Path integration now enjoys a central role in many areas of physics and chemistry. It’s impact has been felt in other fields as well. For example, in seismology the path integral provides a way to deal with partial differential equations, while for finance it is a tool for the analysis of stochastic processes.

With the issuing of this Dover Edition, I am adding a few new topics. They are presented in Sec. I of this supplement. While some are elementary and hopefully of broad interest, I am not providing a comprehensive survey of developments, as might be suitable for a bona fide second edition (which this is not). The coverage ranges from self-contained to telegraphic. Sec. II lists recent texts where a fuller picture can be obtained. Finally, in Sec. III are errata and comments on particular parts of this book. Partly these are factors of two and such (which eluded me, despite the opportunity to correct them on each of Wiley’s dozen or so reprints), but mainly they are comments that were too lengthy to fit on the reprinted page.

References to “Sections” 1 through 32 are to sections of this book (a.k.a. chapters). Similarly, equation numbers with periods, \((n.m)\), refer to equations in the book, while those consisting of a number alone refer to the supplement. Sections of the supplement have labels beginning with Roman numerals.

In preparing this supplement I have had the help of many individuals. Moreover, since the publication of the book, quite a few people have come to me with corrections, some of which were incorporated in previous reprints and some of which appear here. Since these events stretch over 20 years, I hope those whose contributions have slipped my mind will forgive me. Those I recall—and thank, whether for counsel, corrections or both—are A. Auerbach, Y. Avron, C. Dewitt, P. Exner, P. Facci, B. Gaveau, H. Grabert, T. Jacobson, H. Jauslin, G. Junker, M. Kac, S. Kivelson, A. Mann, D. McLaughlin, E. Mihokova, D. Mozyrsky, D. Mugnai, S. Pascazio, P. Pechukas, A. Ranfagni, M. Revzen, G. Roepstorff, M. Roncadelli, A. Scardicchio, L. J. Schulman, D. Tolkunov, U. Weiss, and N. Yamada.
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I. TOPICAL SUPPLEMENTS

A. Path integral in a magnetic field using the Trotter product formula

In Sec. 1 the path integral is derived using the Trotter product formula. This is basically the statement

\[
\left| e^{-i\epsilon H/\hbar} - e^{-i\epsilon p^2/2m\hbar} e^{-i\epsilon V(x)/\hbar} \right| < \mathcal{O}(\epsilon),
\]  

where standard notation, \( H = p^2/2m + V \), etc., is used. Later, in Sec. 4, I turn to a more delicate derivation, the path integral in the presence of a magnetic field. This involves a term \( \dot{x} \cdot A \) in the classical Lagrangian and, as explained in Sec. 5, to the need to evaluate \( A(x) \) at the midpoints along the broken line path within a path integral.

The derivation in Sec. 4 follows the original paper of Feynman. However, by a slight modification of the method of Sec. 1, it is possible to use the Trotter formula. The yet more delicate case of a curved-space metric can also be treated by operator methods.

1. Splitting a sum

As usual, we want the propagator, \( G \), the kernel of the operator \( \exp(-iHt/\hbar) \). With a magnetic field

\[
H = \frac{1}{2m} \left( p - \frac{e}{c} A(x) \right)^2 + V(x),
\]

where \( A \) is the vector potential. For \( a, b \in \mathbb{R}^3 \), \( G \) satisfies the following sequence of identities

\[
G(b, t; a) = (b| \exp(-iHt/\hbar)|a) = (b| [\exp(-iHt/\hbar)]^N |a) \\
= (b| e^{-iH\epsilon/\hbar} \int d^3x_1 |x_1\rangle \langle x_1| e^{-iH\epsilon/\hbar} \int d^3x_2 |x_2\rangle \langle x_2| e^{-iH\epsilon/\hbar} \\
\times \cdots \times \int d^3x_{N-1} |x_{N-1}\rangle \langle x_{N-1}| e^{-iH\epsilon/\hbar} |a\rangle,
\]

where after line (3) I use the definition \( \epsilon \equiv t/N \). Eq. (3-4) can be written concisely as

\[
G(b, t; a) = \int \prod_{k=1}^{N-1} d^3x_k \prod_{\ell=0}^{N-1} G(x_{\ell+1}, \epsilon; x_\ell),
\]
with \( x_N = b \), and \( x_0 = a \). This is the starting point for the Sec. 1 derivation. The smallness of \( \epsilon \) allowed \( H \) to be split into kinetic and potential energy terms. For convergence one must maintain \( O(\epsilon) \) accuracy. That is, \( G(x_{\ell+1}, \epsilon; x_\ell) \) can be replaced by other functions that differ only by terms going to zero faster than \( \epsilon \). For operators this is a more nuanced project and the reader is referred to Sec. 1 and its Notes.

The goal then is to approximate \( G(x, \epsilon; y) \) to first order in \( \epsilon \). It is helpful to phrase this in operator language. For operators \( A \) and \( B \),

\[
\exp[\lambda(A + B)] = \exp(\lambda A) \exp(\lambda B) \exp \left( \frac{\lambda^2}{2} [B, A] + O(\lambda^3) \right). \tag{6}
\]

The propagator is of this form, with \( \lambda = \epsilon \), \( A = -iK/\hbar \), \( B = -iV/\hbar \), and \( K \) the kinetic energy (called “\( T \)” in Sec. 1). For present purposes it is sufficient to know that

\[
\exp[\lambda(A + B)] = \exp(\lambda A) \exp(\lambda B) + O(\epsilon^2). \tag{7}
\]

In Sec. 1, without the vector potential, this quickly led to the path integral. To prepare for the vector potential case, here’s a recap.

Focus on one factor in the integrand of (5). Using Eq. (7) and the fact that \( V \) is diagonal in position we obtain

\[
G(x, \epsilon; y) = \langle x | \exp[-iK\epsilon/\hbar]|y\rangle \exp[-iV(y)\epsilon/\hbar] + O(\epsilon^2). \tag{8}
\]

Let \( |p'\rangle \) be an eigenvector of \( p \) with eigenvalue \( p' \). Inserting \( 1 = \int d^3p' |p'\rangle \langle p'| \) to the left of \( |y\rangle \) yields

\[
\int d^3p' \langle x | e^{-ip'^2\epsilon/2m}|p'\rangle \langle p'| y \rangle = \left[ \frac{m}{2\pi i\hbar\epsilon} \right]^{3/2} \exp \left( \frac{i m(x - y)^2}{\hbar \cdot 2\epsilon} \right). \tag{9}
\]

[Eq. (16) gives \( \langle p|x \rangle \) explicitly.] Combining Eqs. (8) and (9) leads to the expression, which when iterated, gives the path integral in three dimensions:

\[
G(x, \epsilon; y) = \left( \frac{m}{2\pi i\hbar\epsilon} \right)^{3/2} \exp \left[ i \frac{m(x - y)^2}{\hbar \cdot 2\epsilon} - \epsilon V(y) \right] + O(\epsilon^2) \tag{10}
\]

[cf. Eq. (1.24)]. Inserting Eq. (10) in Eq. (5), we obtain the classical action in the exponent. Had we interchanged \( K \) and \( V \) in Eq. (7), the argument of \( V \) in Eq. (10) would be \( x \) rather than \( y \). This changes the short-time propagator by less than \( O(\epsilon) \) and therefore does not change the final result.
2. Splitting a product

When a magnetic field is present, \( K = [p - eA(x)/c]^2/2m \). Inserting a momentum-state resolution of the identity, as in Eq. (9), is inadequate, because \( A \) is a function of \( x \). The way around this is to look at the square root of \( K \), which because it is a sum of \( p \) and \( A \) can be resolved by separately expressing \( p \) and \( A \) in momentum and position space bases. Taking the square root in the exponent is accomplished by introducing an additional integral: uncompleting the square, as in Sec. 32.5. We extend Eq. (32.29) by going to three dimensions and taking multiples of the variables used there, to obtain the identity

\[
\exp\left(-i\frac{\epsilon b^2}{2m}\right) = \left(\frac{1}{\sqrt{2\pi i}}\right)^3 \int d^3 u \exp\left(i\frac{u^2}{2} - i\sqrt{\frac{\epsilon}{m\hbar}} b \cdot u\right). \tag{11}
\]

For convenience, let \( a(x) \equiv eA(x)/c \) and \( \tilde{\epsilon} \equiv \sqrt{\epsilon/m\hbar} \). We want to evaluate Eq. (8) and concentrate on the kinetic energy part, designated \( G_K \). Setting \( b = p - a \) in Eq. (11), this becomes

\[
G_K(x, \epsilon; y) \equiv \langle x|e^{-iK\epsilon/\hbar}|y\rangle = \langle x|\exp\left(-i\frac{\epsilon (p - a)^2}{2m\hbar}\right)|y\rangle = (2\pi i)^{-3/2} \int d^3 u e^{iu^2/2} \langle x|\exp\left(-i\tilde{\epsilon} (p - a) \cdot u\right)|y\rangle. \tag{12}
\]

In Eq. (12), focus on \( \exp\left(-i\tilde{\epsilon} (p - a) \cdot u\right) \). Because the noncommuting \( p \) and \( a(x) \) no longer appear as a product one might expect that this would allow factoring of the exponential, as in the Trotter product formula. Unfortunately, applying Eq. (7) would not provide the needed accuracy, because it is \( \tilde{\epsilon} \) that appears in Eq. (12), not \( \epsilon \). The error is the square of \( \tilde{\epsilon} \), namely \( \epsilon/m\hbar \), which cannot be neglected. This problem occurs whenever an expression mixes non-commuting variables, for example in a curved-space metric (or position-dependent kinetic energy). Generally speaking, this is why path integration does not fix operator ordering. The way to deal with this is to improve on Eq. (7).

3. A more accurate product formula

A variation of Eq. (7) provides \( O(\lambda^3) \) accuracy:

\[
\exp[\lambda(A + B)] = \exp(\lambda B/2) \exp(\lambda A) \exp(\lambda B/2) + O(\lambda^3). \tag{13}
\]
This can be checked by direct expansion of the exponentials [1]. Apply Eq. (13) to \( \langle x | \exp (-i\epsilon (p - a) \cdot u) | y \rangle \) to obtain
\[
\langle x | \exp (-i\epsilon (p - a) \cdot u) | y \rangle = \langle x | e^{i \epsilon a \cdot u/2} e^{-i\epsilon p \cdot u} e^{i \epsilon a \cdot u/2} | y \rangle + O(\epsilon^{3/2}) \\
= e^{i \epsilon a \cdot u/2} \langle x | e^{-i\epsilon p \cdot u} | y \rangle e^{i \epsilon a \cdot u/2} + O(\epsilon^{3/2}).
\] (14)

Errors are \( O(\epsilon^{3/2}) \), larger than \( \epsilon^2 \), but small enough. The operators \( a \), functions of the position operator, appear symmetrically, and are immediately evaluated on the last line as \( a(y) \) and \( a(x) \). This is the step that gives the midpoint rule. For the momentum-dependent portion, a momentum resolution of the identity is inserted:
\[
\langle x | \exp (-i\epsilon p \cdot u) | y \rangle = \int d^3 p' \langle x | \exp (-i\epsilon p \cdot u) | p' \rangle \langle p' | y \rangle.
\] (15)

With
\[
\langle x | p' \rangle = (2\pi \hbar)^{-3/2} \exp(i p' \cdot x/\hbar),
\] (16)
Eq. (15) becomes
\[
\langle x | \exp (-i\epsilon p \cdot u) | y \rangle
= \left( \frac{1}{2\pi \hbar} \right)^3 \int d^3 p' \exp \left[ -i\epsilon p' \cdot u + i p' \cdot (x - y)/\hbar \right] \\
= \frac{1}{\hbar^3} \delta^3 \left( -\epsilon u + \frac{x - y}{\hbar} \right) = \frac{1}{(\epsilon \hbar)^3} \delta^3 \left( -u + \frac{x - y}{\epsilon \hbar} \right).
\] (17)

The integral over \( u \) in Eq. (12) is now trivial, yielding
\[
G_K(x, \epsilon; y)
= \left[ \frac{1}{2\pi i \epsilon \hbar} \right]^3 \int d^3 u e^{iu^2/2} \exp \left[ i\epsilon u \cdot \frac{a(x) + a(y)}{2} \right] \delta^3 \left( \frac{x - y}{\epsilon \hbar} - u \right) \\
= \left[ \frac{m}{2\pi i \epsilon \hbar} \right]^3 \exp \left[ i \frac{m}{\hbar} \frac{(x - y)^2}{2\epsilon} \right] \exp \left[ i \frac{m}{\hbar} (x - y) \cdot \frac{a(x) + a(y)}{2} \right],
\] (18)

where in the last line the original \( \epsilon \) has replaced \( \bar{\epsilon} \). Recall that \( a \equiv eA/c \).

For the classical Lagrangian, a magnetic field contributes \( v \cdot eA(x)/c \). If we multiply and divide by \( \epsilon \) in the last expression in Eq. (18), we obtain
exactly the appropriate term. For completeness, we restore $A$ and $V$, yielding,

$$G(x, \epsilon; y) = \left[ \frac{m}{2\pi i \hbar \epsilon} \right]^\frac{3}{2} \exp \left\{ i \hbar \epsilon \left[ \frac{m}{2} \frac{(x - y)^2}{\epsilon^2} + \frac{x - y}{\epsilon} \left( \frac{A(x) + A(y)}{2} - V(y) \right) \right] \right\} + O(\epsilon^{3/2}).$$

(19)

The argument of the exponent is seen to be $i\epsilon/\hbar$ times the classical Lagrangian.

