## ARTICLES

# Continuous and pulsed observations in the quantum Zeno effect

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(Received 19 May 1997; revised manuscript received 31 October 1997)

The usual formulation of the quantum Zeno effect (QZE) uses a sequence of rapid observations to prevent change. The same effect can be achieved with continuous observation. For partial inhibition of change we provide a quantitative relation between pulsed (at intervals $\delta t$) and continuous (with response time $\tau_O$) observations: the slowdown is the same for $\delta t = 4\tau_O$. Including the continuous observer in the Hamiltonian gives an alternative view of the QZE, showing the appropriateness of the name ''dominated time evolution.'' Finally, the variation on the QZE in which a nonconstant sequence of projections can *force* change is also shown to be achievable by time-varying continuous observation. [S1050-2947(98)01303-1]

PACS number(s): 03.65.Bz, 42.50.Lc.

## I. INTRODUCTION

The usual formulation of the so-called quantum Zeno effect (QZE) is the inhibition of decay due to frequent, intermittent observation. An early question was the issue of *continuous* observation. Many decays let their existence be known rapidly without official monitoring. An $\alpha$ particle from a nuclear decay creates a track, thereby constituting a ''measurement.'' Or an atom decays and sensitive detectors irreversibly note the emitted photon. How can these decays have taken place?

In this paper we make the point that continuous observation is an idealized concept, and any particular form it takes will have a characteristic time scale. When the scale is less than that for slowing the decay, were one to pulse observations at that frequency, then the ''continuous observation'' would hinder the decay. If the time scale is longer, then it would not. There have been investigations of the continuous versus pulsed problem, for example Refs. [1,2], where other works are quoted. Our emphasis in the present paper is on having a more realistic model of the continuous observation. Moreover, the connection we do make has a quantitative formulation, specifically: *Let the time between pulsed observations be $\delta t$, and let the time scale (defined below) for the continuous observation be $\tau_O$. Then as $\delta t \to 0$, the two forms of interference with the decay provide equal degrees of hindrance if $\delta t = 4\tau_O$.*

Thus pulsing is inessential, as already suggested by the characterization, ''watched pot effect'' [2]. In any case, to see significant hindrance of the decay, either of the characteristic times must be below a time scale called the ''jump time'' [3,4]. This will be established both analytically and numerically.

In Sec. II we calculate decay rates under various circumstances. Unmonitored and pulsed-observation decay rates are straightforward. For continuous observation a particular form of observation is given and a modified decay rate computed. In Sec. III the conclusions of Sec. II are checked numerically. Following that, another line of inquiry is pursued: by the incorporation of continuous observation in the quantum Hamiltonian one dispenses with the black magic of ''quantum measurement'' and develops an alternative perspective on the inhibition of decay. With this view it is not so much that one watches the pot, but that the combined system and observer no longer can boil. Section V takes up a dynamical version of the QZE in which the state on which the ''measurement'' projects varies in time, forcing the system to change. For this too we provide a continuous version.

## II. MONITORED DECAY

The system is an atom in an excited state ($|1\rangle$), able to decay to the ground state ($|0\rangle$) by emitting photons. The Hamiltonian is

$$H = \tilde{\omega}_0 |1\rangle\langle 1| + \sum_k \tilde{\omega}_k a_k^\dagger a_k + \sum_k [a_k^\dagger \Phi_k |0\rangle\langle 1| + a_k \Phi_k^* |1\rangle\langle 0|].$$

The atom's ground-state energy is zero, and $|1\rangle$ has energy $\tilde{\omega}_0$, falling in the band $\{\tilde{\omega}_k\}$ associated with the photons. (Although I speak of an ''atom'' and ''photons,'' the formalism is more general.) It is convenient to use matrix notation, and by subtracting $\tilde{\omega}_0$ from the diagonal (and defining $\omega_k \equiv \overline{\omega_k} - \tilde{\omega}_0$) we have

$$H = \begin{pmatrix} 0 & \Phi^\dagger \\ \Phi & \omega \end{pmatrix} \quad \text{and} \quad \psi = \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1}$$

where $\Phi$ and $y$ are column vectors of the same dimension, and $\omega$ is a diagonal matrix. The Schrödinger equation (with $\hbar = 1$) becomes

$$i\dot{x} = \Phi^\dagger y, \quad i\dot{y} = \omega y + \Phi x. \tag{2}$$

 

TABLE I. Times defined.

| | |
|---|---|
| $\tau_L$: | Lifetime—ordinary |
| $\tau_Z$: | Zeno time |
| $\tau_J$: | Jump time |
| $\delta t$: | Time interval for pulsed measurements |
| $\tau_{\rm EP}$: | Effective lifetime when subject to pulsed observations |
| $\tau_O$: | Response time of observer's apparatus |
| $\tau_{\rm EC}$: | Effective lifetime when being observed continuously. |

