1 < \xi_2) = 1 - \exp(-x) \int_0^{z/2} \exp(-t)\, I_0\left(2\sqrt{xt}\right) dt +$$

$$+\frac{1}{2} \exp(-x - z/2) \sum_{k=1}^{M} \left[\left(\frac{z}{2x}\right)^{k/2} (2 - C_{-k,M+1}) - \left(\frac{z}{2x}\right)^{-z/2} C_{k,M+1} \right] I_k\left(2\sqrt{\frac{xz}{2}}\right) -$$

$$+\frac{1}{2} \exp(-x - z/2)\, I_0\left(2\sqrt{\frac{xz}{2}}\right), \tag{76}$$

where the coefficients $C_{k,M+1}$ are given by expression (64).

It follows from determination of Q-function [1,2] that

$$1 - \exp(-x) \int_0^{z/2} \exp(-t)\, I_0\left(z\sqrt{xt}\right)\ dt = Q\left(\sqrt{2z},\ \sqrt{z}\right). \tag{77}$$

Also it is shown in Appendix 2, that

$$C_{-k,M+1} + C_{k,\,M+1} = 2 \tag{78}$$

Taking into account notation (65), we obtain from (76) - (78):

$$\Pr\ (\xi_1 \lhd \xi_2) = Q\left(\sqrt{\frac{\lambda_2}{2}},\ \sqrt{\frac{\lambda_1}{2}}\right) + \frac{1}{2} \exp\left(-\frac{\lambda_1 + \lambda_2}{4}\right) \times$$

$$\times \left[-I_0\left(\frac{1}{2}\sqrt{\lambda_1 \lambda_2}\right) + \sum_{k=0}^{q-1} C_{k,q} \frac{\lambda_1^k - \lambda_2^k}{(\lambda_1 \lambda_2)^{k/2}}\, I_0\left(\frac{1}{2}\sqrt{\lambda_1\, \lambda_2}\right) \right], \tag{79}$$

where

$$C_{k,q} = 2^{-(q+k-1)} \sum_{m=0}^{q-k-1} 2^{-m} \begin{pmatrix} m+q+k-1 \\ m \end{pmatrix}. \tag{80}$$

As (79) identically equals to (23) for each permissible values of parameters λ_1 andλ_2, the validity of (64) is proved.

Appendix 2

Statement 6

Let q and k are integer or half-integer coincidentally such, that $q \geq 1$, ?k? $\leq q$- 1. Then

$$\begin{aligned} C_{k,q} &= 2\frac{B_{1/2}\,(q+k, q-k)}{B\,(q+k, q-k)} = \qquad (81) \\ &= 2^{-(q+k-1)} \sum_{m=0}^{q+k-1} 2^{-m} \begin{pmatrix} m+q+k-1 \\ m \end{pmatrix} \end{aligned}$$

Proof

Denote

$$a = q-k, \quad b = q+k, \quad 1-z = 1/2. \tag{82}$$

According to the known property of incomplete beta-function [7], rewrite the left-side of (81) in the following form

$$C_{k,q} = 2\left(1 - \frac{B_z\,(a,b)}{B\,(a,b)}\right). \tag{83}$$

Use now the expression (57). Then

$$B_z\,(a,b) = \frac{1}{a} z^a\,(1-z)^b\ {}_2F_1\,(1,\ a+b,\ a+1,\ z)\,. \tag{84}$$

Since [7]

$${}_2F_1\,(1,\ l,\ n,\ z) = (n-1)!\,(-z)^{1-n}\,(-1)^{1-\hat{\imath}}\,\frac{1}{(1-l)_n} \times \tag{85}$$

$$\times \left[(1-z)^{n-l-1} - \sum_{m=0}^{n-2} \frac{(l-n+1)_m}{m!} \, z^m \right],$$

where $l = a + b$, $n = a + 1$ are natural ($l \geq n$), after some transformations, (82) takes the following form:

$$B_z\,(a,b) \;=\; B\,(a,b) \left[1 \;-\; (1\;-\;z)^b \sum_{m=0}^{a-1} \left(\begin{array}{c} b+m-1 \\ m \end{array} \right) \; z^m \right]. \qquad (86)$$

Now, if to take into consideration notation (82), then (81) follows from (83) and (86). Also the following property of coefficients $C_{k,q}$ can be readily extracted from (81)- (83):

$$C_{-k,q} + \; C_{k,q} \;=\; 2. \qquad (87)$$

Statement 7
Coefficients $C_{k,q}$ have the following equivavelent to (81) representation:

$$C_{k,q} \;=\; 2^{-2(q-1)} \sum_{m=0}^{q-k-1} \left(\begin{array}{c} 2q\;-\;1 \\ m \end{array} \right). \qquad (88)$$

Proof
Consider the following function:

$$C_{k,q} = f\,(N_1, N_2) \;=\; 2^{-N_2} \sum_{m=0}^{N_1} 2^{-m} \left(\begin{array}{c} m+N_2 \\ m \end{array} \right), \qquad (89)$$

with

$$N_1 = q-k-1, N_2 = q+k-1. \qquad (90)$$

Use the following known connection:

$$\left(\begin{array}{c} m\;+\;N_2 \\ m \end{array} \right) \;=\; \left(\begin{array}{c} m+\;N_2+1 \\ m \end{array} \right) \;-\; \left(\begin{array}{c} m\;+\;N_2 \\ m\;-\;1 \end{array} \right). \qquad (91)$$

After some transformations we obtain

$$f\,(N_1, N_2) \;-\; f\,(N_1 +\; 1, N_2+1) \;=\; 2^{-(N_1+N_2)} \left(\begin{array}{c} N_1+N_2+1 \\ N_2 \end{array} \right). \qquad (92)$$

It is evident, (76) can be represented in following form:

$$f\left(N_1,\,N_2\right)\;=\;\sum_{i=0}^{N_1} f\left(N_1-\,i,\,N_2+\,i\right)\;-\;\sum_{i=0}^{N-1_1} f\left(N_1-i\,-\,1,\,N_2+i+\,1\right). \tag{93}$$

According to (89),

$$f\left(\,-1,\,N_1+\,N_2\right)\;=\;0, \tag{94}$$

and therefore (93) takes the form

$$f\left(N_1,\,N_2\right)\;=\;\sum_{i=0}^{N_1}\left(f\left(N_1-\,i,\,N_2+\,i\right)-\,f\left(N_1-i\,-\,1,\,N_2+i+\,1\right)\right). \tag{95}$$

After substitution (87) into (89) and changing the summation index, we obtain

$$f\left(N_1,N_2\right)\;=\;2^{-(N_1+N_2)}\sum_{m=0}^{N_1}2^{-m}\begin{pmatrix}N_1+N_2+\,1\\ m\end{pmatrix}. \tag{96}$$

The equivalence (81) and (88) follows from (96).

Appendix 3

Evaluation of the integral

$$B\;=\;\int\limits_0^{\infty}x^{k+l+1}\exp\left(\,-px^2\right)I_k\left(ax\right)I_l\left(bx\right)dx, \tag{97}$$

where k, l are integers ($k\,\geq\,0,|l|\,\leq\,k$) and a, b, p a are real positive numbers.

Introduce the following parameters:

$$g\;=\;a^2/4,\;\;h\;=\;b^2/4 \tag{98}$$

and change the integration variable $y\;=\;x^2$,

$$B\;=\;\frac{1}{2}\,g^{k/2}h^{l/2}\int\limits_0^{\infty}\exp\left(\,-py\right)\varphi\left(y,\,k,g\right)\varphi\left(y,\,l,h\right)dy, \tag{99}$$

where

$$\varphi(z,n,c) \;=\; (z/c)^{n/2}\, I_n\left(2\sqrt{cz}\right). \tag{100}$$

Fulfill $(k+l)$ -fold integration by parts. Then

$$B \;=\; A + \frac{1}{2}\, g^{k/2}h^{l/2}p^{-(k+l)} \int\limits_0^\infty \exp\left(\,-py\right)\frac{d^{k+l}}{dy^{k+l}}\left[\varphi\left(y,k,g\right)\,\varphi\left(y,\,l,\,h\right)\right]\,dy, \tag{101}$$

with

$$A \;=-\frac{1}{2}\, g^{k/2}h^{l/2}\sum_{m=1}^{k+l} p^{-m}\left\{\exp\left(\,-py\right)\frac{d^{k+l}}{dy^{k+l}}\left[\varphi\left(y,k,g\right)\,\varphi\left(y,\,l,\,h\right)\right]\right\}_{y=0}^{\infty}. \tag{102}$$

Using (9), it is simple to show, that

$$\frac{d^m}{dz^m}\varphi\left(z,n,c\right) \;=\; \varphi\left(z,n-m,c\right). \tag{103}$$

With usage of (34) and the known asymptotic representation [7]

$$\begin{array}{l} I_n\left(z\right) \;\to\; \frac{1}{\sqrt{2\pi z}}\exp\;(z) \\ z\to\infty \end{array}, \tag{104}$$

we obtain from (101)-(103), that $A = 0$ and

$$\begin{aligned} B \;&=\; \frac{1}{2}\, g^{k/2}h^{l/2}p^{-(k+l)} \sum_{m=0}\left(\begin{array}{c} k+l \\ m \end{array}\right)\times \\ &\times\int\limits_0^\infty \exp\left(-py\right)\varphi\left(y,k-m,g\right)\varphi\left(y,m-k,h\right)\,dy. \end{aligned} \tag{105}$$