Note that what naturally arises is the average of the vector potential, $A$, at the endpoints of the broken line path. The “midpoint” way of attaining the same level of precision uses $A[(x + y)/2]$. The difference between these is essentially a second derivative of $A$ times $(x - y)^2$. The latter is of order $\epsilon$ and in turn multiplies an additional power of $(x - y)$, so that the difference, of order $\epsilon^{3/2}$, can be neglected.

4. Precision and rough paths

The need for precision in “sensitive” path integrals has been emphasized in this book and in many other places. It was known to Feynman and—for the Wiener integral—is what lies behind Ito’s theorem, discussed in Sec. 5. This is not a quirk of the path integral, but is a central feature of quantum mechanics. It is evident in the fact that velocity cannot be defined as the limit of $\Delta x/\Delta t$. Moreover, the same property plays a central role in applications of the Wiener integral, for example, in the derivation of the Black-Scholes formula.

Integration of terms in which the “kinetic energy” is space-dependent is even more sensitive than integration of $\int A \cdot \dot{x} \, dt$. The reason is easy to see. For the vector potential, $A$ multiplies $\Delta x$, which is $O(\sqrt{\epsilon})$. On (e.g.) curved spaces, the metric function $g$ multiplies $(\Delta x)^2/\Delta t$, which is $O(1)$, hence yet more stringent demands arise for the evaluation of $g$. Such dynamics can also be quantized operator methods.

The degree of roughness of the paths, reflected in the nonexistence of the derivatives, can be expressed in other ways. Like the paths that dominate Wiener measure, those that are important for the path integral can be assigned a fractal dimension (which is 2).

**Exercise:** Another way to derive the midpoint rule. Simplify notation by setting $a = eA/c$, $\hbar = 1$ and $m = 1$. You want to eval-
uate $\langle x | e^{-i\epsilon(p-a)^2/2} | y \rangle$. You can write this to order $\epsilon$ as $\langle x | e^{-i\epsilon p^2/2} (1 - i\epsilon(pa + ap)/2) e^{-i\epsilon a^2/2} | y \rangle$. In this expression it’s obvious that one takes $a$’s argument symmetrically. What’s not obvious is that you have to keep it that way, and why. Also in this formulation effort is needed to show that the $a^2$ disappears.

Notes

Feynman’s derivation is in his Rev. Mod. Phys. paper [2]. Here I follow [3], which is in turn based on [4] and [5]. The Trotter formula variation of Eq. (13) has also been used in numerical evaluations of path integrals by Janke and Sauer [6]. The curved space path integral is derived in [4] by operator methods, although I expect that an extended Trotter expansion could be made to work in this case too.

Ref. [4] works with the path integral in a form not usually employed by physicists. The mathematician talking about the path integral will often call it the Feynman-Kac formula (not exactly the usage of Sec. 7) and write down something that looks little like our familiar sum over paths. In fact this form does appear in this book, in the “$E_{xt}$” of Eq. (9.17). In practice the expectation $E_{xt}$ is essentially the expectation over Wiener measure. Usually Wiener measure depends on $t$ alone, but $E_{xt}$ can be recovered by inserting a $\delta$-function to enforce the endpoint condition $x$. These remarks may be too telegraphic to make any sense. At a 1987 Trieste meeting/workshop I gave a path integral derivation in this language [7], although it may be a challenge to track down the notes.

If for some reason one prefers not to use the midpoint rule for a curved-space path integral, a correct propagator can still be obtained by adding a potential. This is discussed in [8].

The work of Nelson has already been mentioned in Sec. 1. The Trotter product formula continues to attract the attention of mathematicians; see for example Ichinose [9].

The Black-Scholes formula is a method for assigning a price to a financial option. The book of Hull [10] derives this formula using the Itô integral. But don’t expect “Black-Scholes” plus your knowledge of path integration to make you rich, not immediately, anyway. Markets are not well described by Gaussian statistics and are closer to being Levy-distributed, for which the standard deviation does not exist. This has been stressed by Mandelbrot [11, 12]. A sampling of other sources is [13–15].

The fractal dimension of the paths that enter the nonrelativistic path
integral has been discussed extensively and it has even been proposed that the dimension could be experimentally measured. See articles by Abbott and Wise [16], Cannata and Ferrari [17], Kroger [18, 19] and Kroger, Lantagne, Moriarty and Plache [20].

**B. Path decomposition expansion**

1. **The formula**

It often happens that a coordinate space breaks up in a natural way, for example the inside and outside of a region surrounded by a high potential energy wall. One may know what the propagator looks like inside and outside the wall, and would like to put together a propagator for the entire space. The answer to an easier question has been known since the beginning of path integration: if the environment changes as a function of time, say there is one potential before a certain time \( t_0 \) and a different one afterward, then the “before” and “after” propagators combine easily, and the full propagator is given by

\[
G(x, t; y) = \int dz \, G_{\text{after}}(x, t - t_0; z) \, G_{\text{before}}(z, t_0; y),
\]

(20)

where \( x, y \) and \( z \), are in the system’s coordinate space, whatever that happens to be.

The path decomposition formula, gives a way to connect different parts of the coordinate space, while summing over times. The simplest example is in one dimension, \( x \in [a, b] \) (where \( -\infty \leq a < b \leq \infty \)). Let \( a < c < b \), then

\[
G(x, t; y) = \int_0^t ds \, G(x, t - s; c) \left[ \frac{i\hbar}{2m} \frac{\partial}{\partial z} G^{(r)}(z, s; y) \right]_{z=c},
\]

(21)

where \( G^{(r)}(z, s; y) \) is the propagator for a particle restricted to the region \( y \in [a, c] \). There is no restriction on the potential except that it be time-independent.

Where does Eq. (21) come from? The idea is that the particle is restricted to \([a, c]\) up until time \( s \), after which it can be anywhere. The integrand is thus the amplitude for this collection of events, that is, it is the sum of \( e^{iS_{\text{after}}/\hbar} \) for paths that stay in \([a, c]\) up until time-\( s \), after which they can be anywhere. The integral over \( s \) is therefore interpreted—as usual—as the sum over the possible times for which this restriction is obeyed. But the astute reader will have noticed that I’ve been glib on
two points: first, the propagator for times \( t \) greater than \( s \) is the full propagator, not the propagator for \( [c, b] \). Second, the propagator for the time interval prior to \( s \) appears with a derivative.

The fact that the full propagator appears for times later than \( s \) can be offset by judicious choice of \( c \). For tunneling applications you would like \( c \) to be such that on one side is the well, on the other side freedom. This works for the energy-dependent propagator, \( \tilde{G}(E) \), if you take \( c \) on the outer slope of the potential, away from the well. For the time-dependent propagator this strategy may still work in an approximate way.

Regarding the second point, the derivative in Eq. (21) had to appear. This is because the restricted propagator \( G^{(r)}(z, s; y) \) vanishes on the boundary, \( z = c \). The path restriction forces Dirichlet boundary conditions on \( G^{(r)}(z, s; y) \), just as for the hard wall case discussed in Sec. 6. The derivative in Eq. (21) represents a current across the boundary. If you take the restricted propagator for a particle confined to a half line [e.g., Eq. (6.43), page 40], apply \( i\hbar \partial_z /2m \), and evaluate at the boundary, you will find the classical velocity times the free propagator.

The result, Eq. (21), has been derived in a number of ways. Most satisfying to me is time slicing [breakup into integrals at times \( t_k = k(t_{\text{final}} - t_{\text{initial}})/N \)], which I present below. But ordinary Green’s function techniques can also be used, and there is another approach, using operators, that allows restrictions more general than those defined by the coordinate space.

2. Proof of the path decomposition expansion

For simplicity take the points \( a \) and \( b \) to be \( \mp\infty \) and \( c \) to be 0. The time interval is \([0, t]\). Let \( t_k = k\epsilon \) with \( \epsilon = t/N \) for large \( N \). Then for each path from \( y \) to \( x \) (with \( y < 0 < x \)) there will be a last time that that path was entirely to the left of the origin. Let that time lie between \( t_m \) and \( t_{m+1} \). Then for all integrals \( \int dx_k \), for \( k \leq m \), the range of integration is restricted to \((-\infty, 0)\). On the other hand, \( x_{m+1} \) only varies from 0 to \( \infty \), since if it took negative values, \( t_m \) would not be the last time the particle was confined to \( x < 0 \) [21]. For all integrals after this one \((\geq m + 2)\), the range of integration is over the entire real line.

The total propagator is the sum over all these possibilities, i.e.,

\[
\sum_m \int_{-\infty}^{\infty} \prod_{k=m+2}^{N-1} dx_k \int_{0}^{\infty} dx_{m+1} \int_{-\infty}^{0} \prod_{k=1}^{m} dx_k A \exp(iS/\hbar), \tag{22}
\]

where the integrand is only indicated schematically. If all the integrals from 1 to \( m - 1 \) are performed one gets the restricted propagator from
y to $x_m$ in time $t_m$. Similarly the integrals beyond the \((m+1)\)th give the full propagator from $x_{m+1}$ to the final point $x$ in time $t - t_{m+1}$. As a result there are only two integrals to perform:

$$G(x, t; y) = \sum_m \int_0^\infty dx_{m+1} \int_{-\infty}^0 dx_m G(x, t - t_{m+1}; x_{m+1})$$

$$\times \left[ \frac{m}{2\pi i\hbar} \right]^{\frac{1}{2}} \exp \left( \frac{im(x_{m+1} - x_m)^2}{2\epsilon} \right) G^{(r)}(x_m, t_m; y). \quad (23)$$

[See [22] concerning the absence of $V(x_m)$ in Eq. (23).] We know that the $(x_{m+1} - x_m)^2/\epsilon$ in the exponent keeps these two variables together, which means they both must be close to zero. Before doing the integrals let me show how we “know” this.

**Notation:** define $u \equiv x_m$, $v \equiv x_{m+1}$, $\tau' \equiv t_{m+1}$, $\tau \equiv t_n$ and $D \equiv \hbar/m$. We also refer to the $m^{th}$ term in the sum over $m$ in Eq. (23) as $\Phi_m$. Finally we define a shorthand for the free propagator, $g(x, t) \equiv \exp(ix^2/2Dt)/\sqrt{2\piDt}$.

Now rewrite the short time propagator

$$g(v - u, \epsilon) = \frac{\lambda}{2\pi} \int_{-\infty}^{\infty} dq \, e^{-iq^2} \exp \left( -i\lambda q v + i\lambda qu \right), \quad (24)$$

with $\lambda \equiv \sqrt{2/D\epsilon}$. With this split, the integral over $u$ can be performed separately. Using the identity $\exp(i\lambda qu) = (1/iq\lambda)(d/du)\exp(i\lambda qu)$, we integrate by parts

$$\int_{-\infty}^0 du \, e^{i\lambda qu} G^{(r)}(u, \tau; y) = \left[ \frac{e^{i\lambda qu}}{i\lambda q} G^{(r)}(u, \tau; y) \right]^{0}_{u=-\infty}$$

$$- \frac{1}{i\lambda q} \int_{-\infty}^0 du \, e^{i\lambda qu} \frac{\partial}{\partial u} G^{(r)}(u, \tau; y) \quad (25)$$

The term in square brackets vanishes for the following reasons. At infinity we assume that there is a small imaginary part to regularize it, while at 0, a restricted propagator always vanishes at the endpoint of its restriction. Thus only the derivative of $G^{(r)}$ contributes. Furthermore, the integration by parts can be done again. The piece arising from the total derivative no longer vanishes, and depends only on the value of the derivative of $G^{(r)}$ at zero. The remaining integral is asymptotically smaller by a factor $\lambda$. (See Sec. ID 2.)

The same argument can be applied to the $v$ integral. In this case the leading term is $G(x, t - \tau'; 0)$, which does not vanish. It will turn
out that the individual contributions, \( \Phi_m \), to the integral in Eq. (23) are \( O(\epsilon) \) (so that \( \sum_m \) turns into \( \int ds \)). Contributions from the derivative of \( G \) or from the second derivative of \( G^{(r)} \) carry an additional power of \( \lambda \), for an overall \( O(\epsilon^{3/2}) \), which can be neglected.

The foregoing observations will be used at a later stage, and we return to the double integral, Eq. (23), undoing the transformation (24). The next step is establish integral relations satisfied by \( g \), the free propagator. These relations are most easily derived using the Laplace transform, and in particular the Faltung theorem.

\textbf{Faltung facts}  The Laplace transform of a function \( f(t) \) is \( \hat{f}(\omega) \equiv \int_0^\infty dt \, e^{-\omega t} f(t) \). The Faltung theorem says that if \( \hat{f} \) and \( \hat{g} \) are the Laplace transforms of \( f \) and \( g \) respectively, then the Laplace transform of \( \int_0^t ds \, f(t-s) g(s) \) is \( \hat{f}(\omega) \hat{g}(\omega) \). The integral of \( f(t-s) g(s) \) is the convolution, or Faltung (folding). Bearing in mind the form of the path decomposition expansion it is not surprising that the Faltung theorem is relevant. Here are the several results that we will use.

