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On generation of bandpass random processes with predetermined univariate density and correlation functions of its envelope

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Abstract

We consider possibilities to adjust the canonical procedure for generation of a bandpass process with predetermined statistical properties of its envelope. The relationships are obtained, necessary for calculations of scheme elements for a corresponding device and examples are quoted herein.
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1 Introduction

In many experimental problems of radio engineering and radiophysics there arises a necessity to form random processes with predetermined statistical properties. As an example we can mention tests of real equipment with the help of communications channels imitators [1, 2], different simulation problems [3] and others.

In principle, any random process is perfectly characterised by its multivariate distribution. However, when applied to the generation problem, these requirements may be found too extensive and difficult for realization due to the following reasons:

- as a rule, volume of available information, on which the requirements imposed on the generated process are based, is limited and in many cases it is just not known by which multivariate distribution the given process should be characterised;

- even as analytical and/or algorithmic solution of the problem are known, technical realisation can meet extensive, frequently insurmountable difficulties.

In these conditions, it is natural to try to solve the formation problem in such a way as to provide at least partial compliance of the formed process to the required statistical structure. As a rule, this is implemented through determination of univariate characteristics (certain number of moments or univariate distribution) and correlation function. Problems of formation of a random process with predetermined univariate density and correlation function have been researched during a long period of time, and a lot of works were dedicated to this subject [4 - 6]. However, the overwhelming majority of these works is related to the baseband processes, i.e. those having continuous power density spectrum spread in the area, of low frequencies down to zero. Together with this, in radio engineering practice there are widely used bandpass processes commonly used as models of signals passed through the communication channel, bandpass interference [7 - 9], etc. Such processes emerge in research and modeling phenomena in non- linear mechanics [10, 11], biomedical instrumentation [12], as well as in other areas, and therefore the formation of such processes represents rather general interest.

The widely accepted formation procedure applied to baseband processes, is fully quoted and analysed, for example, in [6]. The whole idea of the method is to generate some initial, as a rule, normal process, which is than undergoing a non-linear zero-memory transformation. Non-linearity endows univariate distributions with a required form. As concerns the correlation function (spectrum density), preliminary linear filtration provides the spectrum of initial normal process with such properties that the correlation function of output random signal after a non-linear transformation would acquire the desired form.

Thus, the clearly defined procedure for the formation is aligned: generation of an initial random process, forming filtration, and non-linear transformation. However, with particular reference to generation of bandpass processes this procedure is directly inapplicable. It must be followed by another filtration since it is not successful to get the characteristics of the first filter such that the process would retain its bandpass after the non-linear transformation. However, an obstacle arises caused by well-known effect of normalisation when random signals are passing though bandpass linear circuits and, as a result, the process loses with necessity the required form of its univariate density. Unfortunately, up to the date it has not been successful to surmount the contradiction between the necessity to retain the required band and density in spite of numerous attempts to do it. As a result, many specialists came up to the idea of impossibility to resolve the contradiction and solve the problem with the traditional method. Since the demands for working out the problem nevertheless exists, an attempt was undertaken to attack the problem from another position, namely with methods of non-linear mechanics applying closed non-linear oscillatory circuits [13, 14]. In that case the generative circuit is described by a differential equation with nonlinearity, and the realizations of the process are Markov-type ones. In the present work we show that it is possible to retain the general line for solving the problem of generation of bandpass random signals. Such approach promises inplementational advantages, since one can use already existing techniques. We give the ways for determination of characteristics for each unit of the corresponding scheme. It should be noted, that as applied to stationary processes, in majority of cases not the instantaneous values are of interest, but rather the envelope [9]. Therefore, we are considering most actual variant of the problem, in which the univariate function and correlation function of the envelope are determined in advance.