Now, after substitution (100) into (105) and inverse changing $x \;=\; \sqrt{y}$

$$\begin{aligned} B \;&=\; p^{-(k+l)}\sum_{m=0}\left(\begin{array}{c} k+l \\ m \end{array}\right) g^{m/2}h^{(l+k-m)/2}\times \\ &\times\int\limits_0^\infty x\exp\left(\,-px^2\right)\, I_{k-m}\left(2x\sqrt{g}\right)\, I_{m-k}\left(2x\sqrt{h}\right)\,dx. \end{aligned} \tag{106}$$

Since for integer $n, I_n(z) = I_{-n}(z)$, we can use the known formula for the integral in (106) [7]:

$$\int_0^{\infty} x \exp\left(-px^2\right) I_{k-m}\left(2x\sqrt{g}\right) I_{m-k}\left(2x\sqrt{h}\right) dx = \tag{107}$$
$$= \frac{1}{2p} \exp\left(\frac{g+h}{p}\right) I_{m-k}\left(\frac{2}{p}\sqrt{gh}\right).$$

It follows from (106) and (107) with accounting denote (92), that

$$\begin{aligned} B &= (2p)^{-(k+l+1)} \exp\left(\frac{a^2+b^2}{4p}\right) \times \\ &\times \sum_{m=0}^{k+l} \binom{k+l}{m} a^m b^{l+k-m} I_{m-k}\left(\frac{ab}{2p}\right). \end{aligned} \tag{108}$$

Changing the summation index $i = k - m$ and using (107), we obtain

$$\begin{aligned} B &= (2p)^{-(k+l+1)} \exp\left(\frac{a^2+b^2}{4p}\right) a^k b^l \times \\ &\times \sum_{i=-l}^{k} \binom{k+l}{i+l} \left(\frac{b}{a}\right)^i I_i\left(\frac{ab}{2p}\right). \end{aligned} \tag{109}$$

The correctness of (59) follows from (97) and (109).

Appendix 4

Evaluation of the integral

$$A = \int_0^{\infty} x^M \exp\left(-\frac{p^2x^2}{2}\right) I_{M-1}(cx)\, Q_M(b, ax)\, dx \tag{110}$$

where M is arbitrary natural, a, b, c, p, are real positive numbers,

$$Q_M(x,y) = \int_y^{\infty} t \left(\frac{t}{x}\right)^{M-1} \exp\left(-\frac{t^2+x^2}{2}\right) I_{M-1}(xt)\, dt \tag{111}$$

is so-called Q_M -function (Q_1 is Marcum's Q-function).

For Q_M -function the representation exists, following from (111):

$$Q_M(x,y) \;=\; Q\,(x,y) \,+\, \exp\,\left(\frac{x^2\,+\,y^2}{2}\right)\,\sum_{k=1}^{M-1}\left(\frac{y}{x}\right)^k I_k\,(xy)\,. \tag{112}$$

Also the following property of Q_M -function is known, which simply deduced from (112) and (50):

$$Q_M(x,y)\,+Q_M(y,x) \;=\; 1+\exp\left(\frac{x^2\,+\,y^2}{2}\right)\sum_{k=-(M-1)}^{M-1}\left(\frac{x}{y}\right)^k I_k\,(xy)\,. \tag{113}$$

Convey Q_M (b,ax) with use Q_M (ax,b) on the base of (113) and substitute into (110). Then for integral A we can write:

$$A = B - D + E \tag{114}$$

where [5]

$$B \;=\; \int\limits_0^\infty x^M \exp\left(\,-\frac{p^2x^2}{2}\right)\,I_{M-1}\,(cx)\;dx \;=\; \frac{1}{c}\,\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c^2}{2p^2}\right), \tag{115}$$

and

$$\begin{aligned} D \;=\; & \int\limits_0^\infty x^M \exp\left(\,-\tfrac{p^2x^2}{2}\right)\,I_{M-1}\,(cx)\;Q_M\,(ax,b)\;dx \;= \\ =\; & \tfrac{1}{c}\,\left(\tfrac{c}{p^2}\right)^M \exp\left(\tfrac{c^2}{2p^2}\right)\,Q_M\left(\tfrac{ac}{p\sqrt{p^2+a^2}},\;\tfrac{bp}{\sqrt{p^2+a^2}}\right) \end{aligned}\;, \tag{116}$$

$$\begin{aligned} E \;\;=\;\; & \exp\left(\,-\frac{b^2}{2}\right)\sum_{k=-(M-1)}^{M-1}\left(\frac{a}{b}\right)^k \int\limits_0^\infty x^{M+k}\times \\ & \times\exp\left(\,-\frac{x^2\,(p^2\,+\,a^2)}{2}\right)\,I_{M-1}\,(cx)\,I_k\,(abx)\,dx, \end{aligned} \tag{117}$$

The integral from (117) was evaluated in Appendix 3.Thus, by substituting (59) into (117), changing the order of summation and integration and using (107), we obtain

$$
\begin{aligned}
E = {} & \tfrac{1}{c}\left(\tfrac{c}{p^2+a^2}\right)^M \exp\left(\tfrac{c^2-p^2b^2}{2(p^2+a^2)}\right)\sum_{k=-(M-1)}^{M-1}\left(\tfrac{ac}{b(p^2+a^2)}\right)^k \times \\
& I_k\left(\tfrac{abc}{p^2+a^2}\right)\sum_{i=0}^{M-k-1}\begin{pmatrix} M+k+i-1 \\ \\ i \end{pmatrix}\left(\tfrac{a^2}{p^2+a^2}\right)^i .
\end{aligned}
\tag{118}
$$

With usage of (86), (118) can be rewritten as follows:

$$
\begin{aligned}
E = {} & \tfrac{1}{c}\left(\tfrac{c}{p^2}\right)^M \exp\left(\tfrac{c^2-p^2b^2}{2(p^2+a^2)}\right)\sum_{k=-(M-1)}^{M-1}\left(\tfrac{ac}{bp^2}\right)^k I_k\left(\tfrac{abc}{p^2+a^2}\right) \times \\
& \times\left[1-\tfrac{B_z(M-k,M+k)}{B(M-k,M+k)}\right],
\end{aligned}
\tag{119}
$$

where $z=\frac{a^2}{p^2+a^2}$.

It follows from (115) and (116) with accounting (113), that

$$
\begin{aligned}
B-D = {} & \tfrac{1}{c}\left(\tfrac{c}{p^2}\right)^M\left\{\exp\left(\tfrac{c}{p^2}\right)Q_M\left(\tfrac{bp}{\sqrt{p^2+a^2}},\tfrac{ac}{p\sqrt{p^2+a^2}}\right)-\right. \\
& \left.-\exp\left(\tfrac{c^2-p^2b^2}{2(p^2+a^2)}\right)\sum_{k=-(M-1)}^{M-1}\left(\tfrac{ac}{bp^2}\right)^k I_k\left(\tfrac{abc}{p^2+a^2}\right)\right\}.
\end{aligned}
\tag{120}
$$

The substitution of (191) and (120) into (114) brings:

$$
\begin{aligned}
A = {} & \tfrac{1}{c}\left(\tfrac{c}{p^2}\right)^M \exp\left(\tfrac{c}{p^2}\right)\left\{Q_M(U,V)-\exp\left(-\tfrac{U^2+V^2}{2}\right)\times\right. \\
& \left.\times\sum_{k=-(M-1)}^{M-1}\left[1-\tfrac{B_z(M-k,M+k)}{B(M-k,M+k)}\right]\left(\tfrac{V}{U}\right)^k I_k(VU)\right\},
\end{aligned}
\tag{121}
$$

where

$$U=\frac{bp}{\sqrt{p^2+a^2}}, V=\frac{ac}{p\sqrt{p^2+a^2}}, z=\frac{a^2}{p^2+a^2}. \tag{122}$$

The validity of (60) follows from (110), (121) and (122).

Note, that with using of the following property of incomplete beta-function [7], $B_z(x,y) = 1 - B_{(1-z)}(y,x)$, and also (107) and (112), integral (110) can be represented in the following form:

$$\begin{aligned} A &= \frac{1}{c}\left(\frac{c}{p^2}\right)^M \exp\left(\frac{c}{p^2}\right) \times \\ &\times \left[-\beta_0 I_0(UV) + \sum_{k=1}^{M-1} \frac{\alpha_k V^{2k} - U^{2k}}{(UV)^k} I_k(UV) \right] \Bigg\} \end{aligned} \tag{123}$$

where $\alpha_k = \dfrac{B_{(1-z)}(M-k, M+k)}{B(M-k, M+k)}$, $\beta_k = \dfrac{B_z(M-k, M+k)}{B(M-k, M+k)}$ and U, V, z are determined by expression (122).

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A variational approach for correspondence finding between three rectified images

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Abstract

We present a variational approach for the computation of dense correspondence between three rectified images. Using rectification allows reducing the problem to a single parameter functional. Classically, the functional is composed of a data term and a regularization term. We introduce an improved model, where the pixel wise data term is combined with a template matching approach, which is well adapted to the case of rectified images. This modification increases the accuracy significantly.

The minimization is practically handled by solving the Euler-Lagrange equation. In our setting, the Euler-Lagrange equation is replaced by a parabolic evolution equation. Under general assumptions, we prove that this equation has a single asymptotic state, that is obtained by any initial guess.

1 Introduction

Recent works used a variational approach to recover a dense disparity map from a set of two weakly calibrated stereoscopic images [2, 3]. In this context, the computation of this apparent motion, the optical flow, uses the *fundamental matrix* [4], to relate corresponding pixels in the two views. An energy term is minimized, taking into account the geometric constraint defined by the fundamental matrix, as well as the intensity information with an appropriate regularization term. The associated Euler-Lagrange equations allow to solve this minimization problem, and so to find the speed vector.