One generally derives decay rates from Eq. (2) by using the Laplace transform. A quick way to reproduce that result is to assume time dependence $\exp[-i(E-i\Gamma/2)t]$ for both $x$ and $y$. One obtains

$$E-i\,\frac{\Gamma}{2}=\Phi^\dagger\,\frac{1}{E-\omega-i\Gamma/2}\,\Phi\rightarrow\int d\omega\,\frac{\rho(\omega)|\phi(\omega)|^2}{E-\omega-i\Gamma/2},\tag{3}$$

where the arrow indicates a continuum limit, $\rho$ is the density of states, and $\phi$ the appropriate limit of $\Phi$. The usual manipulations now give $\Gamma=2\pi\rho(0)|\phi(0)|^2$, the Fermi-Dirac golden rule. Let $\tau_L\equiv1/\Gamma$, the decay lifetime. We want the short-time transient behavior, and return to Eq. (2). The initial conditions are $x(0)=1$ and $y(0)=0$. We generate successive time-zero derivatives of $x$,

$$Dx(0)=0,\quad D^2x(0)=\Phi^\dagger\Phi,\quad D^3x(0)=i\Phi^\dagger\omega\Phi,$$

$$D^4x(0)=\Phi^\dagger(\omega^2+\Phi\Phi^\dagger)\Phi,$$

with $D\equiv d/dt$. A power-series expansion for $x(t)$ then gives

$$|x(t)|^2=1-(t^2/\tau_Z^2)+O(t^4)\tag{4}$$

where $\tau_Z$, the Zeno time [5], is given by

$$\tau_Z\equiv1/\sqrt{\Phi^\dagger\Phi}.\tag{5}$$

Finally, we define the "jump time," $\tau_J$ [4]. This corresponds in a sense to the minimum time needed to make the transition, analogous to the tunneling time for barrier transmission. The "sense" is that interruptions (as in the QZE) at intervals equal to or less than $\tau_J$ disturb the decay. Less frequent interruptions do not. This quantity is given by [4]

$$\tau_J=\frac{\tau_Z^2}{\tau_L}.\tag{6}$$

We assume throughout that $\tau_Z>0$. If the second moment of $H$, $\Phi^\dagger\Phi$, is infinite, the transient behavior may be different from what appears in Eq. (4) [6], and one may not obtain the QZE.

In Table I is a list of the times used in this paper.

### A. Pulsed interruptions

If a measurement is made at $t=\delta t$, then there is probability $1-(\delta t/\tau_Z)^2$ that the system is in $|1\rangle$. After $N$ successive measurements the probability of finding the system undecayed is

$$\Pr_T^{(N)}(\text{nondecay})=[1-(\delta t)^2/\tau_Z^2]^N,\tag{7}$$

with $T=N\delta t$. For fixed $T$ and large $N$, this gives

$$\Pr_T^{(N)}(\text{nondecay})=\exp(-T^2/N\tau_Z^2),\tag{8}$$

which goes to unity for $N\rightarrow\infty$. This is the standard QZE. Consider, however, finite $N$, or finite $\delta t$. If the leading approximation in Eq. (4) holds, then we obtain an effective decay rate by setting

$$1-(\delta t)^2/\tau_Z^2\approx\exp(-\delta t/\tau_{\rm EP}).\tag{9}$$

In this way we have an effective lifetime, extended by the pulsed measurements and dependent on the pulsing interval $\delta t$, whose value [from Eq. (9)] is

$$\tau_{\rm EP}=\tau_Z^2/\delta t.\tag{10}$$

### B. Continuous observation

The forms taken by continuous observation are many, so that specific conclusions depend on the model of observation. Moreover, traditional thinking about measurement may further hinder discussion, since "measurement" may be viewed as a trauma. Our approach, consistent with Refs. [7–9], is that there is no evolution but quantum evolution. As such it makes sense to consider the combined quantum dynamics of system and apparatus.

One way to monitor the decay is to have a laser shine on the atom at the frequency of a transition from its ground state to some *other* state, one with a short lifetime. In this way, "as soon as" the decay ($|1\rangle\rightarrow|0\rangle$) occurs, the atom is yanked to the other state whose irreversible decay provides the measurement. This is the way that "quantum jumps" were observed [10], and the present paper explains why those observations did not stop the decay. This is also the framework of Ref. [11], where stronger fields are contemplated and the decay can be stopped. Alternatively, $|0\rangle$ might not be the ground state, but is itself metastable with a short lifetime. Yet another possibility is a nearby and rapidly responding counter sensitive to the emitted photon.