2 Formulation of the problem

Structural scheme of signal formation is depicted in Fig. 1. The scheme contains three units. First filter 1 forms a bandpass process $\eta\,(t)$ from the arriving white Gaussian noise $\xi(t)$. Realisations of $\eta(t)$ at the output of the filter 1 can be described according to [7] as follows:

$$\eta((t) = E(t)\cos[\omega_0 t + \theta(t)],$$

where ω_0 is the centre frequency of the band filter 1, and processes $E(t)$ and $\theta(t)$ are varying slowly, statistically independent in corresponding time moments and distributed,respectively, in accordance with the Rayleigh and uniform distributions. After a non-linear transformation $\vartheta = u(\eta)$, output process $\vartheta(t)$ can be described as [7]:

$$\vartheta(t) = \sum_{n=0}^{\infty} R_n(E(t))\cos[n(\omega_0 t + \theta(t))], \tag{1}$$

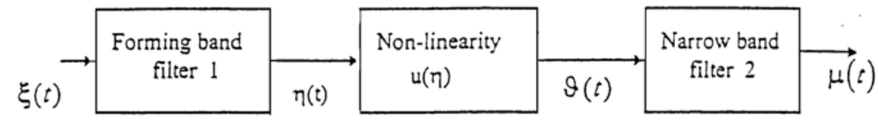

Figure 1: Block-diagram of the narrow-band random process formation.

where $\{ R_n(\cdot) \}$ is a set of functions describing the dependency of the envelope of each of the quasiharmonics in (1) on the envelope $E(t)$ of process $\eta(t)$. The most suitable for our goals form of $R_n(\cdot)$ will be given below. The other filter 2 separates from the sequence (1) only one component (to be specific we would assume below that this is the 1st term of the sequence , i.e. $n = 1$). If $\eta(t)$ is sufficiently bandpass, process $\vartheta(t)$ yields a possibility, that the corresponding quasiharmonic can be selected from it without distortions, and at the same time to exclude influence of other quasiharmonics. Therefore, it is possible not to claim any special requirements regarding the form of frequency characteristics of the filter 2. The sufficient requirement is that it be uniform in the vicinity of ω_0 and vanish in the vicinity of other quasiharmonics. Thus, our objective is to determine the form of non-linearity $u(\cdot)$ and amplitude-frequency characteristics $G(\omega)$ of the filter 1, generating the bandpass Gaussian process $\eta(t)$, which after non-linear transformation and selection by the filter 2, would be transformed into the process $\mu(t)$ with required properties of its envelope. Distinction of the present approach from the widely accepted one is in the fact that both the filter 1 and the non-linearity should endow with the predetermined properties not to the process $\vartheta(t)$ itself at the nonlinearity output, but rather to the envelope $R_1(E(t))$ of its first quasiharmonic $\mu(t)$ selected thereafter by the filter 2.

3 Deduction of computational dependencies

As it has been noted before, we are considering the case, when the density function is determined not for the process itself, but for its envelope. It should be noted, however that for a stationary bandpass process the following relationship is valid, connecting the density function $f_R(R)$ of the envelope and the density $f_\mu(\mu)$ of instantaneous values [15]:

$$f_\mu(\mu) = \frac{1}{\pi}\int_{|\mu|}^{\infty} \frac{f_R(R)}{\sqrt{R^2-\mu^2}}dR = \frac{1}{\pi}\int_0^{\infty} f_R\left(|\mu|\mathrm{ch}x\right)dx \tag{2}$$

Therefore even under our assumption that only $f_R(R)$ is determined, it is always possible to determine the form of $f_\mu(\mu)$, if a necessity arises.

3.1 Determination of the form of the non-linearity

First we shall determine an expression for $R_n(\,\cdot)$ in (1). Using the notion $\varphi = \omega_0\tau + \theta(\tau)$ and assuming $\vartheta = u(\eta)$, we can re-write (1) in the form

$$u(E\cos\varphi) = \sum_{n=0}^{\infty} R_n(E)\cos n\varphi. \tag{3}$$

Let us multiply both parts of the expression obtained above with cos $k\varphi$ $(k = 0, 1, \ldots)$ and integrate by φ from 0 to 2π. Then

$$R_0(E) = \frac{1}{\pi}\int_0^{\pi} u(E\cos\varphi)d\varphi,$$

$$R_k(E) = \frac{2}{\pi}\int_0^{\pi} u(E\cos\varphi)\cos(k\varphi)\,d\varphi,\ (k = 1, 2, \ldots\ \ ). \tag{4}$$