More images bring more information and so, more accuracy in the results. In this work we shall use the generalization of the fundamental matrix to three views, the trilinear tensor [6]. Moreover, the rectification embeds the geometric information provided by the trifocal tensor. We shall investigate more in depth this approach by using a triplet of rectified images, produced by the method of Zhang [7] and his team, after the work of Hartley [8]. In the framework of the variational approach, it allows us to compute only one Euler-Lagrange equation to solve the problem of minimization.

We shall first in Section 2 present the rectification method we used. Then in Section 3, the minimization problem is presented, with the mathematical foundation. Finally, some experiments are shown in Section 4.

2 Rectification

The use of the process of rectification is of major importance on practical applications. It consists in re-sampling pairs of stereo images taken from different viewpoints in order to produce a pair of "matched epipolar projections". In the case of three images, a conventional disposition is a reference bottom image, a second right image, and finally a top image. Concerning the bottom and right images, there are projections in which the epipolar lines run parallel to the x-axis and consequently, disparities between the images are in the x direction only. In practice, the epipolar lines are to be transformed to lines parallel to the x-axis, then the epipoles of each image should be mapped to points at infinity (with a third coordinate equals to 0). Which can be done with the x-axis works obviously with the y-axis, and so, concerning the bottom and top images, we can map the epipoles to the points at infinity in the same way. The question is to find a good projective transformation H to map the epipoles to points at infinity.

We use here the results of Zhang and his team [7], by posing three constraints on the trinocular rectification. The rectified images are denoted as: $\bar{b}$ (bottom, the bar means rectified image), $\bar{r}$ (right) and $\bar{t}$ (top), respectively. The three constraints can be described as follows: images $\bar{b}$, $\bar{r}$ and $\bar{t}$ are rectified if:

1. All the epipolar lines of images $\bar{b}$ and $\bar{r}$ are horizontal and the corresponding points have the same y-coordinate.
2. All the epipolar lines of images $\bar{b}$ and $\bar{t}$ are vertical and the corresponding points have the same x-coordinate.
3. For any triplet of corresponding points ($\bar{p}_b$, $\bar{p}_r$ and $\bar{p}_t$), the disparity u between $\bar{p}_b$,$\bar{p}_r$ and $\bar{p}_b$,$\bar{p}_t$ is equal.

We have so, for $\bar{p}_b = (x, y)$ in $\bar{b}$, the corresponding point $\bar{p}_r = (x + u, y)$ in $\bar{r}$, and the corresponding point $\bar{p}_t = (x, y + u)$ in $\bar{t}$.

For convenience, the canonical fundamental matrices of the rectified images are used to represent the rectification constraints. Considering the three rectified image pairs $\bar{b}\bar{r}$, $\bar{b}\bar{t}$ and $\bar{r}\bar{t}$, the fundamental matrices to obtain are:

$$\bar{F}_{br} \simeq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \bar{F}_{bt} \simeq \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \bar{F}_{rt} \simeq \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix} \quad (1)$$

where $\simeq$ means "equal up to a non-zero scale".

We need three rectification homographies H_b, H_r, H_t that make the fundamental matrices have the form (1) and can be parametrized by a set of independent free parameters over which it is possible to minimize the distortion. Following [7], we therefore have

$$H_r^T \bar{F}_{br} H_b = \lambda_1 F_{br}, \quad H_t^T \bar{F}_{bt} H_b = \lambda_2 F_{bt}, \quad H_t^T \bar{F}_{rt} H_r = \lambda_3 F_{rt}, \qquad (2)$$

for some non-zero λ_1, λ_2,λ_3. Furthermore, the homographies have to map the epipoles to infinity. This leads to direct expression for the last row of each homography matrix. Thus, the equation set (2) is a linear under-constrained problem with 6 degrees of freedom. One can exploit the fact that the set of solutions is infinite in order to enforce additional geometric constraints, that minimizes the distortion generally caused by the rectification (for more details see [5]).

3 Energy minimization and anisotropic diffusion

3.1 Correspondence finding

Finding dense correspondences between images is a difficult problem. One has to find for each pixel in the first image, where the corresponding pixel is in the second (and third image). Therefore a measure of similarity has to be used in order to maximize the similarity between corresponding pixels. If we denote by $I_j(x, y)$ the intensity of the image j at pixel (x, y), the optimization problem can be formulated as the minimization of $\sum_{i,j}(I_j(x', y') - I_k(x, y))^2$, where (x', y') is the corresponding pixel of (x, y).

This minimization problem can be handled either a local approach [9] or a global approach [10]. It is well known that global methods produce a more consistent motion between nearby pixels, but are less efficient for discontinuities preservation. Therefore in order to improve results obtained by global methods at image discontinuities, Nagel and Enkelmann proposed to replace the isotropic regularization term found in [10], by an anisotropic operator [13]. More details are given in Section 3.3.

In the case, two rectified images are used, these ingredients are combined in [2, 3]. In the sequel, we show how the framework introduced in [2] can be generalized to three rectified images with some further improvement. Since our approach consists in minimizing a global functional on the image, we shall first recall some basic facts.

3.2 Energy minimization

Following [11], we summarize the modeling assumptions used in the sequel and define the matching problem in the context of the calculus of variations. We consider two images intensities $I_1^\sigma = I_1 * G_\sigma$ and $I_2^\sigma = I_2 * G_\sigma$ at a given scale σ, i.e. resulting from the convolution of two square-integrable functions $I_1 : \mathbb{R}^2 \to \mathbb{R}$ and $I_2 : \mathbb{R}^2 \to \mathbb{R}$ with a Gaussian kernel of standard deviation σ. Given a region of interest Ω, a bounded region of $\mathbb{R}^2$, (we may require its boundary $\partial\Omega$ to fulfill some regularity constraints, e.g. that of being of class $\mathcal{C}^2$), we look for a function $u : \Omega \to \mathbb{R}^2$ assigning to each point x in Ω a displacement vector $u(x) \in \mathbb{R}^2$. This function is searched for in a set $\mathbf{F}$ of admissible functions such that it minimizes an energy functional $\mathbf{E} : \mathbf{F} \to \mathbb{R}$ of the form :

$$\mathbf{E}(u) = \mathbf{J}(u) + \mathbf{R}(u) \tag{3}$$

where $\mathbf{J}(u)$ measures the "similarity" between I_1^σ and $I_2^\sigma o(Id + u)$ and $\mathbf{R}(u)$ is a measure of the "irregularity" of u (Id is the identity mapping of $\mathbb{R}^2$).

The similarity term will be defined in terms of global or local statistical measures on the intensities of I_1^σ and $I_2^\sigma o(Id+u)$, and the irregularity term will generally be a measure of the variations of u in Ω. In summary, the matching problem is defined as the solution of the following minimization problem:

$$u^* = \arg\min_{u\in F} \mathbf{E}(u) = \arg\min_{u\in \mathbf{F}} (\mathbf{J}(u) + \mathbf{R}(u)) \tag{4}$$

Assuming that $\mathbf{E}$ is sufficiently regular, its first variation at $u \in \mathbf{F}$ in the direction of $h \in \mathbf{F}$ is defined by :

$$\delta\mathbf{E}(u,h) = \lim_{\varepsilon\to 0} \frac{\mathbf{E}(u+\epsilon h) - \mathbf{E}(u)}{\epsilon} = \frac{d\mathbf{E}(u+\epsilon h)}{d\epsilon}\Big|_{\epsilon=0}$$

If a minimizer u^* of $\mathbf{E}$ exists, then $\delta\mathbf{E}(u^*, h) = 0$ must hold for every $h \in \mathbf{F}$. Assuming that $\mathbf{F}$ is a linear subspace of a Hilbert space H, endowed with a scalar product $(.,.)_{\mathrm{H}}$, we define the gradient $\nabla_{\mathrm{H}}\mathbf{E}(u)$ of $\mathbf{E}$ by requiring that

$$\forall h \in \mathbf{F}, \quad \frac{d\mathbf{E}(u+\epsilon h)}{d\epsilon}\Big|_{\epsilon=0} = (\nabla_{\mathrm{H}}\mathbf{E}(u), h)_{\mathrm{H}}$$

The Euler equations are then equivalent to $\nabla_{\mathrm{H}}\mathbf{E}(u^*) = 0$. Rather than solving them directly, the search for a minimizer of $\mathbf{E}$ is done using a "gradient descent" strategy. Given an initial estimate $u_0 \in \mathrm{H}$, we introduce time and a differentiable function, also noted u from the interval $[0,T]$ into H (we say that $u \in \mathcal{C}_1([0,T];\mathrm{H})$) and we solve the following initial value problem:

$$\begin{cases} \frac{du}{dt} = -\nabla_{\mathrm{H}}\mathbf{E}(u) = -(\nabla_{\mathrm{H}}\mathbf{J}(u) + \nabla_{\mathrm{H}}\mathbf{R}(u)) \\ u(0)(.) = u_0(.) \end{cases} \tag{5}$$

That is, we start from the initial field u_0 and follow the gradient of the functional $\mathbf{E}$ (the minus sign is because we are minimizing). The solution of the matching problem is then taken as the asymptotic state (i.e. when $t \to \infty$) of $u(t)$.