A Hamiltonian for this kind of continuous observation will be provided in Sec. IV. For now we take a slightly simplified view and assume that there is *some* form of interaction that removes the system from the Hilbert space that is considered when the Hamiltonian is written as in Eq. (1). Let this removal (by an Observer) have a characteristic time scale $\tau_O$, and let $\gamma\equiv1/\tau_O$. Then we model the action of this apparatus by adding $-i\gamma/2$ to $\omega$. (This is justified in Sec. IV.) Thus each state to which $|1\rangle$ can decay is itself unstable with decay rate $\gamma$. Letting

$$\Omega\equiv\omega-i\gamma/2,\tag{11}$$

the Hamiltonian for continuous observation is

$$H=\begin{pmatrix}0 & \Phi^\dagger\\ \Phi & \Omega\end{pmatrix}.$$

We analyze the extent to which this additional interaction retards decay. We do not look at early transients, but rather at the later exponential decay in the presence of the intense continuous observation [12]. As confirmed numerically, for large $\gamma$ the decay is severely suppressed. To evaluate the lifetime under observation we return to (the discrete form of) Eq. (3) and replace $\omega$ by $\Omega$. The "$\Gamma$" that we now obtain is an effective decay rate under continuous observation. For large $\gamma$ this becomes

$$E - i\,\frac{\Gamma}{2} = \Phi^\dagger\,\frac{1}{i\gamma/2}\,\Phi. \tag{12}$$

This yields $\Gamma = 4\Phi^\dagger\Phi/\gamma$. If $\tau_{EC} \equiv 1/\Gamma$ is called the effective lifetime for continuous observation (when observing with response time $\tau_O$), then, from Eq. (12),

$$\tau_{EC} = \frac{\tau_Z^2}{4\tau_O}. \tag{13}$$

Comparing our earlier expression for $\tau_{EP}$ (effective rate under pulsed observation), we find that for the same suppression of decay (i.e., $\tau_{EC} = \tau_{EP}$) one should have

$$\delta t = 4\tau_O. \tag{14}$$

The response time of the observer is thus shown to be related to a pulse time for an equivalent inhibition of decay.

### III. NUMERICAL CONFIRMATION

Although our limit operations, such as deducing Eq. (8) from Eq. (7), or Eq. (10) from Eq. (9), are justified for sufficiently small times (and providing $\tau_Z > 0$), there is no guarantee that there is a significant time interval within which this behavior can be seen. Nor does a comparison of $\tau_{EC}$ and $\tau_{EP}$ insure that other features are the same. For these reasons we check and extend the features seen analytically in Sec. II.

For reproducibility we give the details of $H$. It is convenient to change Eq. (1) slightly. The band of energies $\omega$ is centered on zero, and a shift of the level from the center of the band is provided by a parameter, $h$. Then

$$H = \begin{pmatrix} h & \Phi^\dagger \\ \Phi & \omega \end{pmatrix}. \tag{15}$$

As found in Ref. [13], although exponential decay appears ubiquitous in Nature, numerically it may be difficult to exhibit. Following Ref. [13], we take

$$\omega_k = \frac{kE}{2N+1}, \quad \Phi_k = \frac{c}{\sqrt{N}}\left[1 - \left(\frac{k}{N+1}\right)^2\right]^p, \quad k = -N,\ldots,N, \tag{16}$$

where $N$ is an integer parameter [so that $H$ is $(2N+2) \times (2N+2)$], $c$ a coupling constant, $E$ the bandwidth, and $p$ a parameter characterizing the band edges.

Figure 1 is a semilog plot of the probability of nondecay as a function of time, for $N = 17$. Other parameters are $c = 0.2$, $h = 0.1$, $E = 5$, and $p = 1$. After dropping to about Pr(nondecay)$\approx 0.012$ for $t \approx 42$, the system recovers, a reflection of the (quantum) Poincaré recurrence. From early times this shows excellent exponential decay.

Fitting the linear regime of the semilog plot gives $\tau_L \approx 9.38$. From the given $\Phi$ we compute the Zeno time to be $\tau_Z \approx 4.705$. Combined with the lifetime, this gives a jump time $\tau_J \approx 2.36$. A discrete approximation to the golden rule formula here gives a rate of $2\pi|\Phi_{k_h}|^2/\Delta E$, where $k_h$ is the index for which $\omega_{k_h} = h$ and $\Delta E$ is the level spacing in $H$. The inverse of this quantity is about 9.66, in agreement with the measured lifetime.