Everything will be stated in real form, with the \( i \)'s inserted later in blind, optimistic analytic continuation. First an already familiar expression:

\[
\int_0^\infty dt \, e^{-\omega t} \frac{1}{\sqrt{\pi t}} e^{-k^2/4t} = \frac{1}{\sqrt{\omega}} e^{-k\sqrt{\omega}}, \tag{26}
\]

for \( k > 0 \). In Sec. 7 this formula took us from the propagator to its energy dependent form. Next, the trivial identity

\[
\frac{1}{\sqrt{\omega}} e^{-k\sqrt{\omega}} \left( -\frac{\partial}{\partial \ell} \right) \frac{1}{\sqrt{\omega}} e^{-\ell\sqrt{\omega}} = \frac{1}{\sqrt{\omega}} e^{-(k+\ell)\sqrt{\omega}}, \tag{27}
\]

and the Faltung theorem imply

\[
\frac{e^{-(k+\ell)^2/4t}}{\sqrt{\pi t}} = \int_0^t ds \, \frac{1}{\sqrt{\pi(t-s)}} \left( -\frac{\partial}{\partial \ell} \right) \frac{e^{-\ell^2/4s}}{\sqrt{\pi s}}, \quad k, \ell > 0. \tag{28}
\]

We will also need to evaluate the above integral without the derivative with respect to \( \ell \). Consider

\[
F(t, k, \ell) \equiv \int_0^t ds \, \frac{e^{-k^2/4(t-s)}}{\sqrt{\pi(t-s)}} e^{-\ell^2/4s} = \int_0^\infty d\omega \, \frac{e^{-k\sqrt{\omega}}}{\sqrt{\omega}} \frac{e^{-\ell\sqrt{\omega}}}{\sqrt{\omega}} \tag{29}
\]

(by the Faltung theorem). To get rid of the extra \( 1/\sqrt{\omega} \) we use \( \omega^{-1/2} = \int_0^\infty d\mu \, e^{-\mu\sqrt{\omega}} \). It follows that

\[
F(t, k, \ell) = \int_0^\infty d\mu \, e^{-(k+\ell+\mu)^2/4t} \frac{1}{\sqrt{\pi t}}. \tag{30}
\]
This is essentially the Error function. We evaluate $F$ in the limit of small $t$. Asymptotically

$$F(t, k, \ell) = \frac{e^{-(k+\ell)^2/4t}}{\sqrt{\pi t}} \int_0^\infty d\mu e^{-\mu(k+\ell)/2t} e^{-\mu^2/4t} \sim \frac{2t}{k + \ell} \frac{e^{-(k+\ell)^2/4t}}{\sqrt{\pi t}}.$$  \hfill (31)

The formulas will be applied with the following substitutions: $t \rightarrow i\epsilon D/2, s \rightarrow i\sigma D/2, k \rightarrow v$ and $\ell \rightarrow -u$.

Eq. (28) and the above substitutions give the following identity for the propagator (recalling that $u < 0$)

$$g(v - u, \epsilon) = iD \int_0^\epsilon d\sigma g(v, \epsilon - \sigma) \left( \frac{\partial}{\partial u} \right) g(u, \sigma). \hfill (32)$$

From the relations (29) and (31) we have in turn

$$\int_0^\epsilon d\sigma g(v, \epsilon - \sigma)g(u, \sigma) \sim \frac{\epsilon}{v - u} g(v - u, \epsilon). \hfill (33)$$

Now we are ready to calculate. Going back to Eq. (23) and recalling that the summands are denoted $\Phi_m$, we have

$$\Phi_m = \int_0^\infty dv \int_{-\infty}^0 du G(x, t - \tau'; v)g(v - u, \epsilon)G^{(r)}(u, \tau; y)$$

$$= \int dv du G(x, t - \tau'; v) \int_0^\epsilon d\sigma g(v, \epsilon - \sigma) \left[ iD \frac{\partial}{\partial u} g(u, \sigma) \right] G^{(r)}(u, \tau; y).$$

Next do an integration by parts with respect to $u$. Since $G^{(r)}(0, \tau; y) = 0$ and there is regularity at infinity, this only amounts to a change of sign and a transferral of the $\partial/\partial u$ to $G^{(r)}$. The integral over $\sigma$ now has the form of Eq. (33), which we use (replacing the asymptotic relation by equality):

$$\Phi_m = -iD \int dv du G(x, t - \tau'; v) \frac{\epsilon}{v - u} g(v - u, \epsilon) \frac{\partial}{\partial u} G^{(r)}(u, \tau; y). \hfill (34)$$

We now set $v = 0$ and $u = 0$ in the arguments of $G$ and $G^{(r)}$, respectively, but not in $g$ and $1/(v - u)$ . To integrate over $u$ and $v$ we go to sum and difference variables. Let $w = v - u$ and $\rho = (v + u)/2$ (with unit Jacobian). By inequalities or by pictures, you can check that the integration range $[v > 0$ and $u < 0]$ corresponds to $[w > 0$ and
This leads to
\[
\Phi_m = \int_0^{\infty} dw \int_{-w/2}^{w/2} d\rho \, G(x, t - \tau'; 0) \frac{-iD\epsilon}{w} \frac{1}{g(w, \epsilon)} \left[ \frac{\partial}{\partial z} G^{(r)}(z, \tau; y) \right]_{z=0}.
\]

The integral over \( \rho \) gives \( w \), cancelling the \( w \) in the denominator. The integral over \( w \) is now a Gaussian and \( g \)'s normalization is such that it would yield unity if integrated over the entire line. Since the integration is only over positive \( w \) the result is 1/2. It follows that
\[
\Phi_m = \epsilon G(x, t - \tau; 0) \left[ \frac{D}{2i} \frac{\partial}{\partial z} G^{(r)}(z, \tau; y) \right]_{z=0},
\]
where I have dropped the \( \tau/\tau' \) distinction. We next sum the terms \( \Phi_m \), and noting the \( \epsilon \) in Eq. (36), recognize that this gives the integral in the path decomposition expansion.

3. More than one dimension

For the higher-dimensional generalization of Eq. (21) one considers a volume surrounded by a surface \( \sigma \) such that prior to time \( s \) the particle was within the volume, at time \( s \) it crossed the surface (for the first time), and thereafter is unrestricted. The formula becomes
\[
G(b, t; a) = \int_0^t ds \int_{\sigma} G(b, t - s; c) \frac{i\hbar}{2m} \frac{\partial}{\partial n_c} \cdot \frac{\partial G^{(r)}(c, s; a)}{\partial n_c} \cdot d\sigma.
\]

\( G^{(r)} \) is the restricted propagator for the interior of the volume and \( \partial/\partial n_c \) the normal derivative (going out of the volume).

The restriction can also be imposed for other time periods, not just the beginning. That is, the formula can easily be adapted to a particle that is anywhere up to some time \( s \), after which it is confined. Multiple restrictions are also possible.

Given the intuitive nature of this formula it is natural that it was already known in probability theory. What is peculiar is that it took until the 1980’s to discover it for quantum mechanics.

A surprising use of this formula arose in relativistic quantum mechanics. Halliwell and Ortiz were puzzled because the composition law for relativistic propagators is more complicated than its nonrelativistic counterpart [which is Eq. (20), with “before” and “after” the same]. In particular, it involves derivatives. What they found was that these
derivatives arise because the relativistic formulas can be phrased as applications of the path decomposition formula. One starts with a path integral with a fifth parameter (Sec. 25) in which particle paths can go forward and backward with respect to physical time. Then one uses Eq. (37) with the intervening surface a space-like surface separating initial and final events.

Notes

The path decomposition expansion was developed by Auerbach, Kivelson and Nicole [23, 24] and was part of Auerbach’s Ph. D. thesis. They used it to deal with quantum tunneling in the articles just cited as well as in collaboration with van Baal [25].

In [26], Goodman evaluates the propagator for a particle bouncing off a hard wall using path integration—in contrast to my Sec. 9 justification, which is basically the method of images. See Sec. III, “comment concerning page 40”, for Goodman’s argument. His way of keeping track of paths is reminiscent of the way Auerbach et al. do the path decomposition formula derivation. Amusingly, Auerbach turned things around and worked out the method of images, starting from the path decomposition expansion [27].

Auerbach et al. also used Green’s function techniques to derive the path decomposition formula, with extensions by van Baal [28]. Another approach, by Halliwell [29], uses operators. Halliwell, together with Ortiz [30] studied relativistic propagators.

Neutralizers are used in some approaches to asymptotics where you want to focus on a particular portion of the range of integration and you multiply your integrand by a $C^\infty$ function that is 1 in your range of interest, 0 far away. Together with Ziolkowski, I used the path decomposition formula to develop this for the path integral [31]. In the same article, we also applied that formula to tunneling-time issues. Yamada has studied quantum tunneling and tunneling time, using the path decomposition formula in his work [32, 33]. In particular he has looked quite deeply into the problem of (quantum) measurements when more than a single time is involved and has pointed out the dangers of careless interpretations of quantum tunneling. Besides [31], Sokolovski and Baskin [34] and Fertig [35] have also applied path integral methods to the tunneling problem.
C. Checkerboard path integral

The Dirac equation for a particle moving in one space dimension is

\[
\frac{i\hbar}{\partial t} \frac{\partial \psi}{\partial t} = mc^2 \sigma_x \psi - ich\sigma_z \frac{\partial \psi}{\partial x},
\]

where \( \psi \) is a two-component spinor and \( \sigma_x \) and \( \sigma_z \) are Pauli spin matrices. The propagator, \( G \), for \( \psi \) is a 2 \( \times \) 2 matrix function of space and time. That is, \( \psi(x, t) = \int dy G(x, t; y)\psi(y, 0) \). Feynman found the following way to compute \( G \): fix an \( N \) and consider all paths of the sort shown in Fig. 1. That is, we allow zig-zag paths that travel upwards at 45° or 135° (i.e., at velocity \( c \)) and which may switch direction at times \( kt/N, k = 1, \ldots, N \). For each such path, let \( R \) be the number of reversals (switches) it suffers. For say, the \( (++) \) element of \( G \), take all paths that start at \( (y, 0) \) moving to the right and that arrive at \( (x, t) \) moving to the right, as illustrated, and use them to compute the following sum

\[
\sum_{\text{zig-zag paths}} \left( \frac{it}{N} \frac{mc^2}{\hbar} \right)^R
\]

For \( N \to \infty \) this becomes \( G_{++} \). From the figure one also sees that these paths are the legal moves of a king in checkers or of a bishop in chess. For this reason the sum is often called the checkerboard or chessboard path integral, although unlike their game board counterparts, these particles only move in one vertical direction (but see the notes for generalizations).

What are we to make of this? It was invented by Feynman in the 1940’s. Presumably, he did not consider it important since he only published it as an exercise in his 1965 book with Hibbs. His goal, he said (privately), had been to start with space only and get spin. Thus in one space dimension his formalism already demands a two-component object. But when he couldn’t do the same thing for three space dimensions he dropped the whole business.

Let me tell you all the things \textit{wrong} with this idea. First the two components in the one-space-dimension Dirac equation have nothing to do with spin—there is no spin in one dimension. They are related to parity and in this light their connection to the left- or right-going paths looks reasonable. In fact for three space dimensions spin only requires two-component spinors, and it is again parity that doubles the number, to give the usual 4-component Dirac spinor. The next thing I never liked about this approach was the absence of any classical action. Perhaps Feynman had found a clever way to compute \( G \), but “clearly” it could
FIG. 1: A zig-zag path on a checkerboard discretization of space time. The dotted rectangle indicates the region of contributing paths. The velocity of light is taken to be 1.

have no relation to his more famous path integral if it had no action. Finally, and this I considered the most devastating observation, the scaling of $\Delta x$ and $\Delta t$ as $N \rightarrow \infty$ was wrong. By taking the paths to have light velocity you get $\Delta x = c\Delta t$ with $\Delta t = t/N$. Thus $\Delta x/\Delta t \sim \text{const}$ for $N \rightarrow \infty$. This is in dramatic contrast to the nonrelativistic path integral, for which $\Delta t$ and the square of $\Delta x$ are of the same size. A more detailed statement is

$$\frac{(\Delta x)^2}{\Delta t} \sim \frac{\hbar}{m}. \quad (40)$$

The quantity $\hbar/m$ is the analogue of the diffusion coefficient. Thus in Eq. (9.6) the diffusion coefficient for Brownian motion emerges from the $\Delta x$ and $\Delta t \rightarrow 0$ limits, while maintaining the ratio $D \equiv (\Delta x)^2/2\Delta t$ constant. This ultimately leads to a density, $\rho = \exp(-x^2/2Dt)/\sqrt{4\pi Dt}$. The \textit{“density”} for the (nonrelativistic) path integral is $\exp(i mx^2/2\hbar t)/\sqrt{2\pi i \hbar t/m}$, so that apart from factors of 2 and $i$, $D$ and $\hbar/m$ have the same role. That not-so-inconsequential $i$ is the reason Eq. (40) has the symbol \(\sim\) rather than a proper equality.