Consequently, equation (5) may be viewed as a first-order ordinary differential equation with values in H. It turns out that studying it from such an abstract viewpoint allows to prove the existence and uniqueness of several types of solutions (mild, strong, classical) of (5), by borrowing tools from functional analysis and the theory of semi-groups of linear operators. In the present Section, we study the generic minimization flow within this abstract framework. The linear operator $-\nabla_{\mathrm{H}}\mathbf{R}(u)$ defined by the regularization term will be simply noted A and the non-linear matching term

$-\nabla_{\mathrm{H}}\mathbf{J}$ will be generically noted F. The unknown of the problem is an H valued function $u : [0, +\infty[\to \mathrm{H}$ defined on $\mathbb{R}^+$.
We want here to establish the properties required by A and F in order for equation (5), which is now written as a semi-linear abstract initial value problem of the form :

$$\begin{cases} \frac{du}{dt} - Au(t) = F(u(t)), \quad t > 0 \\ u(0) = u_0 \in \mathrm{H} \end{cases} \tag{6}$$

to have a unique solution. For this purpose, we shall use the following notations. Let $D(A)$ be the domain of A and S_A be the corresponding semi-group (see [1]).

A continuous solution u of the integral equation

$$u(t) = S_A(t)u_0 + \int_0^t S_A(t-s)F(u(s))ds$$

is called a *mild solution* of the initial value problem (6). The last definition is motivated by the following argument. If (6) has a classical solution then the H valued function $k(s) = S_A(t-s)u(s)$ is differentiable for $0 < s < t$ and, thanks to the theorem claiming that for all $u \in D(A)$, we have $S_A(t)u \in D(A)$ and $\frac{d}{dt}S_A(t)u = AS_A(t)u = S_A(t)Au$ (Theorem 1.2.4 in [12]):

$$\frac{dk}{ds} = -AS_A(t-s)u(s) + S_A(t-s)u'(s)$$

$$= -AS_A(t-s)u(s) + S_A(t-s)Au(s) + S_A(t-s)F(u(s)) = S_A(t-s)F(u(s))$$

If $F \circ u \in \mathcal{L}^1([0, T[, \mathrm{H})$ then $S_A(t-s)F(u(s))$ is integrable and integrating (3.2) from 0 to t yields

$$k(t) - k(0) = u(t) - S_A(t)u_0 = \int_0^t S_A(t-s)F(u(s))ds$$

hence

$$u(t) = S_A(t)u_0 + \int_0^t S_A(t-s)F(u(s))ds$$

The definition of the mild solution is thus natural. Sufficient conditions on A and F for (6) to have a unique mild solution are given by the following theorem.

Theorem 1. *Let $F : H \to H$ be uniformly Lipschitz continuous on H and let $-A$ be a maximal monotone operator. Then the initial value problem* (6) *has a unique mild solution $u \in \mathcal{C}([0, T[, \mathrm{H})$. Moreover, the mapping $u_0 \to u$ is Lipschitz continuous from H into $\mathcal{C}([0, T[, \mathrm{H})$.*

The proof can be found for example in Theorem 6.1.2 of [12]. Since H is a Hilbert space, taking an initial value $u_0 \in D(A)$ suffices to obtain existence and uniqueness of a strong solution.

3.3 Anisotropic Diffusion

The diffusion allows to homogenize a picture. In image processing, the diffusion eliminates the local perturbations of the signal. This is why we use in previous section a convolution with a Gaussian. The inconvenient is that the iterations make the contours more and more blurred. The diffusion has to be uniform far from the contours, and perpendicular to the gradient on the contours, which is meant by "anisotropic". The key idea is so to forbid regularizing and smoothing across the discontinuities. We consider the term $\mathbf{R}(u)$ from equation (3).
The regularization operator we will use was introduced by Nagel and Enkelmann [13], and is expressed as :

$$\mathbf{R}(u) = (\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) \cdot D(\nabla I_1) \cdot \begin{pmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial u}{\partial y} \end{pmatrix} \tag{7}$$

with I_1 the function describing the first image intensity, $D(\nabla I_1)$ a regularized projection matrix perpendicular to ∇I_1:

$$D(\nabla I_1) = \frac{1}{|\nabla I_1|^2 + 2\nu^2} \left[\begin{pmatrix} \frac{\partial I_1}{\partial y} \\ -\frac{\partial I_1}{\partial x} \end{pmatrix} \begin{pmatrix} \frac{\partial I_1}{\partial y} \\ -\frac{\partial I_1}{\partial x} \end{pmatrix}^t + \nu^2 Id \right] \tag{8}$$

Therefore we have the following limit:

$$\lim_{\|\nabla I_1\| \to 0} D(\nabla I_1) = \frac{1}{2} Id$$

On the other hand, when $\|\nabla I_1\| \to \infty$, then $D(\nabla I_1)$ become the projection operator over the direction orthogonal to the gradient.

The resulting effect is that the variation of the displacement is minimized in the direction orthogonal to the gradient of the first image at each pixel. This allows a smoothing effect along the objects boundaries, while preserving these boundaries.

3.4 Euler-Lagrange equation and parabolic equation

We are now in a position to introduce a first instance of the functional that we shall minimize as in the general problem given by (4). An improved model will be introduced in Section 3.5. We consider three rectified images. The rectification assumption is equivalent to assume that the three cameras are organized in a perfect L shape with same distances within the horizontal and vertical pairs. In order to make notation even more concrete and following notations used in Section 2, we shall denote the *rectified* image intensities functions by $\mathtt{b}$, $\mathtt{r}$ and $\mathtt{t}$, which stands for *bottom*, *right* and *top*, the three camera locations in our setting. Since rectification is used, the pixel motion field is described by a single parameter u which a real-valued function of two parameters, as explained in Section 2. Hence the Euler-Lagrange equation will also contain only one unknown. Finally, the energy term to minimize is:

$$\begin{aligned} E(u) = \iint_{\bar{\mathtt{b}}} L(u, \tfrac{\partial u}{\partial x}, \tfrac{\partial u}{\partial y}) dxdy = \quad & \iint_{\bar{\mathtt{b}}} \Big(\|\mathtt{r}(x+u(x,y),y) - \mathtt{b}(x,y)\|^2 \\ & + \|\mathtt{t}(x, y+u(x,y)) - \mathtt{b}(x,y)\|^2 \\ & + \|\mathtt{r}(x+u(x,y),y) - \mathtt{t}(x, y+u(x,y))\|^2 \Big) dxdy \\ & + C \iint_{\bar{\mathtt{b}}} R(u) dxdy, \end{aligned} \tag{9}$$

where C is a constant that and the associated Euler-Lagrange equation is $\frac{\partial L}{\partial u} - \frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial L}{\partial u_y}) = 0$.

The part term $-\frac{\partial}{\partial x}(\frac{\partial L}{\partial u_x}) - \frac{\partial}{\partial y}(\frac{\partial L}{\partial u_y})$ is related to the regularization operator. After computation, it is given by $C.div(D(\nabla \mathtt{b})\nabla u) = 0$. Thus, the following equation has to be solved :

$$\begin{aligned} & 2\tfrac{\partial \mathtt{r}}{\partial x}(x+u,y)\big[2\mathtt{r}(x+u,y) - \mathtt{t}(x,y+u) - \mathtt{b}(x,y)\big] \\ & +2\tfrac{\partial \mathtt{t}}{\partial y}(x,y+u)\big[2\mathtt{t}(x,y+u) - \mathtt{r}(x+u,y) - \mathtt{b}(x,y)\big] \\ & -Cdiv(D(\nabla \mathtt{b})\nabla u) = 0 \end{aligned} \tag{10}$$

We have now to compute the asymptotic state when $t \to \infty$ of the corresponding parabolic equation:

$$\begin{array}{rcl} \frac{\partial u}{\partial t} & = & -2\frac{\partial \mathtt{r}}{\partial x}(x+u,y)\big[2\mathtt{r}(x+u,y)-\mathtt{t}(x,y+u)-\mathtt{b}(x,y)\big] \\ & & -2\frac{\partial \mathtt{t}}{\partial y}(x,y+u)\big[2\mathtt{t}(x,y+u)-\mathtt{r}(x+u,y)-\mathtt{b}(x,y)\big] \\ & & +Cdiv(D(\nabla \mathtt{b})\nabla u) \end{array} \tag{11}$$

We will prove here the convergence to this asymptotic state and its uniqueness, by using the theorem of Section3.2. We shall look for a solution u in $H = L^2(\mathbb{R}^2)$. Let $F : H \longrightarrow H$ be defined as follows:

$$\begin{array}{rcl} F(u) & = & -2\frac{\partial \mathtt{r}^\sigma}{\partial x}(x+u,y)\big[2\mathtt{r}^\sigma(x+u,y)-\mathtt{t}^\sigma(x,y+u)-\mathtt{b}^\sigma(x,y)\big] \\ & & -2\frac{\partial \mathtt{t}^\sigma}{\partial x}(x,y+u)\big[2\mathtt{t}^\sigma(x,y+u)-\mathtt{r}^\sigma(x+u,y)-\mathtt{b}^\sigma(x,y)\big] \end{array}$$

Moreover let $A : \mathcal{D}(A) \subset H \longrightarrow H$ be the differential operator defined as follows:

$$A(u) = -Cdiv(D(\nabla \mathtt{b}^\sigma)\nabla u)$$

Then the system obtained by blurring (11) can be written in a compact form, together with the initial guess, and has actually the form of equation (6):

$$\left\{ \begin{array}{l} \frac{\partial u}{\partial t} + A(u) = F(u) \\ u(0) = u_0 \end{array} \right.$$

The image functions are defined in a bounded domain. For simplicity, we shall consider them as defined on the whole plane. More, we shall consider that $\mathtt{b}, \mathtt{r}, \mathtt{t} \in L^2(\mathbb{R}^2)$, which is physically amply justified.