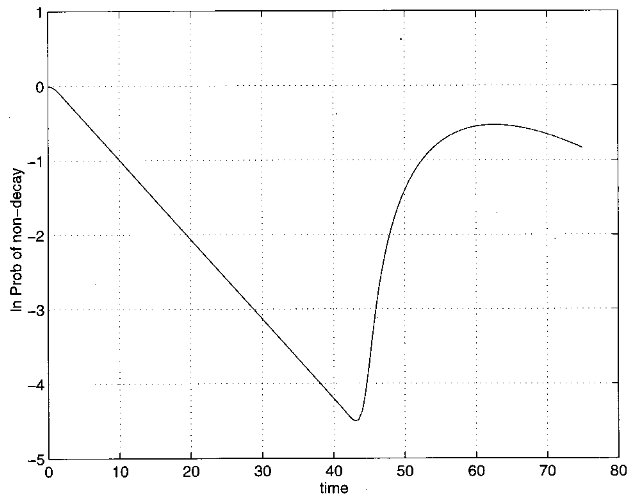


FIG. 1. Semilog plot of the probability of nondecay as a function of time, for $N = 17$, $c = 0.2$, $h = 0.1$, $E = 5$, and $p = 1$. For $3 \lesssim t \lesssim 42$ the decay is to a good approximation exponential. Parameters for the Hamiltonian correspond to the quantities used in Eqs. (15) and (16). The rise after $t \sim 45$ is a (partial) Poincaré recurrence.

Next we explore early transients, $t \lesssim \tau_J$. In Fig. 2 we plot the *square root* of 1 minus the nondecay probability. If the leading approximation to Eq. (4) is accurate, the initial behavior should be a straight line of slope $1/\tau_Z$. Fitting the interval [0,1] gives a slope of 0.205 while the inverse of $\tau_Z$ is 0.2125 (in satisfying agreement). The breakdown of this quadratic dependence on the scale of $\tau_J$ is also evident.

Relation (14) connecting continuous and pulsed observation is only correct asymptotically for short pulses or intense

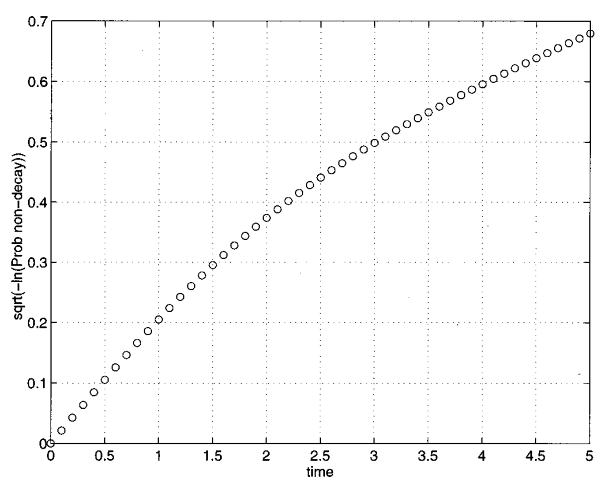


FIG. 2. Plot of $\sqrt{-\ln|x(t)|^2}$, with $x(t)$ the amplitude for nondecay, for $N = 17$, $c = 0.2$, $E = 5$, $h = 0.1$, and $p = 1$. For the quadratic approximation to Eq. (4), the points should lie on a straight line, evidently not a bad approximation for $t \lesssim 1.2$. Note that $\tau_J \approx 2.4$ gives the time scale for the transient era.

TABLE II. Matching hindrances. Values of $\delta t$ and $\gamma$ that provide essentially the same effective lifetimes, and the product $\gamma\delta t$, which by Eq. (14) should have the asymptotic value 4.

| $\delta t$ | $\tau_{EP}$ | $\gamma$ | $\tau_{EC}$ | Product $\gamma\delta t$ |
|---|---|---|---|---|
| 1.466 | 17.46 | 2.084 | 17.46 | 3.056 |
| 1.0 | 23.74 | 3.398 | 23.74 | 3.398 |
| 0.25 | 88.94 | 15.43 | 88.94 | 3.857 |
| 0.05 | 442.78 | 77.86 | 442.76 | 3.893 |

observation. To see that limit approached, we report another numerical investigation. In Table II we show results of several runs in each of which the continuous and pulsed observations were adjusted so that the degree of hindrance, specifically $\tau_{EC}$ and $\tau_{EP}$, were the same. Note that although for lesser degrees of hindrance relation (14) is not obeyed, the product of $\gamma$ ($=1/\tau_O$) and $\delta t$ approaches 4.