Returning to Feynman’s formula, it is obvious that with so great a buildup I am about to tell you why $I$ was wrong. The key to this problem
is to ask the right question. The right question is: for the important contributions to the sum in Eq. (39), how large is $R$?

To answer this, group the terms in the sum according to the number of reversals. For given $N$, let $\phi^{(N)}(R)$ be the number of zig-zag paths with exactly $R$ reversals; we have

$$
G = \sum_{R} \phi^{(N)}(R) \left( \frac{mc^2}{\hbar} \frac{t}{N} \right)^R,
$$

(41)

where $G$ and $\phi^{(N)}(R)$ have implicit subscripts, $(\pm, \pm)$, for path-direction labels. For that same $N$ we want to determine for which $R$ the quantity $\phi^{(N)}(R)(mc^2t/\hbar N)^R$ is maximal. With a bit of combinatorics you can readily derive (see the exercise below) the fact that

$$
R_{\text{max}} = \frac{t mc^2}{\gamma \hbar},
$$

(42)

where

$$
\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad v = \frac{b - a}{t}.
$$

(43)

Remarkably, $R_{\text{max}}$ is independent of $N$. If you stare a bit at Eq. (42) you will see that the typical number of reversals is just the proper time in the particle’s rest frame, measured in time units $\hbar/mc^2$, which is the time for light to cross the particle’s Compton wavelength.

Here is a way to look at this. The particle barrels along at the velocity of light. At any moment it can reverse direction just as when you have a large radioactive sample and at any moment one of the nuclei may decay. That is, there is a rate of reversal just as there is a rate of decay. For either random process, taking a finer mesh for time (increasing “$N$”) does not change the number of reversals or decays per unit physical time. The stochastic process associated with decay is the Poisson process and we have the familiar formula $\text{Prob}(k \text{ decays in unit time for a process with decay rate } r) = r^k e^{-r} / k!$. Formally, Feynman’s one-dimensional electron theory is such a process with one important—and familiar—difference. Decay in time $dt$ has probability $r dt$. Reversal in time $dt$ has probability amplitude, $imc^2 dt/\hbar$, with an $i$. This parallels the real-imaginary correspondence discussed above following Eq. (40).

Why am I stressing this parallel? First to get across the idea that this checkerboard business has a certain richness to it, that it is not just plucked from the air. Second to pay tribute to one of the most important figures in the development of path integration, Mark Kac. It was Kac
who recognized around 1950 that Feynman’s path integral was related by analytic continuation to the Wiener integral used in Brownian motion. As just shown (see also the notes and Footnote [36]) the checkerboard path integral is also an analytic continuation of a stochastic process, and what I wish to mention is that one of those involved in making this more recent connection was the same Mark Kac.

As a mathematical aside, the measure defined by the Poisson process with an imaginary parameter provides a bona fide measure on paths, unlike its nonrelativistic counterpart.

But there is still my main complaint. How can one get $(\Delta x)^2/\Delta t \sim h/m$ from a theory that takes $\Delta x = c\Delta t$? The answer is a satisfying bit of asymptotics. For Brownian motion successive steps are completely uncorrelated. For the relativistic case, even though the electron is allowed to reverse at any moment it usually does not. In fact it has a correlation length $x \sim h/mc$, the Compton wavelength. This means that if I tell you at some point which way it is going, it is likely to be going in the same direction for a distance $\Delta x \sim h/mc$ no matter how finely the time interval is divided. On a short time scale the electron moves at the speed of light. But if you only check its position at widely separated intervals, many reversals will have taken place. Its (net) velocity is less than $c$ and its direction in successive snapshots will be uncorrelated. This is Brownian motion.

Let us estimate the diffusion coefficient for that Brownian motion. The correlation length is $h/mc$, so suppose that on shorter length scales it does not reverse, but that once it has traveled that far it can. For this motion $\Delta x \sim h/mc$. Moreover, while it is travelling without reversing, its velocity is $c$, i.e., $\Delta t = \Delta x/c$. Therefore for this random walk

$$\frac{(\Delta x)^2}{\Delta t} = \frac{(\Delta x)^2}{(\Delta x)/c} = \frac{h}{mc}c = \frac{h}{m}$$

We have recovered the “diffusion coefficient” for the rough paths in the Feynman integral.

Poor time resolution therefore gives the nonrelativistic limit. You could have expected as much. For ordinary Brownian motion the infinite velocities arise from a mathematical idealization. Robert Brown’s grains of pollen move at finite, if large, velocity between the closely spaced blows by water molecules. Similarly we now have insight into the infinite velocities predicted by nonrelativistic quantum mechanics. They are due to smearing over the motion at the smallest scales. As you improve time resolution, velocity increases, but only to its natural maximum, $c$. The checkerboard path integral shows how this transition is accomplished.
This picture of an electron’s motion attributes its mass and lower-than-\(c\) velocity to the random process that makes it change direction; or stated more directly, \((i \text{ times})\) the mass is the rate of flipping. Three dimensional versions of the checkerboard path integral (see below) preserve this property.

**Exercise:** Derive Eq. (42) by evaluating the \(\phi^{(N)}(R)\) of Eq. (41). Don’t forget the subscripts \((\pm, \pm)\) on \(\phi^{(N)}(R)\) and the associated constraints. Hint: it’s a product of two combinatorial coefficients. See [37] for details.

**Notes**

Feynman first published the checkerboard path integral in his book with Hibbs [38], although Schweber, going through boxes of Feynman’s notes, found his early calculations [39]. The checkerboard formulation was independently discovered by Riazanov [40].

The person who turned me around on the checkerboard path integral was Jacobson, who saw in it hints of a fundamental structure of space. Some of the material above is developed in [37], but Jacobson went further and set up a formalism for the full 3-space dimensional object in terms of spinors that pairwise reproduced space [41, 42]. When Kac learned of our results he recalled his own derivation of the telegrapher equation [36, 43] and together we worked out the relation between the Poisson process for the telegrapher equation and the prescription Feynman had developed for the one-dimensional Dirac equation [44].

The checkerboard path integral has attracted enthusiasts over the years, with applications to quantum tunneling, although most work has been devoted to extending the method itself, whether by going to higher space dimension, or by taking more general paths such as those going backward in time. Articles of this sort of which I am aware are by Ord, Jacobson, Ranfagni, DeWitt, Foong and their collaborators [45–54].

Gaveau and I also tried our hand at going to three space dimensions [55–57] and, as for other researchers, the simplicity of Feynman’s one-dimensional method was not recovered. The problem lies in the way the gradient appears in the Dirac equation. Aside from the \(\sigma\) associated with parity, in one dimension the gradient is just \(d/dx\), the generator of spatial translation. By pulling apart the two parity components, one can get this \(d/dx\) to produce paths in ordinary (one-dimensional) space. But in higher dimension one has \(\mathbf{\sigma} \cdot \nabla\), with the \(\mathbf{\sigma}\) now the generator of rotations (and another set of Pauli matrices for parity). (The derivation in [55] allows this to be seen particularly clearly.) This object, \(\mathbf{\sigma} \cdot \nabla\), does not generate translations in 3-space. Instead one introduces other
objects, in our case Grassmann variables, that replace the simple \( dx \) of one dimension. For all these formulations the square of these “other objects” brings you back to ordinary space, which is in itself satisfying, but nevertheless lacks the appeal of the one-dimensional case. Perhaps I should be more positive: the “appeal” of a method is a matter of taste, and taste often needs to be educated. There may come a time when this is considered the natural way to think about the electron.

D. Exact solutions

In the Appendix to Sec. 6 I muse about the known exact solutions, circa 1980. What I mean by “exact” is an analytic expression for \( \langle x | \exp(-iHt/\hbar) | y \rangle \), with \( t \) the physical time [58]. By this definition the list has not grown much, although the thesis that all such solutions correspond to semiclassical results has been knocked out. In the present section I will also mention another class of solutions that are significant contributions to the path integral literature and which are sometimes called “exact” under less stringent criteria.

1. \( \delta \)-function path integral

Consider the Hamiltonian

\[
H = \frac{p^2}{2m} + \lambda \delta(x), \quad x \in \mathbb{R}.
\]

Given the ease with which one can solve for the eigenstates you might have thought the propagator would not present difficulties. Nevertheless, at the time this book was written, no complete form was known. As it turns out, the derivation can be skipped since the result—once you have it—can be verified directly.

Take units with \( \hbar = 1 \) and \( m = 1 \). As usual, the propagator \( G(x, t; y) = \langle x | \exp(-iHt) | y \rangle \). Moreover, like the wave function itself, the propagator will have a discontinuity in its first (space) derivative at \( x = 0 \). Define first the free-particle propagator

\[
G_0(x, t; y) = \sqrt{\frac{1}{2\pi it}} \exp \left( \frac{i(x - y)^2}{2t} \right).
\]

Then the \( \delta \)-function propagator is

\[
G(x, t; y) = G_0(x, t; y) - \lambda \int_0^\infty du e^{-\lambda u} G_0(|x| + |y| + u, t; 0)
\]
Since $G_0$ is a Gaussian integral, $G$ itself is seen to be a Fresnel integral or Error function (if you do an analytic continuation). Verifying that this is indeed a solution involves just a slight subtlety in checking that certain quantities vanish, but I will not go into detail.

This propagator does not agree with what classical mechanics predicts. If you make a square barrier infinitely high but infinitely thin (preserving the product) a particle never gets through. If you make an attractive well infinitely deep and infinitely thin (again preserving the product), the particle passes through with zero contribution to the action (an easy calculation). This is at odds with Eq. (46), which gives the usual partial-transmission and partial-reflection.

Finally, it possible using supersymmetry to generate entire classes of exact solutions, very much like that just presented (and including it as a special example). A reference is given below.

2. Half-plane barrier

Another exact quantum propagator, found since publication of this book, is for the half-plane barrier. In this case the particle is free except that the wave function must vanish on the half-plane defined by $y < 0$, all $z$. Since there is symmetry in the $z$ direction we can drop to two dimensions and look only in the $x$-$y$ plane, with the wave function vanishing on the negative $y$-axis. Using the angles defined in Fig. 2, the propagator, $G(b, t; a)$, is built as follows: Let

$$\omega_1 = (\theta_a + \theta_b)/2, \quad \omega_2 = (\theta_a - \theta_b - \pi)/2,$$

$$\mu = \sqrt{ab/\hbar} \sin \omega_2, \quad \nu = \sqrt{ab/\hbar} \sin \omega_1,$$

where $b = b(\cos \theta_b, \sin \theta_b)$, etc. Define the function $h$ as

$$h(u) \equiv \frac{1}{\sqrt{i\pi}} \int_{-\infty}^{u} \exp(iv^2)dv.$$ (47)

Take the particle’s mass to be $1/2$. Then

$$G(b, t; a) = \frac{1}{4\pi i \hbar t} \exp \left[ \frac{i(a + b)^2}{4\hbar t} \right]$$

$$\times \left\{ \exp(-i\mu^2)h(-\mu) \mp \exp(-i\nu^2)h(-\nu) \right\},$$ (49)

where the upper sign corresponds to Dirichlet boundary conditions (the usual conditions for a barrier), but for completeness the lower sign is also given. The latter would be used for Neumann boundary conditions.
Verification that this is indeed the propagator can be accomplished by taking a lot of derivatives. That’s not the way $G$ was found, but once you have the answer, it’s the quickest. Note that through the function $h$ the Error function or Fresnel integral again (as for the $\delta$-function propagator) make an appearance.

Eq. (48-49) as it stands is not particularly transparent, but it simplifies if one uses the asymptotic form of $h$

\[ h(u) \sim \Theta(u) - \frac{1}{2\sqrt{\pi u}} \exp(iu^2) \sum_{n=0}^{\infty} a_n(iu^2)^{-n}, \quad |u| \text{ large,} \quad (50) \]

with

\[ a_0 = 1, \quad \frac{a_n}{a_{n-1}} = n - \frac{1}{2}, \quad n > 0, \quad \Theta(u) = \begin{cases} 1 & \text{for } u > 0 \\ 0 & \text{for } u < 0 \end{cases}. \quad (51) \]

Geometrically, $h$ will have its unit contribution from $\Theta$ when the points $a$ and $b$ are either mutually visible or when there is a path between them that reflects off the barrier. This allows the usual semiclassical formulas to be recovered. When there is no path, the propagator is smaller, and
in particular there is an additional factor $\sqrt{\hbar}$ multiplying it, as will be seen in detail below.