Theorem 2. *Under the assumption that* $\mathtt{b}, \mathtt{r}, \mathtt{t} \in L^2(\mathbb{R}^2)$*, the function* F *is Lipschitz-continuous, and the Lipschitz constant* L *depends on the functions* $\mathtt{b}, \mathtt{r}, \mathtt{t}$ *and* σ*.*

Proof. Since $\mathtt{b}, \mathtt{r}, \mathtt{t} \in L^2(\mathbb{R}^2)$, we have $\mathtt{r}^\sigma, \mathtt{t}^\sigma \in W^{1,2}(\mathbb{R}^2)$ and $\mathtt{b}^\sigma, \mathtt{t}^\sigma \in L^\infty(\mathbb{R}^2)$, where $W^{1,2}(\mathbb{R}^2)$ denotes the Sobolev space:

$$W^{1,2}(\mathbb{R}^2) = \left\{ \begin{array}{c|l} u \in L^2(\mathbb{R}^2) & \begin{array}{l} \exists g_1, g_2 \in L^2(\mathbb{R}^2), \forall \phi \in C_c^\infty(\mathbb{R}^2), \\ \int_{\mathbb{R}^2} \frac{\partial u}{\partial x}\phi' = -\int_{\mathbb{R}^2} g_1\phi, \\ \int_{\mathbb{R}^2} \frac{\partial u}{\partial y}\phi' = -\int_{\mathbb{R}^2} g_2\phi \end{array} \end{array} \right\},$$

where $C_c^\infty(\mathbb{R}^2)$ denotes the space of infinitely derivablepoints at infinity functions that are zero outside a compact domain.

Thus it clear that $\mathtt{r}^\sigma, \mathtt{t}^\sigma \in W^{1,2}(\mathbb{R}^2)$, because after the smoothing with a Gaussian filter, we obtain infinity derivable functions. Smoothing the images, which are basically bounded, also results in bounded functions. Therefore $\mathtt{b}^\sigma, \mathtt{t}^\sigma \in L^\infty(\mathbb{R}^2)$.

In the sequel, we shall denote $F_1(u) = -2\frac{\partial \mathtt{r}^\sigma}{\partial x}(x+u,y)\big[2\mathtt{r}^\sigma(x+u,y) - \mathtt{t}^\sigma(x,y+u) - \mathtt{b}^\sigma(x,y)\big]$ and $F_2(u) = -2\frac{\partial \mathtt{t}^\sigma}{\partial x}(x,y+u)\big[2\mathtt{t}^\sigma(x,y+u) - \mathtt{r}^\sigma(x+u,y) - \mathtt{b}^\sigma(x,y)\big]$, so that $F(u) = F_1(u) + F_2(u)$.Now, consider $u_1, u_2 \in H$. We have the following estimate (the norm $\parallel . \parallel$ is $\parallel . \parallel_{L^2(\mathbb{R}^2)}$) $\parallel F(u_1) - F(u_2) \parallel \leq \parallel F_1(u_1) - F_1(u_2) \parallel + \parallel F_2(u_1) - F_2(u_2) \parallel$. Therefore let us first exhibit an upper bound of $\parallel F_1(u_1) - F_1(u_2) \parallel$.

$$
\begin{aligned}
&\parallel F_1(u_1) - F_2(u_2) \parallel = \\
&\parallel 2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y)\big[2\mathtt{r}^\sigma(x+u_1,y) - \mathtt{t}^\sigma(x,y+u_1) - \mathtt{b}^\sigma(x,y)\big] - \\
&2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y)\big[2\mathtt{r}^\sigma(x+u_2,y) - \mathtt{t}^\sigma(x,y+u_2) - \mathtt{b}^\sigma(x,y)\big] \parallel \\
&\leq \parallel 4\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y)\mathtt{r}^\sigma(x+u_1,y) - 4\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y)\mathtt{r}^\sigma(x+u_2,y) \parallel \\
&+ \parallel \mathtt{b}^\sigma(x,y)\big[2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y) - 2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y)\big] \parallel \\
&+ \parallel 2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y)\mathtt{t}^\sigma(x,y+u_1) - 2\tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y)\mathtt{t}^\sigma(x,y+u_2) \parallel \\
&\leq 2 \parallel \tfrac{\partial (\mathtt{r}^\sigma)^2}{\partial x}(x+u_1,y) - \tfrac{\partial (\mathtt{r}^\sigma)^2}{\partial x}(x+u_2,y) \parallel \\
&+2 \parallel \mathtt{b}^\sigma \parallel_\infty \parallel \tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y) - \tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y) \parallel \\
&+2 \parallel \mathtt{t}^\sigma \parallel_\infty \parallel \tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_1,y) - \tfrac{\partial \mathtt{r}^\sigma}{\partial x}(x+u_2,y) \parallel \\
&\leq 2C_{lip}(\tfrac{\partial (\mathtt{r}^\sigma)^2}{\partial x}) \parallel u_1 - u_2 \parallel +2 \parallel \mathtt{b}^\sigma \parallel_\infty C_{lip}(\tfrac{\partial \mathtt{r}^\sigma}{\partial x}) \parallel u_1 - u_2 \parallel \\
&+2 \parallel \mathtt{t}^\sigma \parallel_\infty C_{lip}(\tfrac{\partial \mathtt{r}^\sigma}{\partial x}) \parallel u_1 - u_2 \parallel \\
&\leq 2(C_{lip}(\tfrac{\partial (\mathtt{r}^\sigma)^2}{\partial x}) + \parallel \mathtt{b}^\sigma \parallel_\infty C_{lip}(\tfrac{\partial \mathtt{r}^\sigma}{\partial x}) + \parallel \mathtt{t}^\sigma \parallel_\infty C_{lip}(\tfrac{\partial \mathtt{r}^\sigma}{\partial x})) \parallel u_1 - u_2 \parallel,
\end{aligned}
$$

where $C_{lip}(f)$ is the Lipschitz constant of the function f. It is clear that $C_{lip}(\frac{\partial (\mathtt{r}^\sigma)^2}{\partial x})$ and $C_{lip}(\frac{\partial \mathtt{r}^\sigma}{\partial x})$ exists since $\frac{\partial (\mathtt{r}^\sigma)^2}{\partial x}, \frac{\partial \mathtt{r}^\sigma}{\partial x} \in W^{1,2}(\mathbb{R}^2)$.

Eventually, a Lipschitz constant of F_1 is given by :

$$
C_{lip}(F) = 2(C_{lip}(\frac{\partial (\mathtt{r}^\sigma)^2}{\partial x}) + \parallel \mathtt{b}^\sigma \parallel_\infty C_{lip}(\frac{\partial \mathtt{r}^\sigma}{\partial x}) + \parallel \mathtt{t}^\sigma \parallel_\infty C_{lip}(\frac{\partial \mathtt{r}^\sigma}{\partial x})).
$$

Similarly, we could found a Lipschitz constant of F_2: points at infinity

$$C_{lip}(F) = 2(C_{lip}(\frac{\partial(\mathtt{t}^\sigma)^2}{\partial x}) + \| \mathtt{b}^\sigma \|_\infty C_{lip}(\frac{\partial \mathtt{t}^\sigma}{\partial x}) + \| \mathtt{r}^\sigma \|_\infty C_{lip}(\frac{\partial \mathtt{t}^\sigma}{\partial x})).$$

A Lipschitz constant for F is simply the sum of the two previous constant. $\square$

Since $\mathtt{b} \in L^2(\mathbb{R}^2)$, then $\mathtt{b}^\sigma \in W^{1,\infty}(\mathbb{R}^2)$. Thus $\nabla \mathtt{b}^\sigma$ is bounded and the eigenvalues of $D(\nabla \mathtt{b}^\sigma)$ are strictly positive. Therefore since $C > 0$, the operator A is a maximal monotone operator. Then using, theorem 1, we can conclude by the following result.

Theorem 3. *The system* (6) *has a unique mild solution for any initial guess.*

3.5 An improved model

The energy defined by expression (9) uses a mere difference between the intensities of corresponding pixels. In order to make the correspondence finding more robust, the use of a local region around each pixel is necessary. In [14], this is handled by requiring not only the intensities constancy but also the gradient constancy. The drawback of this approach is that it requires the computation of second order image derivatives in the Euler-Lagrange equation. Here we adopt another approach, the data term

$$\begin{array}{l} \iint_{\bar{b}} \Big(\|\mathtt{r}(x+u(x,y),y) - \mathtt{b}(x,y)\|^2 + \|\mathtt{t}(x,y+u(x,y)) - \mathtt{b}(x,y)\|^2 \\ + \|\mathtt{r}(x+u(x,y),y) - \mathtt{t}(x,y+u(x,y))\|^2 \Big) dxdy \end{array}$$

is replaced by

$$\begin{array}{l} \iint_{\bar{b}} \sum_{i=-n}^{i=n} \sum_{j=-n}^{j=n} \Big(\|\mathtt{r}(x+u(x,y)+i,y+j) - \mathtt{b}(x+i,y+j)\|^2 \\ + \|\mathtt{t}(x+i,y+u(x,y)+j) - \mathtt{b}(x+i,y+j)\|^2 \\ + \|\mathtt{r}(x+u(x,y)+i,y+j) - \mathtt{t}(x+i,y+u(x,y)+j)\|^2 \Big) dxdy \end{array},$$

where $2n+1$ is the size of the square neighborhood of each pixel. We found that using $2n+1 = 3$ yields optimal results. It is straight forward to see that the proof in theorem 2 also holds for this improved model. Therefore theorem 3 is also valid.