## IV. ANOTHER PERSPECTIVE: ''OBSERVING'' VERSUS ''DISTORTING''

In studying ''continuous'' observation, one pulls away the curtain that hides the measurement process and makes the apparatus part of the quantum calculation. In Sec. II the ''detector'' for continuous observation was represented by an additional piece of Hamiltonian ($-i\gamma/2$) making the ''$y$'' levels unstable. We now insert physical degrees of freedom that lie behind $\gamma$. This allows a second perspective on continuous observation. The inclusion of the apparatus changes the quantum system. The halting of decay can then be described as modification of the Hamiltonian, stabilizing the original system. This picture conforms to the name ''dominated time evolution'' advocated in Ref. [5], and is supported as well by Ref. [11]—especially its Fig. 3. This will also allow us to estimate the transient period for a combined decay system and measurement apparatus.

We enlarge our ''universe'' to include the continuous measuring apparatus. The job of this instrument is to notice when the system has decayed to $y$, and to pull it away irreversibly, with a time constant $\tau_O$. Instead of the Hamiltonian given in Eq. (15), we use

$$H=\begin{pmatrix} h & \Phi^\dagger & 0 \\ \Phi & \omega & \Theta^\dagger \\ 0 & \Theta & W \end{pmatrix}. \tag{17}$$

The $M$ additional levels $\{W\}$ represent the apparatus, and we assume the coupling ''$\Theta$'' is strong. Also the levels are numerous enough and so distributed that the transition induced by this coupling is effectively irreversible.

We want the new, combined object, eigenvalue structure. The coupled equations analogous to Eq. (2) are easily written down, and with the substitution $i\partial/\partial t \to z$ (''$z$'' is a possibly complex energy), we find that $z$ must satisfy

$$z-h=\Phi^\dagger \frac{1}{z-\omega-\Theta^\dagger \dfrac{1}{z-W}\,\Theta}\,\Phi. \tag{18}$$

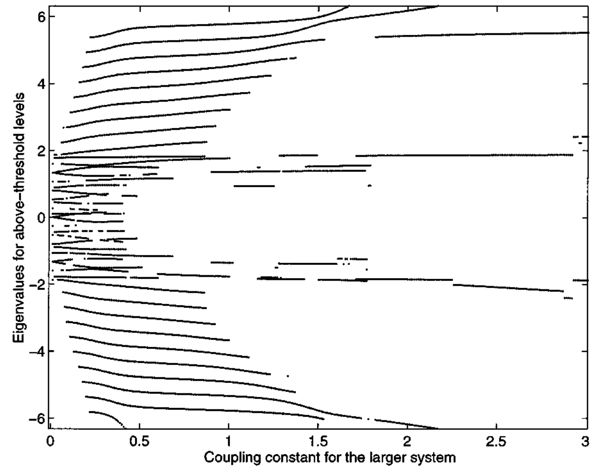


FIG. 3. Energy levels of system plus apparatus for continuous observation, with $N=7$, and $M=13$. The coupling constent in $\Phi$ [of Eq. (17)] is $Cn=0.1$. The energy spread for $\omega$ is 4, and that for $W$ is 12. The undecayed system has energy $h=0.1$. For $\Theta=0$ the decay levels are the original ''$\omega$'''s (forming a quasicontinuum). These are plotted vertically on the far left. As $\Theta$ increases, they change, and ultimately there remain *no* quasicontinuum levels for decay. For $\Theta$ we use the parametrization in Eq. (16); in particular, there is an overall factor ''$c$,'' which is our abscissa. Only those levels of the total Hamiltonian are plotted whose coupling to the original level exceeds 20% of the original coupling.

We approach Eq. (18) in two ways. First, we replace $\Theta$ by a continuum coupling and justify our replacement of $\omega$ by $\omega - i\gamma/2$ in Sec. II. Second, we deal with Eq. (18) as a real eigenvalue equation and study its levels as $|\Theta|$ increases.

### A. Treating the apparatus as an effective damping

We first evaluate $\Theta^\dagger(z-W)^{-1}\Theta$. With the assumption, to be self-consistently verified, that Im $z$ is small, this is

$$\Theta^\dagger \frac{1}{z-W}\,\Theta \to \int dW\,\frac{\rho(W)|\theta(W)|^2}{z-W}$$
$$\simeq \mathrm{P}\int dW\,\frac{\rho(W)|\theta(W)|^2}{h-W} - i\pi\rho(h)|\theta(h)|^2, \tag{19}$$

with $\rho$ and $\theta$ continuum quantities [cf. Eq. (3)]. Substituting in Eq. (18),

$$z-h=\Phi^\dagger \frac{1}{z-\omega-\Delta E+i\pi\rho(h)|\theta(h)|^2}\,\Phi,$$

with $\Delta E$ the real principal value integral. Comparing this to Eq. (11), we identify $\gamma=2\pi\rho(h)|\theta(h)|^2$. Thus the effective observation time used in Sec. II is the decay or relaxation rate for transfer of the system out of the states of interest for the decay of the original level. Also the self-consistency assumption is justified. Im $z$ is small, provided $\Theta$ (hence $\gamma$) is large.