Nevertheless, for the case of no classical path, the term that does remain, cut down by $\sqrt{\hbar}$, has an interesting interpretation. Consider the path (with time variable $s$)

$$\xi(s) = \begin{cases} a (t_0 - s)/t_0 & \text{for } s \leq t_0 \\ b (s - t_0)/(t - t_0) & \text{for } s \geq t_0 \end{cases}. \quad (52)$$

This is the broken line $a\cdot O\cdot b$ (with $O$ the origin). The time $t_0 \equiv at/(a + b)$ is chosen so there is no change in speed at the origin. In a sense it’s the best you can do, given that you want to go from $a$ to $b$ and can’t pass through the negative $y$-axis. How close is this to being a good classical path? By the Euler-Lagrange equations the first functional derivative of the action, $S = (1/4) \int \dot{x}^2 \, ds$, should be zero. Instead, for $\xi(s)$ it is

$$\frac{\delta S}{\delta x(s)} \bigg|_{\xi(\cdot)} = -\frac{(a + b)}{t} \left( \frac{\hat{a} + \hat{b}}{t} \right) \delta (s - t_0) \neq 0 \quad (53)$$

($\hat{a} = a/|a|$, etc.). But $\xi(s)$ does have an extremal characterization: it minimizes $S$ subject to the nonholonomic constraint forbidding transit through the barrier. It is easy to evaluate the action along $\xi$:

$$S[\xi(\cdot)] = \frac{1}{4} \left[ t_0 \left( \frac{a}{t_0} \right)^2 + (t - t_0) \left( \frac{b}{t - t_0} \right)^2 \right] = \frac{1}{4} \frac{(a + b)^2}{t}. \quad (54)$$

Comparing this to the half-plane propagator, Eq. (48-49), one sees that the phase on line (48) is the action of a path from $a$ to the origin to $b$. Moreover, the power of $\hbar$ appearing on the left of line (48) is the correct power for two dimensions. But the point that I consider of greatest interest arises from the rest of $G$. For the case that there is neither a direct nor a reflected path, the last factor in $G$ is asymptotically

$$\begin{bmatrix} \text{Shadow} \\ \text{correction} \end{bmatrix} \equiv \exp(-i\mu^2\hbar(-\mu)) \mp \exp(-i\nu^2\hbar(-\nu))$$

$$\sim -\frac{1}{2} \sqrt{i\hbar} \frac{1}{\pi ab} \left[ \sec \left( \frac{\theta_a - \theta_b}{2} \right) \mp \sec \left( \frac{\theta_a + \theta_b}{2} \right) \right].$$

This has the additional factor $\sqrt{\hbar}$, mentioned earlier, complete with a geometrical factor. At this point you might be tempted to say, “Great! I know how to deal with flawed paths: hit them with an extra $\sqrt{\hbar}$ and
work out a geometrical term that has information about how deeply shadowed the endpoints are from each other.” That would indeed be an excellent idea, and you would have rediscovered the beginnings of the “Geometric theory of diffraction,” due to Keller, which gives prescriptions for a variety of flawed paths. The context is not quantum mechanics, but electromagnetic wave propagation, and the power of the theory is not that you produce the rare exact solution, but rather that for geometrical configurations that approximate the exact layouts this prescription works asymptotically. Thus for a curved barrier with a sharp edge, you would calculate the field, or amplitude, in the shadow by using the geometrical diffraction theory prescription, except that you would have an asymptotic result. As long as the scale of the differences between the true barrier and the ideal one is larger than the asymptotic parameter (wavelength in Keller’s case), the method works.

**Infinite dimensional perspective.** The result (48-49) was first derived in a peculiar way: a third point, call it $c$, was defined in the plane, mutually visible from $a$ and $b$. Next, the semiclassical propagators from $a$ to $c$ and from $c$ to $b$ were multiplied by one another and the result integrated over all appropriate $c$. (Reflection off the barrier was also included.) The result, after several asymptotic approximations, was Eq. (48-49), which is an exact solution. I won’t focus on the good fortune of unjustifiably landing on an exact solution, but instead will elaborate on the asymptotic features. First recall simple facts about asymptotics.

Consider the integral

$$ F(\lambda) = \int_0^\infty e^{i\lambda f(x)} g(x) \, dx, \quad x \in \mathbb{R}, \quad \lambda \text{ large.} \tag{55} $$

Usually (e.g., Sec. 11) one studies the “stationary phase approximation,” where for some $x_0 > 0$, $f'(x_0) = 0$. In that case

$$ F(\lambda) \sim \sqrt{\frac{2i\pi}{\lambda f''(x_0)}} e^{i\lambda f(x_0)} g(x_0) \tag{56} $$

But if there is no stationary point, that is, $|f'(x)| > 0, \forall x \geq 0$, one can do an integration by parts:

$$ F(\lambda) = \frac{i}{\lambda f'(0)} g(0) - \frac{1}{i\lambda} \int_0^\infty dx e^{i\lambda f} \frac{d}{dx} \left( \frac{g}{f'} \right) \tag{57} $$

The new integral on the right in Eq. (57) is like the old one, but carries an extra $1/\lambda$. Therefore, asymptotically

$$ F(\lambda) \sim \frac{ig(0)}{\lambda f'(0)}. \tag{58} $$
Two features of this result should be emphasized: first, the asymptotic value of the integral is determined by the boundary of integration, in this case \( x = 0 \). Second, the overall scale is cut down by a factor \( \sqrt{\lambda} \) relative to its value in the stationary phase case [Eq. (56)].

Returning to the path integral, we take the brave perspective that it is a sum over paths, an infinite dimensional integral,

\[
G(b, t; a) = \int_{\Omega} D x(\cdot) e^{iS[x(\cdot)]/\hbar},
\]

where \( \Omega \) is the domain of functional integration. In the half-plane barrier case, \( \Omega \) should not include paths crossing the negative \( y \)-axis (now down to 2 dim.). Therefore, depending on \( a \) and \( b \), there may not be stationary “points” of the action, \( S \) in \( \Omega \). But Eq. (48-49) tells us what to do in that case: go to the best point on the boundary of \( \Omega \), in this case a path that just nicks the tip of the barrier, evaluate \( S \) for this path, and cut down the entire result by the square root of the asymptotic parameter—exactly what we did for one-dimensional asymptotics.

I have been tantalized by this connection for more than 20 years, and off-and-on have tried to put it all together. I will mention a bit more of what I believe must go into the connection, and hope that some enterprising reader can succeed. (Presumably one should first do this for the Wiener integral, where Eq. (59) has rigorous meaning, but I will continue to speak of the path integral and stationary phase approximations.) The \( n \)-dimensional generalization of Eq. (57) is

\[
\int_{\Omega} d^n x g(x) e^{i\lambda f(x)} = \frac{1}{i\lambda} \int_{\partial \Omega} d^{n-1} x \frac{g(x) \nabla f}{|\nabla f|^2} e^{i\lambda f} \tag{60}
\]

This formula is valid only if \( \nabla f \neq 0 \) throughout \( \Omega \). If \( \nabla f \) had vanished in \( \Omega \) there would be a \( 1/\sqrt{\lambda} \) multiplying the leading term, rather than \( 1/\lambda \). Moreover, as before, the first term on the right of Eq. (60) is larger by a factor \( \lambda \) than the second term. Thus the step from 1 dimension to \( n \) does not substantially change the asymptotics. The tough part is the case \( n = \infty \) and establishing that the surface integral gives the desired term. One feature of Eq. (60) that only appears in dimension greater than one is that the “surface integral” (over \( \partial \Omega \)) is also subject to the stationary phase approximation, which is why in our half-plane example the flaw in the path \( \xi(t) \) occurred only on the boundary. As a last remark I mention that the concept of neutralizer, so useful in asymptotics, can also be defined for the path integral [31].
3. **Coulomb potential and related solutions**

In classical mechanics there is a way to make the Kepler (or Coulomb) problem into a harmonic oscillator. It is known as the Kustaanheimo-Stiefel transformation and involves both a change of spatial coordinates and a change of time variable. The new time is well known in classical mechanics and is called the “eccentric anomaly” [59]. It is explicitly a function of a particular path: \( s(t) \equiv \int_0^t \frac{d\tau}{r(\tau)} \), with \( r \) the radial coordinate of the particle. The change of coordinates is more complicated (in three dimensions) and one must ascend to a 4-dimensional coordinate space. Nevertheless, it brings considerable simplification to the problem and has been extensively used in classical contexts. Duru and Kleinert [60, 61] had the lovely idea that once you had a path integral you could take advantage of the Kustaanheimo-Stiefel transformation since, thanks to Feynman, there were paths, and functions \( r(t) \), to work with inside the path integral. Thus for each path you go over to the new time. With the additional spatial variable, the dynamics becomes that of the harmonic oscillator, and can be done explicitly.

What these authors and others derived from this formalism is mainly information about the energy-dependent Green’s function. This is because the harmonic oscillator propagator that emerges is for paths in the “new” time, \( s \). Since there is a different \( s \) for each path, they cannot all be gathered into a single physical-time-dependent propagator. For this reason the goal of an explicit function \( G(r'' \, t; r) \equiv \langle r'' | \exp \left( -\frac{iHt}{\hbar} \right) | r \rangle \) has still not been attained with \( H \) the Coulomb Hamiltonian.

**Notes**

The propagator for the \( \delta \)-function potential given in Sec. ID 1 was found by Gaveau and me [62] using two methods. The simpler was an expansion over eigenstates; the other was probabilistic and used the path decomposition expansion. Ref. [63] provides the verification (mentioned in the text) that the given propagator satisfies the time-dependent Schrödinger equation. Some years earlier Goovaerts, Babcenco and Devreese [64] had worked out a perturbation expansion for the path integral and applied it to the \( \delta \)-function potential. They had a particular projection of the full propagator which nevertheless sufficed for the recovery eigenfunctions and scattering states.

The supersymmetry-based generation of exact propagators was found by Jauslin [65] and includes as a special case the \( \delta \)-function.

In more than one dimension the \( \delta \)-function is invisible; nevertheless,
people in nuclear physics long ago developed singular objects for idealized point scattering centers. Explicit propagators for such “potentials” were found by Scarlatti and Teta [66].

In 1982 there was a conference on “wave-particle duality” in honor of de Broglie’s ninetieth birthday. Keller’s geometrical diffraction theory [67] provides counterpoint to that theme: historically diffraction was a deciding factor in calling light a wave, but here was Keller with a way to calculate diffractive fields using rays, something you might be tempted to call particle paths. Because of the importance of paths in Keller’s story it seemed a good idea to derive his results from the path integral (cf. page 169). Ref. [68] uses a semiclassical approximation to derive the knife-edge (or half-plane) propagator, and noting factors \( \sqrt{\hbar} \) and the dominance of the boundary term (of the range of functional integration) I conjectured that this was part of a larger asymptotic theory. The later realization that the result was exact led to an additional publication [69]. Apparently the exactness arises because the plane-with-barrier is the projection of a two-sheeted Riemann surface with the cut located along the barrier [70]. On this surface the motion looks free. This was exploited by Sommerfeld and for the heat kernel by Carslaw. For some of the history of this problem see [71].

As to getting exact results after lots of approximations, my only reaction is to recall the witticism (of Dyson?) that a good physicist is one who makes an even number of mistakes.

Following the original papers by Duru and Kleinert [60, 61] there was a great burst of activity, some offering alternative demonstrations, and many extending the technique to other systems. A small part of the literature is Refs. [72–77].

E. Dissipation and other forced-oscillator applications

You could make the case that the most powerful, effective, and famous applications of the path integral are based on the forced harmonic oscillator. The explicit path integral is given in Eqs. (6.41) and (6.42). As seen in Sec. 21 on the polaron, this comes into play when you have a complicated degree of freedom in contact with a bunch of oscillators, notably the electromagnetic or phonon field, and the coupling to those oscillators is linear in the oscillator coordinate. Call the “complicated” degree of freedom \( r \) and the oscillator coordinates \( Q_n, n = 1, \ldots \). Then the coupling in the Lagrangian will be \( \sum f_n(r)Q_n \), with the form of \( f_n \) depending on the specific problem. Because \( \int \mathcal{D}r(\cdot) \) is performed last, as far as \( Q_n \) is concerned \( f_n(r(\cdot)) \) is just an external force.
At the beginning of the 1980’s a new application of this idea sprung to life and has been going strong ever since. The physical system first studied was the Josephson junction, for which the state of a large number of electrons is subsumed into a single degree of freedom, the trapped flux in a superconducting ring interrupted by a resistor. At a phenomenological level this flux variable, \( \phi \), satisfies an equation of motion

\[
M \ddot{\phi} + \eta \dot{\phi} = -\frac{dV}{d\phi} + F_{\text{ext}}(t),
\]

where the various parameters, the potential and the external force characterize the system, and in particular \( \eta \) is a measure of dissipation. The case of particular interest is the “SQUID” (superconducting quantum interference device) for which the potential \( V \) can be made to be a double well. For the usual situation, having the system variable \( \phi \) be in one or the other well means that the supercurrent is flowing in one direction or the other around the loop that constitutes the SQUID. At the quantum level an amazing possibility opened up. This variable could tunnel, by quantum tunneling, between the wells. This would mean that a collective variable for perhaps \( 10^{11} \) electrons would behave in a purely quantum fashion.

The theoretical framework for studying this tunneling was to model the source of the dissipation, namely to put in the coordinates of the phonons that provided the resistance in the junction. Thus one would have a quantum treatment of the whole system: energy would go from the principal degree of freedom to the resistor, but that degree of freedom could itself go from being in one state (localized on one well) to another (the other well).