Since we conditioned before that filter 2 selects from (1) the first quasi-harmonic, we take from (4) the integral with k = l. Then, after substitution $y = E\cos\varphi$ and some elementary transformations, the following integral equation is obtained for so far unknown function $u(y)$:

$$R(E) = \frac{2}{\pi E}\int_{-E}^{E} y\frac{u(y)}{\sqrt{E^2-y^2}}dy. \tag{5}$$

From simple physical reasons it follows that function $u\,(y)$ should be odd, since both half -waves of the process (1) should be transformed in identical way. Then function $yu(y)$ is even, and (5) can be re-written as :

$$R(E) = \frac{4}{\pi}\int_0^{E} y\frac{u(y)}{\sqrt{E^2-y^2}}dy.$$

Thereafter we shall treat only the upper half of function $u(\cdot)$. Further on, by substitution $t = y^2,\ x = E^2$, this expression can be transformed to the Abel integral equation with the known solution [16]:

$$u(y) = \frac{1}{2y}\frac{d}{dy}\int_0^y \frac{E^2 R(E)}{\sqrt{y^2 - E^2}}\, dE. \tag{6}$$

On the other hand, it follows from (1) that function $R(E)$ can be considered a non-linear non-inertial transformation of the envelope $E(t)$ for the normal process $\eta(t)$, distributed, as it is known, in accordance with the Rayleigh law. Therefore, for values of the transformed functions to have the predetermined density $f_R(R)$ with the distribution function $F_R(R) = \int_0^R f_R(x)dx$, the transformation should be described by the function dependency [6]:

$$R(E) = F_R^{(-1)}\left(1 - \exp\left(-\frac{E^2}{2\sigma^2}\right)\right), \tag{7}$$

where $F_R^{(-1)}(\cdot)$ is the function inverse to $F_R(\cdot)$, σ^2 is the variance of $\eta(t)$.

By substitution (7) into (6), we obtain:

$$u(y) = \frac{1}{2y}\frac{d}{dy}\int_0^y \frac{E^2 F_R^{(-1)}\left(1 - \exp\left(-\frac{E^2}{2\sigma^2}\right)\right)}{\sqrt{y^2 - E^2}}\, dE. \tag{8}$$

With the help of simple transformations, the dependence (6) can be represented in some equivalent form as follows:

$$u(y) = \frac{1}{2y}\frac{d}{dy}\int_0^y \sqrt{y^2 - x^2} R\left(\sqrt{y^2 - x^2}\right) dx = \tag{9}$$

$$\frac{1}{2y}\int_0^y \frac{d}{dy}\left(\sqrt{y^2 - x^2} R\left(\sqrt{y^2 - x^2}\right)\right) dx = \tag{10}$$

$$\int_0^y \left[\frac{R\left(\sqrt{y^2 - x^2}\right)}{\sqrt{y^2 - x^2}} + R_E' R_E\left(\sqrt{y^2 - x^2}\right)\right] dx, \tag{11}$$

where $R_E'\left(\sqrt{y^2 - x^2}\right)$ is $\frac{d}{dE}R(E)\big|_{E=\sqrt{y^2-x^2}}$.

Formula (11) after substitution of (7) yields

$$u(y) = \int_0^y \{ \frac{F_R^{(-1)}\Big(1-\exp\Big(-\frac{y^2-x^2}{2\sigma^2}\Big)\Big)}{\sqrt{y^2-x^2}} + \frac{\sqrt{y^2-x^2}\exp\Big(-\frac{y^2-x^2}{2\sigma^2}\Big)}{\sigma^2 f_R\Big[F_R^{(-1)}\Big(1-\exp\Big(-\frac{y^2-x^2}{2\sigma^2}\Big)\Big)\Big]} \} dx. \tag{12}$$

The last formula is convenient in the case when it is impossible to obtain $F_R^{(-1)}(\,\cdot)$ analytically, because it allows to avoid a numerical differentiation. The set of formulas (8)-(12) yields solution of the problem for finding non-linear transformation of instaneous values of the process $\eta(t)$.