### B. True eigenvalues in the presence of the apparatus

We return to Eq. (18) with no continuum approximation, and diagonalize the full $H$ numerically. For Fig. 3 the form of $\Theta$ is the same as in Eq. (16), except that the maximum for each row of $\Theta^\dagger$ is centered on the corresponding diagonal entry. Again using ''$c$'' for the overall coupling strength, we find that for small $c$ the quasicontinuum into which the original level decays remains intact and decay occurs. (For some cases the exponential time evolution of the decay was also checked.) However, as $c$ increases the levels of interest move away from the original level. Ultimately there is no continuum, hence no decay. In the figure this occurs for $c \gtrsim 0.5$. Note that the *total* Hamiltonian continues to have levels matching those of the system. However, most of these do not substantially couple to the original level. For the plot in Fig. 3 only those levels were used for which there was a coupling of at least 20% of the original coupling, that is, the original ''$\Phi$.'' This substantially agrees with the results of Ref. [11].

Using the Hamiltonian in Eq. (17) we can also examine the transients of the combined object, system plus apparatus. We address the following question. The QZE is often phrased as the observation of a decay system during its transient, quadratic decay. In our continuous formulation it is the new exponential decay lifetime that we calculate (because we contemplate long-term observation). But one can still ask about transients for the combined object. We now show that they are substantially shortened.

An estimate of the duration of the quadratic regime is the jump time, $\tau_J$, given by $\tau_Z^2/\tau_L$ [4]. Thus we evaluate—for the *combined* object—the values of $\tau_Z$ and $\tau_L$. But this is easy: $\tau_Z$ is unchanged, while $\tau_L$ is what Sec. II was concerned with, and in particular was denoted $\tau_O$ in that section. [Also, the calculation of Sec. II was related in the present section to the full Hamiltonian of Eq. (17).] To see that $\tau_Z$ is unchanged, recall [5] that an alternative phrasing of $\tau_Z$ is $\tau_Z^2 = 1/\langle\psi|(H-\langle\psi|H|\psi\rangle)^2|\psi\rangle$. ''$\psi$'' is the initial undecayed state, in our case a vector with 1 in the first entry, zeros elsewhere. All that must be done is to set $h=0$ in Eq. (17), square the resulting object, and look at its 1-1 component. The answer is $\Phi^\dagger\Phi$, as in Eq. (5). It follows that the duration of the quadratic decay regime scales like the lifetimes.

In Sec. VI we further comment on the perspective presented here. Viewing the QZE as a restructuring of the total Hamiltonian results from including the ''apparatus'' in the Hamiltonian. The complaint that this kind of continuous observation is *only* a matter of changing the system, not the ''true'' QZE, is in my view an artifact of the traditional but no longer tenable separation of the world into apparatus and system.

## V. DYNAMIC CONTINUOUS OBSERVATION

The name quantum Zeno effect suggests a relation to Zeno's paradox, and the name ''watched pot effect'' relates to the ironic maxim that such pots do not boil. We now demonstrate how the principle underlying the QZE can *force* the pot to boil. However, one could still call this the ''QZE,'' if one imagined the tortoise on a train passing the sleeping Achilles. Zeno's nonmotion argument would then imply that the train and the sleeper had no relative motion, so that Achilles must be in motion.

Let $\psi$ evolve with $H$; in the absence of observation, $\psi(t)=\exp(-iHt)\psi(0)$. Let there be a family of states $\phi_k$, $k=1,\ldots,N$, such that $\phi_1=\psi(0)$, and such that successive states differ little from one another (i.e., $|\langle\phi_{k+1}|\phi_k\rangle|$ is nearly 1). Now let $\delta t = T/N$ and at $t=k\,\delta t$ project the evolving wave function on $\phi_k$. Then for sufficiently large $N, \psi_T \approx \phi_N$. (The usual QZE is the case $\phi_k=\phi_0=\psi(0)\forall k$.) In this section we produce the same result with *continuous* observation. Consider a two-level system, similar to that on which Rabi oscillation experiments [14] were done, first confirming the QZE. As a preliminary, the pulsed observation result is established.