The coupling of the phonons to the system degree of freedom could indeed be set up in the desired form (linear forcing) and the tunneling calculated. The Lagrangian used by Caldeira and Leggett [78] was

\[
L = \frac{1}{2}M \dot{q}^2 + V(q) + \frac{1}{2} \sum_\alpha m_\alpha \dot{x}_\alpha^2 + \frac{1}{2} \sum_\alpha m_\alpha \omega_\alpha^2 x_\alpha^2 + q \sum_\alpha c_\alpha x_\alpha,
\]

with \( q \) the dynamical variable (ultimately identified as \( \phi \)), \( \{x_\alpha\} \) the phonon coordinates, and \( \{c_\alpha\} \) characterizing the coupling. One of the important physical properties that played a role in the results was the phonon spectrum, embodied in \( \{c_\alpha\} \) and implicit in \( \sum_\alpha \). Simplifications were also introduced, for example replacing \( q \) by a two-state variable, giving the entire system the name “spin-boson model.” As for the polaron, one first does the path integral for the \( \{x_\alpha\} \), then sandwiches the propagator between states of the oscillators, typically the ground states. After this the functional integral involves the original Lagrangian in \( q \).
plus the result of all the completed integrations, which can now depend only on $q$. This gives an action contribution (the imaginary part of the logarithm of the result of all those integrals), which is an effective self-interaction of $q$. This self-interaction is non-local in time (as for the polaron) and represents the back-reaction of the field of oscillators on the motion of the variable of interest, $q$.

By now there are books and review papers that not only describe the SQUID application but also the many other situations where a system interacts with a heat bath, and nevertheless retains quantum properties. Typically, but not always, energy concentrated in the principal degree of freedom leaks into the oscillator modes, i.e., there is dissipation. I will not document the history of this application, but only cite sources where the reader can get into the subject [79–83]. More on this (and other) literature appears in Sec. II.

Meanwhile, other applications of Feynman’s forced-oscillator integration continue to flourish. The polaron has historically had a special role: even those content to go through life without the pleasures of path integration will concede that for the polaron you can compute things using the path integral that are difficult to get by other methods. A review by Devreese [84] summarizes the polaron situation as of 1996. His group in Antwerp continues to work in this area [85, 86] and to use the path integral for other physical systems as well [87].

I recently used the forced oscillator technique for a condensed matter application, the quantization of discrete breathers. These are lattice excitations involving strong nonlinearity. Classically such localized oscillations are a kind of stationary soliton and can live forever [88]. By massaging the system it can be brought to a form suitable for the forced oscillator machinery, and the quantum stability of the system can be studied [89]. Let me pass on one lesson I learned from this: trust no one (including me). Even if you take Eqs. (6.41) and (6.42) as correct (which I believe they are), you still have several Gaussian integrals to do. Then there’s usually an analytic continuation and a double integral variously written $\int_0^T dt \int_0^T ds$ or $\int_0^T dt \int_0^T ds$, which brings in another factor two. For the serious people, I trust them to have gotten all these signs and factors right in their research papers—but not in their lectures! In my work, [89], I wasn’t sure of all these factors and signs until I found a way in which the effective action could be used to recover the spectrum of the original system, and when it agreed with that spectrum calculated by other means, I stopped worrying. By the way, symbolic manipulation programs, at least in the hands of my collaborators, were of no help. (See also the “comment on page 173” in Sec. III.)
Making proper mathematics out of the path integral has been a dream since Feynman’s first paper on the subject. Like Newton and Dirac, mathematical innovators in quest of physical knowledge, Feynman knew how to get things right even if not all the mathematical i’s were dotted. And perhaps, as for the inventions of his predecessors, a full mathematical theory will follow. In Sec. 9 I mentioned work of DeWitt, Albeverio, Hoegh-Krohn and Truman, all of them having this goal in mind. Cameron (cited in the Sec. 9 notes) had found that the path integral could not provide a bona fide measure. But if you weaken the demands, redefine “bona fide,” new possibilities open. For example, Tepper and Zachary [90] have relaxed the need for countable additivity in the measure and find a useful object nevertheless. Another tack has been taken by Fujiwara and Kumano-go [91, 92], who focus on the properties of the time-sliced expression for the propagator, establishing classes of functions for which that is meaningful, and developing asymptotics in that context as well. Yet another approach is by Nakamura [93], who has been using non-standard analysis [94], taking advantage of that framework’s ability to with deal with explicit infinities and infinitesimals.

One avenue that appeals to me physically (although it has received only slight attention [95]) would start from the checkerboard path integral. For this there is a well defined measure. Then find its non-relativistic limit, which as discussed in Sec. IC, is the usual path integral.

G. Electromagnetism and other PDE’s

In Sec. 20 I showed how the path integral could be applied to geometrical optics, basically to the wave equation, where the big step is to get rid of the \( \frac{\partial^2}{\partial t^2} \) by looking at a fixed frequency. Then, by hand, I inserted an artificial time variable, so as to make it look like the Schrödinger equation. My goal was aesthetic, to show the common features of Fermat’s principle of optics and the semiclassical approximation of quantum mechanics.

It turns out that there’s a practical side to this resemblance as well. People really do use the path integral for the solution of partial differential equations other than the Schrödinger and diffusion equations. Furthermore, in some of these applications the “time” variable has a clear physical meaning as the spatial variable along the direction of propagation (analogous to the eikonal approximation). Quite a few papers on
sound transmission in channels have appeared and Refs. [96–98] should help you find your way into this field. More recently, Schlottmann [99] pioneered the use path integral methods in geology. This is a difficult wave transmission problem, since not only is the medium nonuniform, but it’s that uniformity that you are interested in. Impressive results of in this area have been obtained by Fishman [100, 101], whose path integral methods for the Helmholtz equation can provide a detailed picture of geological structure, especially useful for oil exploration.

For electromagnetism path integral methods have been used to find the low order modes of complicated waveguides [102, 103]. This is not the time-dependent propagator, but an application of the method of Sec. 7, taking advantage of the dominance of the lowest mode for large imaginary times. As a practical method this was first proposed by Donsker and Kac in the 1950’s, and I believe they even did numerical calculations (punch cards and all). There have also been significant extensions of ray tracing techniques [104].

H. Homotopy

For diverse reasons, there are physical systems where the homotopy of the coordinate space is not trivial. As described in Sec. 23, the path integral provides an intuitive approach to multiple connectedness, since homotopy theory, like the path integral, focuses on paths, with its starting point being an equivalence relation among them.

The phase that can be associated with winding number, discussed in Sec. 23.1, has as its most common application the Aharonov-Bohm effect. The path integral/homotopy perspective goes back, as far as I know, to my thesis and its extension (for the Aharonov-Bohm effect), Ref. [105]. A physical arena where this frequently arises is the mesoscopic superconducting ring, a recent paper on this being [106]. See also [107, 108]. A closely related way to think about the richness permitted by the multiple connectedness is in terms of the Berry phase.

These phenomena also are present in non-Abelian gauge theories and have been studied by Sundrum and Tassie [109].

A somewhat different application came up in connection with the concept of anyons. As Laidlaw and DeWitt realized (Sec. 23.3), in two dimensions you can get additional kinds of quantum statistics, not just fermions and bosons. Such excitations were proposed in the attempts to understand high temperature superconductivity [110, 111]. (See also Canright’s lectures in [112].) Ultimately, in the realm of superconductivity these ideas were put aside, but other applications have been proposed.
I. And more . . .

Sec. 32 of this book is entitled “Omissions, Miscellany and Prejudices,” and represented my effort to get just a bit more material under the wire of a publication deadline. In this respect, little has changed in 20+ years. Here too I present just the briefest guide to a number of developments that have caught my attention.

1. Variational corrections to the classical partition function

An approximation for the partition function was independently developed by Feynman and Kleinert [113] and by Giachetti and Tognetti [114] in the mid 1980’s. They sought quantum corrections to the semi-classical partition function, making use of quadratic approximations to the potential energy function. These approximations were not merely the lowest quadratic fit, but were themselves found through a variational principle. In both cases, the initial publication was followed by several others of which a small sample is [115, 116].

2. A particular numerical approach

There has been an enormous amount of numerical work using the path integral (usually its real-valued analytic continuation), including cases where fermion statistics create a severe sign problem. I will not review this, although lectures by Pollock can be found in [112] and other literature is given in Sec. II. Here I want to mention one scheme that has not been pursued on an industrial scale and which at first sounds as if it shouldn’t work.

Since the path integral is the sum \( G = \int \mathcal{D}x \exp(iS/\hbar) \), you could imagine a change of variable to a one dimensional integral \( G = \int dS \Omega(S) \exp(iS/\hbar) \), where \( \Omega(S) \) is the number of paths with action \( S \). It’s an ambitious change of variables and requires real belief in the path-sum concept. By this method all the trouble is transferred to the evaluation of \( \Omega(S) \). Creswick [117] proposed this idea and worked out a number of examples with numerical and analytic estimates for \( \Omega(S) \). My first impression was that the roughness of the significant paths mitigated against having any reasonable limit for \( \Omega(S) \) as the time-slicing mesh went to zero. However, although the limit may not exist, for any particular time-slicing (say \( N \) “slices”) \( \Omega \) can be evaluated, and as a function of \( N \) it is possible to define sensible properties. In a way this
reflects the way I believe one should deal with path integral subtleties: go back to the multiple integral, \( \int \prod_{n=1}^{N} dx_n \). Or as Kac is reputed to have said: when in doubt, discretize.

3. Semiclassical approximations for coherent states

In molecular systems there occasionally develop large magnetic moments. Such a moment may have more than one preferred orientation and there will be a quantum tunneling amplitude for transiting between classically degenerate states. Given the largeness of the moments it would be reasonable to expect semiclassical tunneling calculations to be helpful. One would also expect that the coherent state formalism, which has often been used to represent such degrees of freedom, would be a good vehicle for such a calculation, especially since its phase space interpretation brings it close to a semiclassical language. Nevertheless, considerable difficulties were encountered. In recent work more reliable techniques have been developed and I refer the reader to [118] as an entry into this literature.

4. Chaos and order

The use of the path integral in studying quantum chaos was established with Gutzwiller’s trace formula [119]. In many places in this book [e.g., Eq. (13.25)] the propagator is approximated by a sum over classical paths, which means that for given endpoints in \( G(b, t; a) \) there may be several solutions of the classical equations of motion. The propagator is then given by their sum, with interesting possibilities for interference and other quantum phenomena.

For a classically chaotic system the number of solutions to the two-time boundary value problem associated with \( G(b, t; a) \) generally grows exponentially with \( t \). If \( t \) is long enough you’d therefore expect a breakdown in the semiclassical approximation. This is not what happens. Even for long times a great deal of useful information can be extracted from the propagator.

By taking a trace of the propagator and applying a stationary phase approximation, Gutzwiller focused on the periodic orbits. This is because the trace forces initial and final positions to be the same while the stationary phase approximation takes care of matching the momentum. These periodic orbits can be quite long; nevertheless, their contribution to the propagator reliably gives information about the quantum system,
for example concerning the density of states. The literature on this subject extensive. A convenient source, with many references, is [120] by Cvitanovic et al., which is directly available on the web (the Gutzwiller trace formula is in Chapter 30).

Better than expected accuracy in the path sum for a classically ergodic system was also seen by Heller and Tomsovic [121]. They studied a “stadium billiard” (a free particle in a kind of elongated oval) and found that even when there were 30000 paths to add, a great deal of quantum information survived intact. An important technical point is that they were not actually looking at the propagator as a function of its position endpoints; rather they sandwiched it between coherent states, thereby smoothing the singularities that would otherwise arise from caustics of the motion. One feature that helped me understand this discovery was that although in the two-dimensional coordinate space the paths crossed and re-crossed each other, in phase space there is lot more room [122]. Another path-integral oriented approach to the smoothing of caustic singularities, especially in chaotic situations, is by Takatsuka [123].
II. LITERATURE

The library of path integral books has grown substantially since the present volume was first published. There are textbooks, conference proceedings and review papers. Books focussed on other subjects, for example quantum field theory, will often have introductions to the path integral. Below is a partial list of publications since this book appeared. I have not tried to be comprehensive.

A. Texts and review articles

Roepstroff, *Path Integral Approach to Quantum Physics: An Introduction* [124]. Mathematically careful, but not at the expense of good instruction.


Kac, *Integration in Function Spaces and Some of Its Applications* [125]. Notes for lectures Kac gave in Italy. Marked by his usual lively style, although they may be hard to locate.

Wiegel, *Introduction to Path-Integral Methods in Physics and Polymer Science* [71]. Emphasizes polymer applications and is a readable and informative volume.

Gutzwiller, *Chaos in Classical and Quantum Mechanics* [119]. Extensive coverage of quantum chaos, with the path integral serving as a (very effective) tool.

Glimm and Jaffe, *Quantum Physics: a Functional Integral Point of View* [126]. The work of these authors played a part in the development of “constructive quantum field theory,” a subject for which the functional integral is used in a mathematically rigorous way.

Weiss, *Quantum Dissipative Systems* [127]. An introduction with many applications to the use of functional integration in handling dissipation in quantum systems.

Dittrich, Hänggi, Ingold, Kramer, Schön and Zwerger, *Quantum transport and dissipation* [83]. A collection of articles offering introductions to the fields indicated by the title.
Cerdeira, Lundqvist, Mugnai, Ranfagni, Sa-yakanit and Schulman, *Lectures on Path Integration: Trieste 1991* [112]. A collection of diverse lectures at a pedagogical level, given at a workshop at the ICTP, Trieste. As a co-organizer I wanted the topical coverage to be an update of sorts for this book.