3.6 Numeric scheme

For the numerical solution for the system (6), we shall use a finite difference scheme. In this section, we shall denote by (i, j) the grid vertexes on which we compute the flow. The derivative with respect to time is given by:

$$\frac{u_{ij}^{k+1} - u_{ij}^{k}}{\tau},$$

where τ is the time step. Following [2], we use the following scheme for the regularization term:

$$div(D(\nabla \mathtt{b}^\sigma)\nabla u) \approx \begin{array}{l} \frac{a_{i+1,j}+a_{i,j}}{2}\frac{u_{i+1,j}^{k+1}-u_{i,j}^{k+1}}{h_1^2} + \frac{a_{i-1,j}+a_{i,j}}{2}\frac{u_{i-1,j}^{k+1}-u_{i,j}^{k+1}}{h_1^2} \\ +\frac{c_{i,j+1}+c_{i,j}}{2}\frac{u_{i,j+1}^{k+1}-u_{i,j}^{k+1}}{h_2^2} + \frac{c_{i,j-1}+c_{i,j}}{2}\frac{u_{i,j-1}^{k+1}-u_{i,j}^{k+1}}{h_2^2} \\ +\frac{b_{i+1,j+1}+b_{i,j}}{2}\frac{u_{i+1,j+1}^{k+1}-u_{i,j}^{k+1}}{2h_1h_2} + \frac{b_{i-1,j-1}+b_{i,j}}{2}\frac{u_{i-1,j-1}^{k+1}-u_{i,j}^{k+1}}{2h_1h_2} \\ -\frac{b_{i+1,j-1}+b_{i,j}}{2}\frac{u_{i+1,j-1}^{k+1}-u_{i,j}^{k+1}}{2h_1h_2} - \frac{b_{i-1,j+1}+b_{i,j}}{2}\frac{u_{i-1,j+1}^{k+1}-u_{i,j}^{k+1}}{2h_1h_2} \end{array},$$

where $D(\nabla \mathtt{b}^\sigma) = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$.

Each component of the data similarity term in the first component term is approximated as follows:

$$\begin{array}{l} -2\frac{\partial \mathtt{r}^\sigma}{\partial x}(x+u,y)\big[2\mathtt{r}^\sigma(x+u,y) - t^\sigma(x,y+u) - \mathtt{b}^\sigma(x,y)\big] \approx \\ -2\frac{\partial \mathtt{r}^\sigma}{\partial x}(i+u_{ij}^k,j)\big[2\mathtt{r}^\sigma(i+u_{ij}^k,j) + 2(u_{ij}^{k+1}-u_{ij}^k)\frac{\partial \mathtt{r}^\sigma}{\partial x}(i+u_{ij}^k,j) \\ -\mathtt{t}(i,j+u_{ij}^k) - (u_{ij}^{k+1}-u_{ij}^k)\frac{\partial \mathtt{t}}{\partial y}(i,j+u_{ij}^k) - \mathtt{b}^\sigma(i,j)\big] \end{array}$$

The values for the images and their derivatives outside of the grid points are computed by interpolation. Eventually, we get an implicit scheme that leads to solve a large sparse linear system: $Ax = b$. For solving this system, since the matrix A is typically very large and sparse, an iterative method is used. We use the Gauss-Seidel iteration. If $A = L + D + U$, where L (respectively U) is a strictly lower (respectively upper) triangular matrix, while D is a diagonal matrix, then the iteration is performed as follows:

$$x^{n+1} = -(D-L)^{-1}Ux^n + (D-L)^{-1}b$$

Moreover in order to speed up the computations and to handle large displacement, we embed the scheme into a mutliresolution approach [10].

4 Experiments

We present here the results obtained with the improved model over three rectified images. Fig. 1 presents the three rectified images.

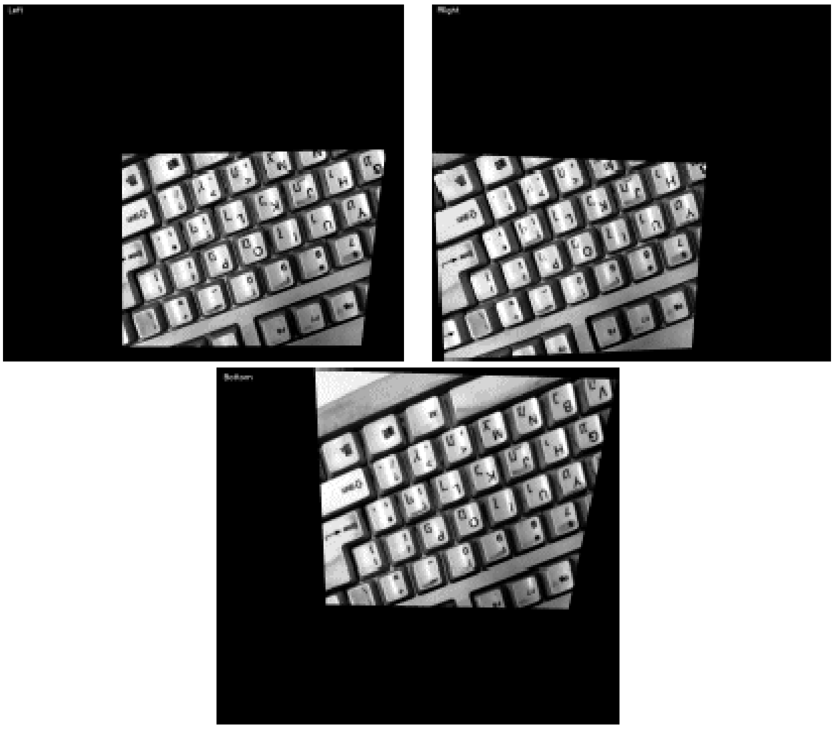

Figure 1: Three rectified images.

In Fig. 2 we show the results. Fig. 2a, the disparity is rendered as a gray-level images, while Figs. 2b and 2c shows the respectively the second and the third image after backward warping toward the first image, using the computed disparity.

The quality of these results is more precisely handled by some statistical data. In the following measures, the images are normalized so that the gray-levels values are between 0 and 1. Thus when we come to compare the original image 1 and the re-sampling of images 2 and 3 based the computed disparity, the two difference images have values between -1 and 1 at most. For the difference computed with the first and the second image, we found the following values: the minimum is -0.992157, the maximum

0.996078, while the mean value and the standard deviation are respectively $\mu_{12} = 0.0119701$ and $\sigma_{12} = 0.121034$. For the difference computed using the first and third images , we found similar values: the minimum and maximum are stricly identical, while the mean value and the standard deviation are respectively $\mu_{13} = 0.0138003$ and $\sigma_{13} = 0.102947$. In both cases, the statistical measures show that the computed disparity yields a high quality resampling.

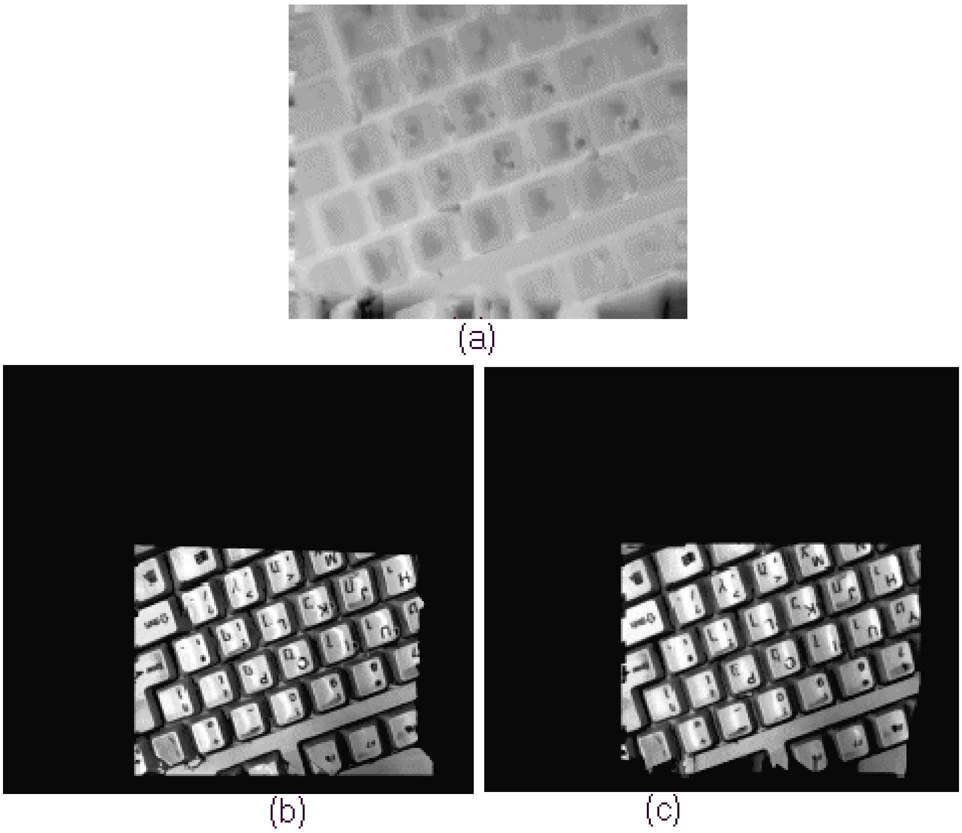

Figure 2: The results computed with the improved model. (a) The rendered disperity; (b) the warping of the second image; (c) the warping of the third image.