Let $H=\mu\sigma_x$. A system started with spin-up has the undisturbed evolution

$$\psi(t)=\exp(-i\mu\sigma_x t)\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}\cos\mu t\\ -i\sin\mu t\end{pmatrix}.$$

Let the family of states for projection be

$$\phi_k \equiv \exp(-i\theta_k\sigma_y)\begin{pmatrix}1\\0\end{pmatrix} \quad \text{with} \quad \theta_k\equiv\frac{ak}{N}, \quad k=1,\ldots,N. \tag{20}$$

Defining $P_0 \equiv \left(\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right)$, the $2\times2$ projection operator at stage $k$ is

$$P_k=|\phi_k\rangle\langle\phi_k|=\begin{pmatrix}\cos^2\theta_k & \cos\theta_k\sin\theta_k\\ \cos\theta_k\sin\theta_k & \sin^2\theta_k\end{pmatrix}.$$

Now allow the system to evolve for a time $T$ with projections at times $k\,\delta t$ ($\delta t=T/N$), and evolution under $H$ in between. Then

$$\psi(T)=\mathcal{T}\left[\prod_{k=1}^{N}\left[P_k\exp(-i\mu\,\delta t\,\sigma_x)\right]\right]\begin{pmatrix}1\\0\end{pmatrix}, \tag{21}$$

where $\mathcal{T}$ indicates the time-ordered product. We wish to show that for $N\to\infty$, $\psi(T)$ is an eigenfunction of the *final* projection operator, $P_N$. Using $P_0^2=P_0$, Eq. (21) can be rewritten

$$\psi(T)=\exp(-ia\sigma_y)\mathcal{T}\left[\prod_{k=1}^{N}B_k\right]\begin{pmatrix}1\\0\end{pmatrix}, \tag{22}$$

with $B_k\equiv P_0\exp(i\theta_k\sigma_y)\exp(-i\mu\delta t\sigma_x)\exp(-i\theta_{k-1}\sigma_y)P_0$. Letting $\delta\theta\equiv a/N$, it follows that

$$B_k=P_0\exp(i\,\delta\theta\sigma_y)\exp(-i\mu\,\delta t\,\tilde{\sigma})P_0,$$

with $\tilde{\sigma}=\sigma_x\cos 2\theta_{k-1}+\sigma_z\sin 2\theta_{k-1}$, a standard piece of SU(2) manipulation. The succeeding steps are similarly straightforward and we find

$$B_k=P_0\begin{pmatrix}1-i\,\delta t\mu\sin 2\theta_{k-1} & \delta\theta-i\,\delta t\mu\cos 2\theta_{k-1}\\ -\delta\theta-i\,\delta t\mu\cos 2\theta_{k-1} & i\,\delta t\mu\sin 2\theta_{k-1}\end{pmatrix}P_0 + O\left(\frac{1}{N^2}\right). \tag{23}$$

To order $1/N$, this is

$$B_k=\begin{pmatrix}1-i\,\delta t\mu\sin 2\theta_{k-1} & 0\\ 0 & 0\end{pmatrix}. \tag{24}$$

Note that $O(1/N^2)$ terms drop out in Eq. (22). Therefore,

$$\psi(T)=\exp\left(-i\,\frac{\mu T}{2a}\,(\cos 2a-1)\right)\exp(-ia\sigma_y)\begin{pmatrix}1\\0\end{pmatrix}. \tag{25}$$

This is an eigenstate of $P_N$. The only effect of the interaction is a phase factor, and the pulsed monitoring with a time-varying projection leaves the wave function in the twisted state. For $a=\pi$ this yields a geometrical or Berry phase [15], and provides an explicit example of the work of Aharonov and Anandan [16]. In this case the ''dynamical'' phase [in the first exponential of Eq. (25)] is zero, while $\exp(-ia\sigma_y)$ is just $-1$, and gives a geometrical phase $\pi$.

### Continuous observation

As for the static QZE, there is no unique way to implement continuous observation. We use a time-dependent Hamiltonian, which for any particular value of the time—if time stood still—would tend to select the state $\phi_k$ given in Eq. (20). This is achieved by adding to $H$

$$V(t)\equiv i\,\ln(P_k+\epsilon Q_k)\quad\text{with}\quad \kappa=tN\quad\text{and}\quad Q_k=1-P_k,$$

The ''$\epsilon$'' is mathematically necessary for the finiteness of $V$, and we ultimately take $\epsilon\to 0$. It possesses an (incomplete) analogy with the finiteness of ''$\gamma$,'' the observer's rate constant of Secs. II and III. The total Hamiltonian is

$$\widetilde{H}=\mu\sigma_x+V(t). \tag{26}$$

Instead of Eq. (21), we have

$$\psi(T)=\mathcal{T}\exp\left(-i\int_0^T\widetilde{H}(t)dt\right)\begin{pmatrix}1\\0\end{pmatrix}$$

$$=\lim_{N\to\infty}\mathcal{T}\left\{\prod_{j=1}^{N}\exp\left[\frac{-i}{N}\widetilde{H}\left(\frac{j}{N}\right)\right]\right\}\begin{pmatrix}1\\0\end{pmatrix}. \tag{27}$$