3.2 Determination of frequency characteristics of the first filter

Since, by our assumption, the input process is a white noise with a uniform spectral density for all (used) frequencies, it is enough to find the spectrum of power $P_\eta(\omega)$ for the process $\eta(t)$ since up to a constant factor,

$$|G_\eta(\omega)|^2 \cong P_\eta(\omega). \tag{13}$$

In turn, to determine the form of $P_\eta(\omega)$, it is enough to find envelope $\rho_\eta(\tau)$ of the normalised correlation function of $\eta(t)$. In accordance with the Wiener-Chinchin theorem [7], functions $\rho_\eta(\tau)$ and $G_\eta(\omega)$ are related to each other as follows:

$$|G(\omega)|^2 = 4\int_0^\infty \rho_\eta(\tau)\cos(\omega_0\tau)\,\cos(\omega\tau)\,d\tau, \tag{14}$$

representing the Fourier cosine-transformation of the correlation function $\rho_\eta(\tau)\,\cos(\omega_0\tau)$ of the bandpass process $\eta(t)$. It follows from (8), that the non-linear transformation $u(\eta)$ followed by the filtration, corresponds to the transformation of its envelope $E(t)$, according to the law determined by the function (8). Therefore, using the expression for bivariate Rayleigh density we can write down the expression relating correlation function $B_R(\tau)$ of the envelope of the output process $\eta(t)$ to the function $\rho_\eta(\tau)$:

$$B_R(\tau) = M(RR_\tau) - M^2(R), \tag{15}$$

$$M(R) = \int_0^\infty \frac{R(E)E}{\sigma^2}\exp\Big(-\frac{E^2}{2\sigma^2}\Big)\,dE, \tag{16}$$

$$M(RR_\tau) = \int\limits_0^\infty \int\limits_0^\infty \frac{E_1 E_2 R(E_1) R(E_2)}{\sigma^4(1-\rho_\eta^2(\tau))} \exp\left\{ -\frac{E_1^2+E_2^2}{2\sigma^2(1-\rho_\eta^2(\tau))} \right\} \times$$

$$I_0\left(\frac{\rho_\eta(\tau) E_1 E_2}{\sigma^2(1-\rho_\eta^2(\tau))} \right) dE_1 dE_2. \tag{17}$$

Substitution of (17) and (16) into (15) yields the integral equation, concatenating to given function $B_R(\tau)$ with the sought function $\rho_\eta(\tau)$. Evaluating the integral, we can find $\rho_\eta(\tau)$.

3.3 Order of evaluation

Thus, the order of determination of the filter 1 characteristics and the non-linearity *is* as follows:

(a) according to the predetermined density function $f_R(R)$ of the envelope (distribution function $F_R(R)$) and with the help of expressions (8), the form of non-linearity $u(\cdot)$ is determined;

(b) with the help of expressions (15)-(17), the relationship between the predetermined correlation function $B_R(\tau)$ and the required function $\rho_\eta(\tau)$ is established analitically or numerically;

(c) with the help of expression (14), the form of frequency characteristics $G(\omega)$ of the filter 1 is determined.

4 Example

Since we are interested only in the forms of the appropriate functional dependencies, we assume for simplicity $\sigma = 1$.

Let us generate bandpass stationary random process with the exponential envelope density $f_R(R) = \alpha\, e^{-\alpha R}$ and correlation function $B_R(\tau) = \exp\left(-\frac{\tau^2}{2\tau_0^2} \right)$.

4.1 Evaluations

(a). The distribution function for this case is $F_R(R) = 1 - \exp(-\alpha R)$ with the inverse function $F_R^{(-1)}(z) = \frac{1}{\alpha} \ln(1-z)$. According to (8), we obtain

$$R(E) = F_R^{(-1)}\left(1 - e^{-\frac{E^2}{\alpha}}\right) = \frac{E^2}{2\alpha}. \tag{18}$$