5 Future work and discussion

We have present a complete and mathematical consistent way to handle the problem of correspondence finding between three images. Future work will incorporate more efficient numeric techniques like multigrid [15]. We shall also investigate the way to deal with more than three images at once.

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Optimal feature space for representation of finite dimensional data

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Abstract

The paper develops a recent Coifman's approach to the problems of Signal and Image Processing, based on spectral theory of operators in Hilbert spaces. New notion of *canvas* is introduced. This allows to use powerful tools of Operator Ideal Theory which, in turn, leads to a numerical algorithm for restoring the original image.

1 Introduction

Recently R. Coifman, see in particular, [1], proposed a new approach to reducing dimension of data appeared in image processing. He used for this goal a new function space generated by a few geometric harmonics, i.e., eigenfunctions of some positive integral operator acting in an infinite-dimensional Hilbert space. These eigenfunctions can be found as extreme points of the corresponding quadratic form on the Hilbert space. In the present paper we suggest a modification of the above Coifman idea in order to find the function spaces for data representation with optimal, in some sense, dimension. We make this by solving a series of extreme points problems which increase the require dimension step to step. To avoid technical complications we consider here only a simple model related to the space of functions continuous on $\mathbb{R}^n$.

2 A feature space or canvas

Definition 1. *A Hilbert space H embedded into the space $C(\mathbb{R}^n)$ of bounded continuous functions on $\mathbb{R}^n$ is called a canvas.*

Every canvas generates a mapping from $\mathbb{R}^n$ to the space H as follows. Each $q \in \mathbb{R}^n$ generates the valuation functional $\delta_q \in C(\mathbb{R}^n)^*$ whose restriction to H can be represented by the F.Riesz theorem as

$$f(q) = \delta_q(f) = \langle f, a(q)\rangle_H, \quad f \in H,$$

here $a : \mathbb{R}^n \mapsto H$ is a weak continuous bounded vector function, which gives the required mapping. Conversely, every such a generates some canvas; namely, the required imbedding of H to $C(\mathbb{R}^n)$ is defined by sending an element $h \in H$ to the continuous function $q \mapsto \langle h, a(q)\rangle_H$. Hence, $a(q)$ may be seen as δ-function on the space H.

Examples of canvases.

1^o. The most natural example of a canvas is the Sobolev space $W_2^s(\mathbb{R}^n)$ with the smoothness $s > n/2$, since $W_2^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$.

2^o. R. Coifman [1] considers the space of bandlimited functions $L_2^\delta(\mathbb{R}^n)$ consisting of square integrable functions on $\mathbb{R}^n$, whose Fourier transforms are supported by the ball of radius δ. Clearly, this space is a canvas.

3^o. A finite dimensional subspace in $C(\mathbb{R}^n)$ can be regarded as a canvas. For this end it is necessary to fix some scalar product on such a space.

4^o. The following example is also due to R. Coifman. Let $K(s,t)$ be positive definite bounded continuous function on $\mathbb{R}^n \times \mathbb{R}^n$. Consider the linear span H_K of the family of functions $\{K(\cdot,t) : t \in \mathbb{R}^n\}$. This can be related to a bilinear form by the following formula.

If $x(q) = \sum_{i=1}^n \lambda_i K(q, s_i)$ and $y(q) = \sum_{j=1}^m \mu_j K(q, t_j)$ then

$$\langle x, y\rangle = \langle \sum_{i=1}^n \lambda_i K(\cdot, s_i), \sum_{j=1}^m \mu_j K(\cdot, t_j)\rangle$$

$$= \sum_{i,j=1}^{n,m} \lambda_i \mu_j K(s_i, t_j).$$

It is easily seen that this definition is well-defined and that $\langle \cdot, \cdot \rangle_H$ is a semidefinite.

Let us show that actually we obtain a scalar product on H_K and that H_K is continuously imbedded into $C(\mathbb{R}^n)$.

Consider a vector-valued function $a : \mathbb{R}^n \to H_K$ defined by

$$a(q)(\cdot) = K(q, \cdot),$$

where $q \in \mathbb{R}^n$. By the definition, $\|a(q)\|^2_{H_K} = \langle K(q, \cdot), K(q, \cdot)\rangle = K(q, q) \leq C$. The space H_K is embedded into $C(\mathbb{R}^n)$ by $x \mapsto \langle x, a(q)\rangle$ with the estimate

$$\|\langle x, a(\cdot)\rangle\|_{C(\mathbb{R}^n)} \leq C\|x\|_{H_K}.$$

which follows from the Schwartz inequality. Moreover, the map $x \mapsto \langle x, a(q)\rangle$ is the identity. Indeed,

$$\langle x, a(q)\rangle = \langle \sum_{i=1}^n \lambda_i K(\cdot, s_i), K(\cdot, q)\rangle = \sum_{i=1}^n \lambda_i K(q, s_i) = x(q).$$

Hence, this bilinear form is not degenerate and actually defines the scalar product on H_K. We denote the completion of H_K with respect to the corresponding norm by the same symbol H_K.

The example 4^o is actually universal because any continuous bounded vector-valued function $a : \mathbb{R}^n \to H$ determines the function $K(s, t) = \langle a(s), a(t)\rangle$ which is a positive definite continuous and bounded on $\mathbb{R}^n \times \mathbb{R}^n$. The corresponding space H_K is isometrically isomorphic to the linear span of the family of vectors $\{a(q)\}$ in H, and corresponding vectors are identified with the same functions of $C(\mathbb{R}^n)$ as follows. Since

$$\|\sum_{i=1}^n \lambda_i a(s_i)\|^2_H = \sum_{i,j=1}^n \lambda_i \lambda_j \langle a(s_i), a(s_j)\rangle$$

$$= \sum_{i,j=1}^n \lambda_i \lambda_j K(s_i, s_j)$$

$$= \langle \sum_{i=1}^n \lambda_i K(\cdot, s_i), \sum_{j=1}^n \lambda_i K(\cdot, s_j)\rangle = \|\sum_{i=1}^n \lambda_i K(\cdot, s_i)\|^2_{H_K},$$

we identify $\sum_{i=1}^n \lambda_i a(s_i)$ with $\sum_{i=1}^n \lambda_i K(\cdot, s_i)$.

3 Representation of data

We present the data as the Radon measure μ on the set Γ.

Let Γ be a compact set in $\mathbb{R}^n$ and μ be a finite Radon measure on Γ. As usual we denote by $L_2(\mu)$ the Hilbert space of measurable functions on Γ such that

$$\int_\Gamma |f(q)|^2 d\mu(q) < \infty.$$

Let H be a canvas. The canvas H is naturally mapped into $L_2(\mu)$ through $L_\infty(\mu)$. We denote this map by $I_\mu : H \to L_2(\mu)$. If we identify the elements of H with the corresponding functions from $C(\mathbb{R}^n)$, then I_μ is the operator of restriction of functions to Γ.

Thus we have

$$I_\mu : H \to L_\infty(\mu) \subset L_2(\mu) \tag{1}$$

The embedding (1) implies that I_μ is a Hilbert–Schmidt operator (see [3]).

The adjoint operator

$$I_\mu^* : L_2(\mu) = L_2(\mu)^* \subset L_\infty(\mu)^* \to H^* = H$$

is also a Hilbert–Schmidt operator. Hence the product

$$H_\mu = I_\mu I_\mu^* : L_2(\mu) \subset L_\infty(\mu)^* \to H \subset L_\infty(\mu) \to L_2(\mu)$$

is a positive trace-class operator in $L_2(\mu)$.

Denote by G_μ the product

$$G_\mu = I_\mu^* I_\mu : H \to L_\infty(\mu) \subset L_2(\mu) \subset L_\infty(\mu)^* \to H,$$

which is also a trace-class operator. Note that if I_μ is an embedding, then Ker $G_\mu = 0$.

Recall that any canvas is generated by a vector-valued function $a(q)$ or a positive definite kernel $K(s,t) = \langle a(s), a(t)\rangle = a(s)(t)$. Our aim now is to present the operators H_μ and G_μ in terms of these functions.

By definition I_μ takes h to $\langle h, a(q)\rangle$ in $L_2(\mu)$, hence I_μ^* takes each function $f(q) \in L_2(\mu)$ to the element which generates the functional

$$\alpha(h) = \int_\Gamma \langle h, a(q)\rangle f(q) d\mu.$$

Hence

$$I_\mu^*(f) = \int\limits_\Gamma f(q)a(q)d\mu(q) \qquad \text{(weak convergence).}$$

Thereby

$$H_\mu(f)(t) = I_\mu I_\mu^*(f) = \langle \int\limits_\Gamma f(s)a(s)d\mu(s), a(t)\rangle \tag{2}$$

$$= \int\limits_\Gamma \langle a(s), a(t)\rangle f(s)d\mu(s) = \int\limits_\Gamma K(s,t)f(s)d\mu(s).$$

From the other side

$$\langle G_\mu(h), g\rangle = \langle I_\mu^* I_\mu(h), g\rangle = \langle I_\mu(h), I_\mu(g)\rangle_{L_2(\mu)}$$

$$= \int\limits_\Gamma \langle a(s), h\rangle \langle a(s), g\rangle d\mu(s)$$

$$= \int\limits_\Gamma \langle (a(s) \otimes a(s))h, g\rangle d\mu(s),$$

where $a \otimes b$ denotes the rank one operator in the space H defined by $(a \otimes b)(h) = \langle h, a\rangle b$.

In other words

$$G_\mu = \int\limits_\Gamma a(s) \otimes a(s)d\mu(s) \qquad \text{(weak convergence),} \tag{3}$$

i.e, G_μ is an integral of rank one operators.