By the Trotter product formula the summands of $\widetilde{H}$ can be pulled apart, and we need only consider a product of terms

$$[P_j+\epsilon Q_j]^{1/N}\exp(-i\mu\sigma_x/N). \tag{28}$$

We emphasize that one must take $N\to\infty$ before $\epsilon\to 0$. Once $\widetilde{H}$ has been defined (fixing $\epsilon$), the time-ordered product and other constructs merely constitute a way to integrate the differential equation. The ''$N$'' in Eq. (27) bears no relation to the number of projections in the pulsed QZE [cf. Eq. (20)]. Expression (28) can be written

$$\left[\exp(-i\theta_j\sigma_y)\begin{pmatrix}1&0\\0&\epsilon^{1/N}\end{pmatrix}\exp(i\theta_j\sigma_y)\right]\exp(-i\mu\sigma_x/N).$$

As before, the product can be regrouped and $\psi(T)$ has the form it did in Eq. (22), but with

$$B_j\equiv\begin{pmatrix}1&0\\0&\epsilon^{1/2N}\end{pmatrix}\exp(i\theta_j\sigma_y)\exp(-i\mu\delta t\sigma_x)\exp(-i\theta_{j-1}\sigma_y)\times\begin{pmatrix}1&0\\0&\epsilon^{1/2N}\end{pmatrix}.$$

($\sqrt{\epsilon^{1/N}}$ appears, since one no longer has a projection.) Because of $\epsilon$ one no longer obtains the simplification in going from Eq. (23) to Eq. (24). Instead,

$$B_j=\begin{pmatrix}1-i\delta t\mu\,\sin 2\theta_{j-1} & \epsilon^{1/2N}[\delta\theta-i\delta t\mu\,\cos 2\theta_{j-1}]\\ \epsilon^{1/2N}[-\delta\theta-i\delta t\mu\,\cos 2\theta_{j-1}] & \epsilon^{1/N}[1+i\delta t\mu\,\sin 2\theta_{j-1}]\end{pmatrix}+O\left(\frac{1}{N^2}\right).$$

We next require the product of the $N$ $B_j$s. To do this we use a spectral decomposition. Consider the calculation of $B_j$'s eigenvalues. The off-diagonal terms of $B_j$ enter only as the product of each other; this is $O(1/N^2)$ and drops out for $N\to\infty$. Similarly, the eigenvectors differ from $\binom{1}{0}$ and $\binom{0}{1}$ by terms of the same negligible order. [Note that $B_{j-1}-B_j=O(1/N^2)$.] Therefore, for this limit, $B_j$ can be replaced by its diagonal. The product now gives a term proportional to $\epsilon$ in the 2-2 position, which drops out for $\epsilon\to 0$. We are then left with exactly what appears in Eq. (25). We have thus shown that the effect of a dynamic set of projections—*making* the pot boil—can be implemented with continuous observation.

## VI. DISCUSSION

Although early formulations of the so-called quantum Zeno effect were phrased in terms of repetitive projections, it was soon realized that the same phenomenon could take the form of continuous and even time-varying observations. Here we have presented a precise connection between the pulsing time for intermittent observation and the response time for continuous observation. One asks, how can an atom decay if I look at it? The answer is that the response time of your eye (or whatever) is far longer than the times needed to stop that decay via the intermittent QZE. (That time is the ''jump time'' of Ref. [4].) This also explains why observations of quantum jumps [10] did not seriously affect the lifetime.

In establishing this result the apparatus was treated as a quantum system. This allowed an alternative view of the QZE, in which the observer's halting of decay can be phrased as the system-cum-apparatus ceasing to be unstable. This may obscure the issue of whether a given observational scheme is or is not an example of the QZE. However, this is a matter of semantics and has no effect on the behavior of the physical systems. In fact, to allay confusion it would be better to call the QZE ''dominated time evolution,'' as advocated in Ref. [5].

Another reason to prefer the foregoing name (unless you want to put the tortoise on a train, as in Sec. V) is the fact that halting change is only one manifestation of the effect. By varying the projections, it has long been known that the system may be forced to change. In the present paper we also present a continuous observation scheme to accomplish the same result.

## ACKNOWLEDGMENTS

This work was supported in part by National Science Foundation Grant No. PHY 93 16681 and by U.S. Air Force Grant No. F30602-97-2-0089.

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