HAIT Journal of Science and Engineering A, Volume 5, Issues 1-2, pp. 140-142

A two-parameter generalization of the complete elliptic integral of the second kind

M. Lawrence Glasser

Department of Physics and Department of Mathematics and Computer Science
Clarkson University, Potsdam, NY 13699-5820, USA
e-mail: laryg@tds.net
Received 15 September 2007

Abstract

The double integral

$$E(a,b) = \int_0^\pi dx \int_0^\pi dy \sqrt{1 + a\cos\ x + b\cos\ y}$$

is evaluated in terms of complete elliptic integrals.

PACS: 02.30.Gp, 05.50.+q
Keywords: Double elliptic integral, hypergeometric function

V. Bârsan[1] has recently investigated a double integral equivalent to

$$E(a,b) = \int_0^\pi dx \int_0^\pi dy \sqrt{1 + a\cos\ x + b\cos\ y} \tag{1}$$

which, by means of an ingenious procedure, he expressed as a derivative of a hypergeometric function of two variables. In this note we show that (1) can be reduced in a relatively direct manner to a simple combination of complete elliptic integrals. let us assume that $0 \leq a + b \leq 1$ and initially that $Re\ s > 0$. Then, in the usual way one has

$$I = \int_0^\pi dx \int_0^\pi dy (1 + a\cos\ x + b\cos\ y)^{-s} =$$
$$\frac{1}{\Gamma(s)} \int_0^\infty ds t^{s-1} e^{-t} \int_0^\pi dx \int_0^\pi dy e^{-at\cos\ x - bt\cos\ y} =$$
$$\frac{\pi^2}{\Gamma(s)} \int_0^\infty t^{s-1} e^{-t} I_0(at) I_0(bt) dt. \tag{2}$$

The latter is a tabulated Laplace transform[2] yielding

$$I = \pi^2 F_4(s/2, (s+1)/2; 1, 1; u(1-v), v(1-u)) \tag{3}$$

with

$$u = \frac{1}{2}[1 + a^2 - b^2 - \sqrt{(1 + a^2 - b^2)^2 - 4a^2}]$$
$$v = \frac{1}{2}[1 - a^2 + b^2 - \sqrt{(1 - a^2 + b^2)^2 - 4b^2}]. \tag{4}$$

(Note that $u(1-v) = a^2$, $v(1-u) = b^2$). Now, by analytic continuation we can take $s = -1/2$. Next, by L. Slater's reduction formula[3], one has

$$E(a,b) = \pi^2[\ {}_2F_1(-1/4, 1/4; 1; u)\ {}_2F_1(-1/4, 1/4; 1; v) +$$
$$\frac{1}{16} uv\ {}_2F_1(3/4, 5/4; 2; u)\ {}_2F_1(3/4, 5/4; 2; v)]. \tag{5}$$

Finally, since[4]

$${}_2F_1(-1/4, 1/4; 1; z^2) = \frac{2}{\pi}\sqrt{1+z}\mathbf{E}(k)$$
$${}_2F_1(3/4, 5/4; 2; z^2) = \frac{8}{\pi z^2 \sqrt{1+z}}[\mathbf{K}(k) - (1+z)\mathbf{E}(k)] \tag{6}$$
$$k = \sqrt{\frac{2z}{1+z}}$$

we have the desired expression

$$\frac{1}{4}E(a,b) = 2\sqrt{(1+\sqrt{u})(1+\sqrt{v})}\mathbf{E}[k(\sqrt{u})]\mathbf{E}[k(\sqrt{v})]+$$

$$\frac{\mathbf{K}[k(\sqrt{u})]\mathbf{K}[k(\sqrt{v})]}{\sqrt{(1+\sqrt{u})(1+\sqrt{v})}} - \sqrt{\frac{1+\sqrt{u}}{1+\sqrt{v}}}\mathbf{E}[k(\sqrt{u})]\mathbf{K}[k(\sqrt{v})]-$$

$$\sqrt{\frac{1+\sqrt{v}}{1+\sqrt{u}}}\mathbf{E}[k(\sqrt{v})]\mathbf{K}[k(\sqrt{u})] \tag{7}$$

For the case $a = b$ (3) and (7) simplify to

$$\frac{1}{4}\int_0^\pi \int_0^\pi \sqrt{1+a(\cos\ x + \cos\ y)}dxdy = \frac{\pi^2}{4}\,{}_3F_2(-1/4, 1/4, 1/2; 1, 1; 4a^2) =$$

$$2(1+\sqrt{u})\mathbf{E}^2(k) + (1+\sqrt{u})^{-1}\mathbf{K}^2(k) - 2\mathbf{E}(k)\mathbf{K}(k), \tag{8}$$

where $u = (1-\sqrt{1-4a^2})/2$, $k = \sqrt{2\sqrt{u}/(1+\sqrt{u})}$.

References

[1] V. Bârsan, *A two-parameter generalization of the complete elliptic integral of second kind*, arXiv:0708.2325, v1 (2007).

[2] A.P. Prudnikov, Yu. A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 2 (Nauka, Moscow, 1981).

[3] L. Slater, *Generalized Hypergeometric Functions* (Cambridge University Press, 1966).

[4] A.P. Prudnikov, Yu. A. Brychkov, and O.I. Marichev, *Integrals and Series*, Vol. 3 (Nauka, Moscow, 1981).