Roncadelli and Defendi, *I Camini di Feynman* [128]. A pedagogical work in Italian, also reflecting contributions of the authors to the field.

Choquet-Bruhat and Dewitt-Morette, *Analysis, Manifolds and Physics* [129]. Introduction to many mathematical methods in physics, with in-depth treatment of the path integral.


Freidlin and Wentzell, *Random Perturbations of Dynamical Systems* [131]. Functional integrals as well as other probabilistic methods, with emphasis on asymptotics.

Steiner and Grosche, *A Table of Feynman Path Integrals* [132]. A useful compendium.

**Books on quantum field theory.** As the reader may have guessed, quantum field theory is not my favorite arena for path integration. But for lots of other people it is (“Let a hundred flowers bloom, . . .”). A few of the more well-known texts that make extensive use of path integration are Popov [133], Ramond [134], Itzykson and Zuber [135], Faddeev and Slavnov [136], and Rivers [137].

**Review articles**

Leggett, Chakravarty, Dorsey, Fisher, Garg and Zwerger, *Dynamics of the dissipative two-state system* [81]. Justification and use of the spin-boson model.

Grabert, Schramm and Ingold, *Quantum Brownian Motion: The Functional Integral Approach* [82]. Pedagogical presentation at an advanced level.

Ceperley, *Path integrals in the theory of condensed helium* [138]. Reviews an extensive literature in which Feynman’s original idea of how to deal with Helium (as a classical model of interacting “polymers”) is implemented with intensive numerical computation.
**Khandekar and Lawande**, *Feynman Path Integrals: Some Exact Results and Applications* [139]. Quite a few explicit path integrals including some that are nonlocal in time.

**Gaspard, Alonso and Burghardt**, *New Ways of Understanding Semiclassical Quantization* [140]. Extensive background on semiclassical methods, with the path integral playing a prominent role. Application to quantum chemistry, including reactions and chaotic scattering. Higher order (in \( h \)) corrections to the Gutzwiller trace formula.

**Littlejohn**, *The Semiclassical Evolution of Wave Packets* [141]. Not really path integration, but a trove of information and perspectives on semiclassical approaches.

**Shankar**, *Renormalization-group approach to interacting fermions* [142]. Extensive use of the path integral for fermions (something not covered in this book) to study interaction between them. The fermion path integral uses Grassmann variables and these too are introduced in this article.

### B. Conference proceedings

In 1983 the first of the “Path integrals from meV to MeV” conferences took place, organized by Akira Inomata, in Albany, New York. Since then, every three or four years there has been another such meeting, although the name of the conference has evolved. The original title was selected to emphasize physical phenomena, in contrast to several mathematically oriented meetings in the then recent past. Today we also have mathematicians coming and the limits meV and MeV have variously been replaced by quarks, peV, quantum information, TeV, galaxies and cosmology. The proceedings give an overview of the range of applications, although the mode of publication has varied from meeting to meeting and some of the proceedings may be hard to track down. The meetings: Albany 1983 [143], Bielefeld 1986 [144], Bangkok 1989 [145], Tutzing 1992 [146], Dubna 1995 [147], Florence 1998 [148], Antwerp 2002 [149], Prague 2005 [150].

Besides the meV/MeV series, there have been conferences and workshops where the path integral played a significant role. These can be a useful way to survey activity in the field, although these days people are more inclined to search the web than to get off their chairs and go to the library. A non-exhaustive list of conferences is [151–154].
III. PAGE BY PAGE COMMENTS AND ERRATA

Conventions: references to “Sections” 1 through 32 are to sections of this book (a.k.a. chapters). Equation numbers with periods, \( (n \cdot m) \), refer to equations in the book; those consisting of a number alone refer to the supplement. Sections of the supplement have labels beginning with Roman numerals.

- **Sec. 1, page 4, Eq. (1.11); elaboration of the associated footnote.**
  
  Show that
  
  \[ f(\lambda) \equiv e^{\lambda A}e^{-\lambda(A+B)}e^{\lambda B} = 1 - \frac{1}{2}\lambda^2[A, B] + \ldots. \]

  Evaluate \( df/d\lambda = Af - e^{\lambda A}(A+B)e^{-\lambda A}f + fB. \) Clearly \( f'(0) = 0. \) The second derivative is \( f'' = Af' - e^{\lambda A}A(A+B)e^{-\lambda A}f + e^{\lambda A}(A+B)Ae^{-\lambda A}f + f' B. \) Since \( f'(0) = 0, \) the only term surviving is \( f''(0) = [B, A]. \)

- **Sec. 6, page 39, solution to part (b), exercise at top of page.**
  
  The correct form of the propagator is
  
  \[ G(x_2, t; x_1, 0) = \left[ \frac{m}{2\pi it} \right]^{3/2} \exp \left\{ \frac{im}{2} \left[ \frac{\omega t}{2} \cot \left( \vec{B} \times \vec{r} \right) \right]^2 \right. \]
  
  \[ \left. -\omega \vec{B} \cdot \vec{x}_2 \times \vec{x}_1 + \frac{1}{t} \left( \vec{B} \cdot \vec{r} \right)^2 \right\} \]

  For more on this propagator, see Glasser [155]. When there are constant electric and magnetic fields on a lattice, although the fields share the underlying periodicity, the potentials giving rise to those fields do not. See [156] for a path integral treatment of this situation.

- **Sec. 6, page 40, discussion following Eq. (6.43).**
  
  Eq. (6.43) asserts
  
  \[ G(b, t; a) = \sqrt{\frac{1}{2\pi it}} \left[ e^{i(b-a)^2/2t} - e^{i(b+a)^2/2t} \right], \quad \text{Sec. 6, Eq. (6.43)} \]

  where \( G \) is the propagator for a particle confined to \( x > 0, \ x \in \mathbb{R} \) but otherwise free (\( V = 0 \)). The justification (in Sec. 9) uses what is essentially the method of images. I complain on page 40 that it is an “embarrassment to the purist” that I did not present a sum-over-paths derivation. Shortly after the book was published, Goodman [26] gave the following purely path integral argument.
As usual write
\[ G(b, t; a) = \sum \exp(iS/\hbar), \]
with the sum taken over \( x(\cdot) \) such that \( x(0) = a, \ x(t) = b \) and \( x(s) > 0, \) for \( 0 \leq s \leq t. \) We also define the free propagator, denoted \( G_0(b, t; a) \), which is the sum over paths from \( a \) to \( b \) in time \( t \), but without the restriction. The difference between \( G \) and \( G_0 \) is the sum over paths that have at least once entered the region \( x < 0 \), but returned to \( b \) nevertheless. Call these “forbidden” paths. Then \( G(b, t; a) = G_0(b, t; a) - \sum_{\text{forbidden paths}} \exp(iS/\hbar). \)

Consider a forbidden path, \( x(\cdot) \). There will be a last time, \( t_\ell \), for which \( x(t_\ell) < 0. \) Define the path \( \bar{x}(\cdot) \) as follows
\[ \bar{x}(s) = \begin{cases} x(s) & \text{for } s < t_\ell \\ -x(s) & \text{for } s > t_\ell \end{cases}. \]
The final position of \( \bar{x}(\cdot) \) is thus \(-b\) rather than \( b. \) The difference between \( S(x(\cdot)) \) and \( S(\bar{x}(\cdot)) \) goes to zero as the mesh for time-slicing goes to zero, because the flip in direction at \( t_\ell \) has negligible impact on \( S. \) On the other hand, the set of paths \( \bar{x}(\cdot) \) includes all paths from \( a \) to \(-b\) in time \( t. \) As a result the sum, \( \sum_{\text{forbidden paths}} \exp(iS/\hbar), \) is in fact equal to the free propagator from \( a \) to \(-b). \) Therefore \( G(b, t; a) = G_0(b, t; a) - G_0(-b, t; a). \)

- Sec. 7, page 47, Eq. (7.22).

An even easier way to do the integral. You want to evaluate
\[ f(c) = \frac{1}{\sqrt{b}} \int_0^\infty dy \exp\left(-\frac{c^2}{y^2} - y^2\right), \quad \text{Sec. 7, Eq. (7.22)} \]
Rewrite the integrand slightly,
\[ f(c) = \frac{1}{\sqrt{b}} \int_0^\infty dy \exp\left[-\left(y - \frac{c}{y}\right)^2 - 2c\right] = \frac{1}{\sqrt{b}} e^{-2c \psi(c)}, \quad (62) \]
where \( \psi(c) \equiv (1/\sqrt{b}) \int_0^\infty dy \exp\left[-(y - c/y)^2\right]. \) Now compute
\[ \frac{\partial \psi}{\partial c} = \int dy \ 2 \left[1 - c/y^2\right] e^{-(y-c/y)^2}. \quad (63) \]
But this derivative is zero, since the individual terms in square bracket give the same integral, as can be seen from the substitution \( x = c/y. \) To finish, set \( c = 0 \) to find \( \psi(c) = \psi(0) = \sqrt{\pi}. \)
• Sec. 9, page 54, prior to Eq. (9.4).
The calculation of $u(j, N)$ of Eq. (9.4) uses the assumptions $\alpha^2/N = O(1)$ and $j^2/N = O(1)$. Note too that for finite $N$ Eq. (9.4) does not hold in the extreme tail of the distribution.

• Sec. 9, page 63, new exercises.
Exercise: You enter a casino with $100, intent upon walking out with either nothing or $1000. Assuming you place a series of $1 bets using a fair coin (tell me where to find this casino . . . ), use the methods of this section to compute the probability of each final state.
Exercise: Evaluate $E \left( \frac{|x(t+\delta)-x(t)|^2}{\delta^2} \right)$.

• Sec. 12, page 81, Eq. (12.9).
For the specific case $L = m\dot{x}^2/2 - V$, Eq. (12.9) takes the simple form $\dddot{\phi} + V'' \phi + \lambda \phi = 0$. See Eq. (12.18).

• Sec. 12, page 82, Eqs. (12.17) and (12.18).
The approximation Eq. (12.17) is appropriate for the intended demonstration. But it could be sharpened if for example one wanted to better estimate the interval during which the path is a true minimum. Eq. (12.18) still holds (with $x(t)$ an Euler-Lagrange equation solution), but Eq. (12.19) need not.

Justification of the approximation: If the magnitude of the force, $F(x) \equiv -dV/dx$, is bounded (say $|F| \leq F_{\text{max}}$) then it is easy to show that Eq. (12.17) is good to $O(T^2)$. The equation to solve is $\ddot{x} = -dV/dx$ with $x(0) = a$, $x(T) = b$. Let the true solution be $x(t) = a + (b-a)t/T + u(t)$, so that $u(0) = u(T) = 0$ and $\ddot{u} = F(x)$. We want to bound $u$. First integrate to $t \leq T$. $\ddot{u}(t) = \dot{u}(0) + \int_0^t F(x(s))ds$ (lower limits of integration are 0 throughout). This implies $|\dot{u}(t)| \leq |\dot{u}(0)| + tF_{\text{max}} \leq |\dot{u}(0)| + TF_{\text{max}}$. Since $u(t) = \int_0^t \dot{u}, |u(t)| \leq (|\dot{u}(0)| + TF_{\text{max}})T$. It remains to bound $\dot{u}(0)$. Integrate the equation for $u$ twice: $u(t) = t\dot{u}(0) + \int_0^t dt' \int_0^{t'} dt'' F(x)$. Set $t = T$ and use $u(T) = 0$. This implies $|\dot{u}(0)| \leq (1/T) \int_0^T dt' \int_0^{t'} dt'' F_{\text{max}} = F_{\text{max}} T/2$, which shows that $u = O(T^2)$.

• Sec. 12, page 90, “THEOREM.”
In quoting the theorem I neglected to define “strong minimum.” The actual terminology used in Bliss [157] is “strong relative minimum.” The distinction between strong and weak refers to how broad is the class of paths relative to which this is a minimum. Earlier on page 90 I allude to the pitfalls of lapses from full rigor in the subject of variational principles, so that thus warned I won’t try to convey all the subtleties of Bliss’s presentation in these pages. His book, [157],
is part of a series, the Carus Mathematical Monographs, published by the Mathematical Association of America, which offers readable and enjoyable introductions to a variety of topics.

- **Sec. 13, page 94, material following Eq. (13.8).**
  If Secs. 11 and 12 have not been read at this point, it is sufficient to begin at the paragraph preceding Eq. (12.29) and continue through the paragraph containing Eq. (12.45).

- **Sec. 13, page 99, discussion following Eq. (13.20).**
  Fig. 3 may be useful for following the argument on page 99. The integrand in (13.20), with \( \alpha = e^{i\phi} \), can be made to look like that in (13.12), but with the path of integration along the ray 0 to \( \infty \cdot e^{-i\phi} \) (the “New contour”).