Since $h(s) = \langle h, a(s)\rangle$ and $g(s) = \langle g, a(s)\rangle$ we also have

$$\langle G_\mu(h), g\rangle = \int\limits_\Gamma h(s)\overline{g(s)}d\mu(s). \tag{4}$$

It is well known that $H_\mu = I_\mu I_\mu^*$ and $G_\mu = I_\mu^* I_\mu$ are similar or metrically equivalent (non-zero eigenvalues of H_μ and G_μ are equal and have the same multiplicity).

Both these operators may be used for estimation of the quality of representation of the data given by the measure μ on the canvas H. However if we want to compare two representations on two different canvases H' and H'', we compare the operators H'_μ and H''_μ. If we want to compare representations of two different data μ' and μ'', we use $G_{\mu'}$ and $G_{\mu''}$.

4 Geometric harmonics

The eigenvectors of the operator G_μ, corresponding to non-zero eigenvalues, were called by R. Coifman by geometric harmonics. Denote by λ_j the eigenvalues of the operator G_μ indexed by the standard manner such that $\lambda_1 \geq \dots \geq \lambda_j \geq \lambda_{j+1} \geq \dots$ including multiplicity. The corresponding orthogonal unit eigenvectors will be denoted by ψ_j. Recall that ψ_j are functions from $C(\mathbb{R}^n)$ since $H \subset C(\mathbb{R}^n)$. Since G_μ is a trace class operator, we have

$$\sum_{j=1}^{\infty} \lambda_j = \mathrm{tr}(G_\mu) = \int\limits_\Gamma \mathrm{tr}(a(s) \otimes a(s)) d\mu(s) = \int\limits_\Gamma K(s,s) d\mu(s) < \infty$$

because of (3).

The operator H_μ has the same non-zero eigenvalues λ_j and the corresponding orthogonal unit eigenvectors will be denoted by $\varphi_j \in L_2(\mu)$.

Consider the Schmidt expansion of I_μ

$$I_\mu = \sum_{j=1}^{\mathrm{rank}\ I_\mu} \lambda_j^{1/2} \langle \cdot, \psi_j \rangle \varphi_j, \tag{5}$$

where ψ_j is an orthogonal unit sequence in H, and φ_j is orthogonal unit sequence in $L_2(\mu)$ (see [2]) .

Hence $I_\mu(\psi_j) = \lambda_j^{1/2} \varphi_j$, which means that the restriction of ψ_j on Γ is multiple of φ_j. Thus ψ_j/λ_j may be considered as an extension of φ_j from Γ to $\mathbb{R}^n$.

Since the Schmidt expansion for the adjoint operator I_μ^* has a form

$$I_\mu^* = \sum_{j=1}^{\mathrm{rank}\ I_\mu} \lambda_j^{1/2} \langle \cdot, \varphi_j \rangle_{L_2(\mu)} \psi_j, \tag{6}$$

we have $I_\mu^*(\varphi_j) = \lambda_j^{1/2} \psi_j$. Hence

$$\psi_j = \frac{1}{\lambda_j^{1/2}} I_\mu^*(\varphi_j) = \frac{1}{\lambda_j^{1/2}} \int\limits_\Gamma \varphi_j(s) a(s) d\mu(s),$$

or

$$\psi_j(q) = \frac{1}{\lambda_j^{1/2}} \int\limits_\Gamma \varphi_j(s) a(s)(q) d\mu(s) = \frac{1}{\lambda_j^{1/2}} \int\limits_\Gamma \varphi_j(s) \langle a(s), a(q) \rangle d\mu(s)$$

$$= \frac{1}{\lambda_j^{1/2}} \int\limits_{\Gamma} \varphi_j(s) K(s,q) d\mu(s),$$

which gives the formula for the extension of $\varphi_j(s)$.

5 Reduction of the canvas dimension

In this section we turn to the problem of low dimension representation of the data. Given a fixed representation on the canvas H we try to find a new canvas F of the smallest dimension such that distortion between the new representation and the old one doesn't exceed fixed number $\varepsilon > 0$. We are going to measure the distance between two representations as the operator norm of the difference $H_\mu - F_\mu$.

Theorem 1. *Let H be an arbitrary canvas and $\varepsilon > 0$. The linear span F of the first geometric harmonics ψ_j, corresponding to the eigenvalues $\lambda_j > \varepsilon$, is the canvas of the smallest dimension such that*

$$\|H_\mu - F_\mu\|_{L(L_2(\mu))} \leq \varepsilon.$$

Proof. Denote by n_ε the smallest j such that $\lambda_{n_\varepsilon+1} \leq \varepsilon$. Then $\lambda_1 \geq \cdots \lambda_{n_\varepsilon} > \varepsilon$ and F is the linear span of $\psi_1, \cdots, \psi_{n_\varepsilon}$. Denote by $a_F(q)$ representing function, corresponding to the canvas F.

We evidently have

$$a_F(q) = \sum_{j=1}^{n_\varepsilon} \langle a_F(q), \psi_j \rangle \psi_j,$$

and since $\langle \psi_j, a_F(q) \rangle = \psi_j(q)$ we conclude

$$a_F(q)(s) = \sum_{j=1}^{n_\varepsilon} \langle a_F(q), \psi_j \rangle \psi_j(s) = \sum_{j=1}^{n_\varepsilon} \psi_j(q) \psi_j(s).$$

Thus the kernel, corresponding to the canvas F, denoted by $K_F(s,t)$ is equal to

$$\sum_{j=1}^{n_\varepsilon} \psi(t) \psi_j(s),$$

and

$$F_\mu(f)(q) = \int\limits_{\Gamma} \sum_{j=1}^{n_\varepsilon} \psi_j(q) \psi_j(s) f(s) d\mu(s),$$

because of (1).

Since $\lambda_j^{1/2}\psi_j|_\Gamma = \varphi_j$ we obtain

$$F_\mu(f)(q) = \int\limits_\Gamma \sum_{j=1}^{n_\varepsilon} \lambda_j \varphi_j(q)\varphi_j(s) f(s) d\mu(s).$$

Therefore

$$[H_\mu - F_\mu](f)(q) = \int\limits_\Gamma \sum_{j=n_\varepsilon+1}^{\text{rank } I_\mu} \lambda_j \varphi_j(q)\varphi_j(s) f(s) d\mu(s),$$

and we conclude that

$$\|H_\mu - F_\mu\|_{L(L_2(\mu))} \le \varepsilon.$$

Recall now that eigenvalues of the operator H_μ satisfy to the approximation property due to G.Allahverdiev (see [2])

$$\lambda_{j+1} = \min \|H_\mu - K\|, \tag{7}$$

where minimum is taken over all bounded linear operators K with the rank less than or equal to j.

Let now $\widetilde{F}$ be a canvas of dimension n such that $\|H_\mu - \widetilde{F}_\mu\| \le \varepsilon$. The rank of $\widetilde{F}_\mu$ is less than or equal to n, therefore (7) yields $\lambda_{n+1} \le \varepsilon$. Hence by our definition of n_ε we obtain $n \ge n_\varepsilon$. Thus n_ε is the smallest dimension for representation of the measure μ with the distortion ε. Theorem is proved.

Thus we see that the problem of finding an optimal finite dimensional representation is reduced to the problem of finding the geometric harmonics and the corresponding eigenvalues of the operator H_μ.

If we have opportunity to find the norm of a positive compact operator and the corresponding extreme point we can find the optimal canvas F such that $\|H_\mu - F_\mu\| \le \varepsilon$ for any $\varepsilon > 0$ by the following procedure.

Step 1. Set $F = 0$.

Step 2. Find

$$m := \max_{f \in H \ominus F,\ \|f\|=1} \langle G_\mu f, f \rangle = \|f\|^2_{L_2(\mu)}$$

and a maximum point $g \in H \ominus F$.

Step 3. If $m \le \varepsilon$, then STOP. If $m > \varepsilon$, then $F := F \oplus \{\lambda g\}$.

Step 4. GOTO Step 1.

The final space F is generated by the first harmonics $\psi_1, \cdots \psi_{n_\varepsilon}$, since m consequently gives us λ_j, and g gives us the corresponding ψ_j.

If we have an opportunity for any given ε to find a unit vector g such that $\|Ag\| > \varepsilon$, or to show that

$$\max_{f \in H,\ \|f\|=1} \|Af\| \leq \varepsilon,$$

then we can find the dimension of the optimal canvas with the help of the following algorithm.

Again we denote by F a finite dimensional subspace in H.

Step 1. Set $F = 0$.

Step 2. If we find unit vector $g \in H \ominus F$ such that

$$\|g\|_{L_2(\mu)} > \varepsilon,$$

then $F := F \oplus \{\lambda g\}$, if for all $f \in H \ominus F$, we have $\|f\|_{L_2(\mu)} \leq \varepsilon$, then STOP.

Step 3. Find

$$m := \min_{f \in G,\ \|f\|=1} \|f\|_{L_2(\mu)}$$

and a minimal point g.

Step 4. If $m > \varepsilon$ GOTO Step 1. If $m \leq \varepsilon$, then $F := F \ominus \{\lambda g\}$.

Step 5. GOTO Step 1.

Thus for the final space F we have $\|x\| > \varepsilon$ if $\|x\| = 1,\ x \in F$, and $\|y\| \leq \varepsilon$ for all unit vectors y from the orthogonal complement of F. The same property has the space generated by the first geometric harmonics. Therefore neither of these two spaces contains vectors orthogonal to the other space. Hence these spaces are isomorphic. Thus we find the dimension of the optimal canvas.

References

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