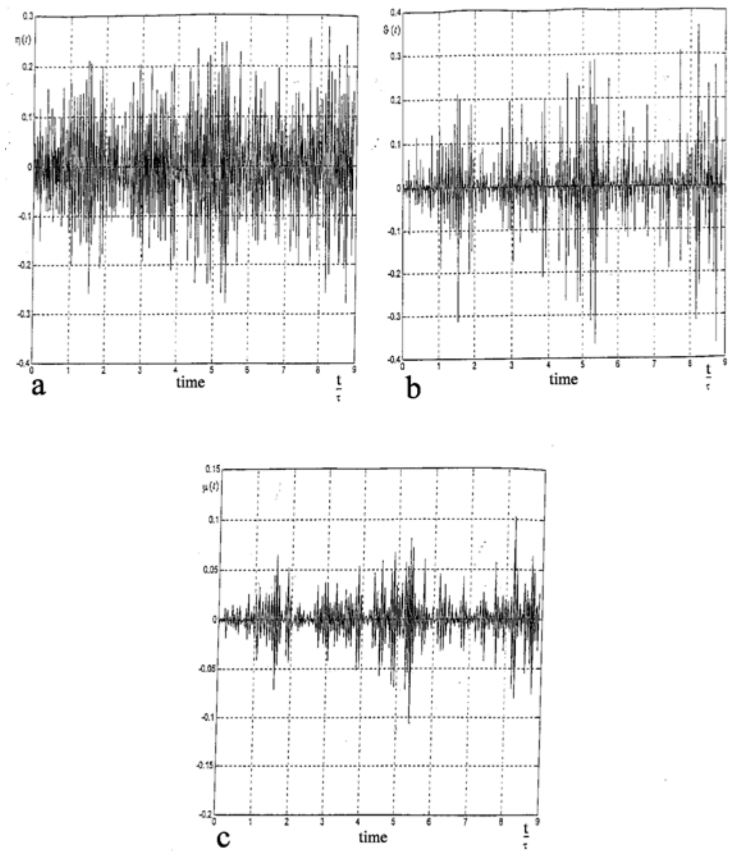

Figure 2: Realization of a random process after each consequent transformation (a, b and c respectively).

With the use of (10), we obtain the expression for the non-linearity:

$$u(y) = \frac{1}{2y}\int_0^y \frac{d}{dy}\frac{(y^2 - x^2)^{\frac{3}{2}}}{2\alpha}dx = \text{sign}(y)\frac{3\pi y^2}{2}. \qquad (19)$$

(b). According to (15) - (17), with the use of elementary methods of integration, we obtain :

$M(R) = \frac{1}{\alpha}$, $M(RR_\tau) = \frac{1+\rho_\eta^2(\tau)}{\alpha^2}$, $B_R(\tau) = \frac{\rho_\eta^2(\tau)}{\alpha^2}$, such as $\rho_\eta^2(\tau) = \alpha^2 \exp\left(-\frac{\tau^2}{2\tau_0^2}\right)$.

(c). From (14) we deduce [17]:

$$|G(\omega)|^2 = \sqrt{2\pi}\tau_0 \exp\{-(\omega - \omega_0)^2\tau_0^2\}. \qquad (20)$$

Thus, to generate the demanded process, a white normal noise must be filtered by a filter with the frequency characteristics (17) and subjected to odd-parabolic transformation, followed by filtration to separate the first quasiharmonic.

4.2 Simulation

According to the structural scheme (Fig. 1) and with the use of the expressions (11), (12) for non-linearity $u(y)$ and amplitude-frequency characteristics $G(\omega)$ of the forming band filter 1, a simulation model for the example was built. Realizations of the processes after each transformation are shown in Fig. 2 (a,b,c). Agreement of the envelope distribution function with the required function was estimated using the χ^2 criterion. The result of evaluation on 50 independent samples of the envelope confirmed that with the confidence not less 0.98, the results of simulation do no contradict to the hypothesis that the envelope is distributed according exponential low.

5 Conclusion

The canonical procedure for generation of random process with prescribed statistical properties is extended to the case of a bandpass process. The peculiarity of the procedure is in the fact that properties of the envelope, rather then instantaneous values, are given. The methods of evaluation of characteristics for the generator are brought. An example with exponential distribution of envelope is given. The findings seem to be useful for several implementations, and especially for the design of testing generators and simulators.

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