- **Sec. 13, page 104, paragraph following Eq. (13.42).**
  Concerning the slope of the curves in Fig. 13.1, for small \( E \) and for the lowest curve, \( T(E) \) is given by \( \frac{2\pi}{\omega} \left[ 1 - \frac{3}{4}(\lambda E/\omega)^4 + O(E^2) \right] \). The \( T \) versus \( E \) curve can be much richer than is illustrated in Fig. 13.1, and can bend around, connecting paths with different numbers of turning points. Doing the Exercise on page 105 will illustrate this point. Examples are given in [158].
• Sec. 14, page 110, Eq. (14.12).
The $\sigma$ defined here and that defined in Sec. 6 differ by a factor $\epsilon/\hbar$, i.e.,

$$\sigma_{\text{Sec. 6}} = \frac{\epsilon}{i\hbar} \sigma_{\text{Sec. 14}}.$$ 

• Sec. 18, page 155, Eqs. (18.27), (18.28a) and (18.28b).
A trivial example with which to see the interplay of path absence and imaginary time is the free particle. Eq. (7.18) has the same form as Eq. (18.27). An exact integration yields [Eq. (7.20)]

$$\tilde{G}(E) = \sqrt{m/2E} \exp\left(i|x|\sqrt{2mE/\hbar}\right),$$

but one could as well arrive at this result by a stationary phase approximation, with the stationary point in time [for Eq. (18.28)] given by $t_{\text{stationary}} = -x\sqrt{m/2E}$. Now consider $E < 0$, for which the free particle does not have classical paths. This gives us what we want: pure imaginary “stationary” time and exponential dropoff in space.

• Sec. 21, page 173.
Integrals of the form $\int_0^T \int_0^T$ and $\int_0^T \int_0^t$ make their appearance on this and nearby pages. Moreover they come with factors $\pm 2^{\pm n}$. A confession: I cannot promise you that I got it all right. I recently did a calculation of this sort and found that not only could I not rely on my own presentation, but as far as I could tell Feynman gave different versions in different places, and that other authors generally are not explicit about these terms. In my own recent calculation I could only be sure when I found a different way of getting the system’s spectrum and it agreed with whatever signs and factors I’d finally convinced myself to use (it also helped that Denis Tolkunov worked with me on this).

• Sec. 21, page 181
The reference given for the work of Donsker and Varadhan may not be easily accessible. A web search for their names plus “large deviations” will turn up many sources. The fourth in their series of articles is in the journal Comm. Pure Appl. Math. vol. 36 (1983), 183–212. I believe a significant stimulus to the large deviation quest was work of Friedberg and Luttinger [159] on disordered systems, which, at a heuristic level, arrived at results similar to those of Donsker and Varadhan.

• Sec. 22, page 188, just after the first paragraph.
Feynman suggested the use of time-ordered products for the Dirac equation in [160]. But he apparently did not consider them practical since according to [161] they would demand delving into “the geometrical mysteries involved in the superposition of hypercomplex numbers.”
• Sec. 23, page 195, paragraph beginning, “The degree of pathology...”
In this paragraph, “entertaining” properties of the Jacobi \( \theta \)-function are exhibited, based on its zeros. But you need to know that there is at least one zero per quasi-periodic cell:

**Exercise:** Show that \( \theta_3(z_0, t) = 0 \), with \( z_0 = \pi(1 + t)/2 \).

The fundamental theorem for \( \theta \)-functions is Eq. (23.13). It is worth going through its derivation:

**Exercise:** The Poisson summation formula (for appropriate \( f \)) is

\[
\sum_{n=-\infty}^{\infty} f(x + n) = \sum_{k=-\infty}^{\infty} \exp(2\pi ikx) \int_{-\infty}^{\infty} dy f(y) \exp(-2\pi iky).
\]

Find a function, \( f \), such that this formula leads to Eq. (23.13).

For the solution to this exercise as well as a great deal more entertainment, I recommend the book of Bellman [162], where the field of elliptic functions is heralded not only as entertaining, but enchanting and wondrous as well. (See also the end of Sec. III.)

• Sec. 23, page 197, after the theorem.
In the proof I show one of the properties of an equivalence relation, transitivity. The other properties are reflexivity, \( f \sim f \), and symmetry, \( f \sim g \Rightarrow g \sim f \). If this is your first exposure to homotopy theory, it would be worth writing down the steps to prove these other properties.

• Sec. 23, page 203, **Definition** and **Theorem**.
For an illustration of a covering projection see Fig. 23.1, page 193.

• Sec. 23, page 211, second paragraph of Sec. 23.4.
I needed to be more careful in my statement. The condition that the function must vanish at the endpoints of the interval (plus certain regularity conditions) does define \( H \) sufficiently to make it self adjoint. See [163], §119. But when you do not demand that the function vanish, there emerges the additional richness discussed in this section, in particular the absence of essential self adjointness and the possibility of defining the operator in many ways.

• Sec. 24, page 220, following Eq. (24.25)
For the record, the Laplacian on SU(2) is

\[
\Delta = \frac{\partial^2}{\partial \theta^2} + \cot \theta \frac{\partial}{\partial \theta} + \frac{1}{\sin^2 \theta} \left( \frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial \psi^2} - 2 \cos \theta \frac{\partial^2}{\partial \phi \partial \psi} \right).
\]
• Sec. 25, page 234, after line 5 (end of continuing paragraph).
Continuing the discussion of this paragraph, it’s worth being more explicit. For Kruskal coordinates, $u \sim \cosh(t/4M)$ and $v \sim \sinh(t/4M)$. Consider $\exp(t/4M)$ with $t \to t - i\pi/\kappa$. Then $\exp(t/4M) \to -\exp(t/4M)$, since $\kappa = 1/4M$. Thus $u \to -u$ and $v \to -v$. The point $(-u,-v)$ is $B'$ in Fig. 25.4.

• Sec. 27, page 244, Eq. (27.9).
Here is a proof of Eq. (27.9): Let $[[A, B], A] = [[A, B], B] = 0$. Then
\[
e^{A+B} = e^A e^B e^{-[A,B]/2} = e^B e^A e^{[B,A]/2}
\]
Sec. 27, Eq. (27.9)

Lemma 1. $[A, B^n] = nCB^{n-1}$, where $C \equiv [A, B]$.
Proof by induction: $[A, B^{n+1}] = B[A, B^n] + [A, B]B^n$, which by the inductive hypothesis is $BnCB^n + [A, B]B^n$, and which by our assumptions and definitions is $(n+1)CB^n$. The case $n = 1$ is obviously true. Note that $[B, A^n] = -nCA^{n-1}$, by interchange of letters.

Lemma 2. $[A, e^{\lambda B}] = \lambda Ce^{\lambda B}$
To prove this, expand the exponential, use Lemma 1, and re-sum.

Now define
\[
g(x) \equiv e^{-xA} e^{x(A+B)} e^{-xB}
\]
Take $dg/dx$. By straightforward manipulations and the use of Lemma 2 this becomes $dg/dx = e^{-xA} [B, e^{x(A+B)}] e^{-xB}$, which by Lemma 2 (with an interchange of letters) leads to $dg/dx = -xCg$ (also using the fact that $C$ commutes with everything). Now integrate the differential equation from 0 to 1 to get $g(1) = \exp(-C/2)$. This is the desired result (proof attributed to Glauber).

I think of Eq. (27.9) in terms of the Baker-Campbell-Hausdorff series. This is an expression for $\exp(\lambda(A + B))$ in the case where $[A, B]$ need not commute with $A$ and $B$. In this series, terms with powers higher than $\lambda^2$ all involve commutators of $A$ and $B$ with $[A, B]$. So Eq. (27.9) is simply a truncation of the Baker-Campbell-Hausdorff series. For information on (including a derivation of) this series see [164–166].

• Sec. 27, page 245, Exercise.
Solution to the exercise: One possibility is $v(z) = 1 - |z|^2/2$. By Eq. (27.12) it is only necessary to show that $\int \exp(-|z|^2/2) z^n v(z) d^2z = 0$, for all $n$. Polar coordinates will help.
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• Sec. 27, page 247, discussion following Eq. (27.21).
  Cross reference: the propagator “obtained earlier in this volume” is Eq. (6.38). Note that the “very easy” way to do this path integral only works for the harmonic oscillator.

• Sec. 27, page 248, just after Eq. (27.28).
  Eq. (27.28) is often written without explicit “time-slicing” as
  \[ G(z_f, t; z_i) = \int_{z_i,0}^{z_f,t} d^2z(\cdot) \exp\left\{ i \int_0^t d\tau \left[ \frac{i}{2} \left( z^* \frac{dz}{d\tau} - \frac{dz^*}{d\tau} z \right) - H(z^*, z) \right] \right\} \]

• Sec. 28, Eq. (28.3).
  For completeness I give an indication of how Eq. (28.3), the expected number of particles in each state, is derived. For a fixed number, \( N \), of particles, the canonical partition function is \( Z_N = \sum_{\alpha} \exp(-\beta E_\alpha) \), where \( \alpha \) labels states and I have dropped the superscript \( (C) \) specifying the sphere-configuration that appears in Eq. (28.3). (Recall that the canonical partition function pertains to a system in contact with a heat reservoir at temperature \( 1/\beta \).) Since this is an ideal gas (no interaction between particles), its states can be labeled \( \alpha = (n_1, \ldots, n_k, \ldots) \), where \( n_k \) is the number of atoms in the one-particle state \( k \). The total energy is \( E_\alpha = \sum_k n_k \epsilon_k \), with \( \epsilon_k \) the \( k \)th one-particle energy. The constraint \( N = \sum_k n_k \) makes the partition function sum difficult so one goes to the grand canonical ensemble. For the latter ensemble one has a reservoir of particles with chemical potential \( \mu \) and partition function \( Z^{GC} = \sum_N Z_N e^{\beta \mu N} \). The probability of finding a particular state and a particular number of particles is thus \( e^{-\beta E_\alpha + \beta \mu N} / Z^{GC} \). The chemical potential is then fixed so that the weaker requirement, \( N = \sum_k \langle n_k \rangle \), is satisfied, with the expectation taken with respect to the probability just indicated. One can now calculate the expected occupation of a particular one-particle state:
  \[ \langle n_k \rangle = \sum_{N,\alpha} n_k \Pr (\text{state}-k \text{ occupied by } n_k \text{ particles}) = \frac{-1}{\beta} \frac{\partial}{\partial \epsilon_k} \log Z^{GC}. \]

  All sums are geometric and one gets
  \[ \langle n_k \rangle \approx \frac{1}{\beta} \frac{\partial}{\partial \epsilon_k} \log \left( 1 - e^{-\beta(\epsilon_k - \mu)} \right) = \frac{1}{e^{\beta(\epsilon_k - \mu)} - 1}. \]

• Sec. 23, p. 195 (out of order).
  Solution to the new exercise for page 195: \( f(x) = \exp(-ix^2/\pi t) \).
Those with a mathematician’s aesthetic will recoil at the idea of going through a tedious expansion to derive Eq. (13), after having done a similar calculation to derive Eq. (6). The product on the right-hand-side of Eq. (13) can be written $\exp(\lambda B/2) \exp(\lambda A/2) \times \exp(\lambda A/2) \exp(\lambda B/2)$, with the first and second pairs giving $\exp(\lambda (A + B)/2)$ by Eq. (6), and with corrections of order $\lambda^2$. Because the order of $A$ and $B$ is reversed in the two pairs, these corrections will be equal but of opposite sign. Putting the corrections next to each other, which can be done to this level of precision, allows the $O(\lambda^2)$ corrections to cancel, leaving an error that is $O(\lambda^3)$.


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[21] It may be overkill, but let me explain this point in detail. Imagine that for fixed $N$ (the time-slicing parameter) we also discretize space and write the path integral as an enormous sum. For small enough mesh (and large enough spatial cutoff) this sum approaches the integral, because a finite slicing of the path integral is an ordinary Riemann integral. Each contribution to the sum is labeled by $N$ positions and one time. We want to define sets of labels such that the last time the particle is to the left of 0 is $t_m$. Then this set must have its $x_{m+1}$ values to the right of 0. This breakup partitions the entire set of labels into disjoint sets whose union is the entire set of labels.

[22] In Eq. (23), I omit the factor $\exp(iV\epsilon/\hbar)$. A single $O(\epsilon)$ contribution can be discarded without affecting the result.


[28] P. van Baal, Tunneling and the Path Decomposition Expansion, in [112].


Kac’s telegrapher equation model consists of a particle on a line that moves with fixed speed, |v|, but at random, Poisson distributed, times reverses the sign of its velocity. This process is identical to that of the Dirac equation, à la Feynman, except for the imaginary rate. The telegrapher equation is second order in time, with dissipation. It corresponds to the second order equation one gets by “squaring” the Dirac equation (with an imaginary transition rate there is no dissipation).


D. Mugnai, A. Ranfagni, and R. Ruggeri, *Path-Integral Solution of the


[58] As to what “analytic expression” means, well that’s more difficult. It should mean expressible in simple and well-known functions, which leaves you to define “simple” . . .


A similar construction can be made for opaque wedges when the wedge angle is 2π/integer. The corresponding propagator is also solvable [168].


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[145] V. Sa-yakanit, W. Sritrakool, J.-O. Berananda, M. C. Gutzwiller, A. In-


[150] The eighth conference has already taken place and the proceedings are slated to be published by the Joint Inst. Nucl. Res., Dubna.


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See pages 349–359 for the indexes to this book.

The table of contents for the supplement is on page 362.