MEASURE OF ENTANGLEMENT

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ABSTRACT

The extent to which a given wave function, ψ, is entangled is measured by minimizing the norm of ψ minus all possible unentangled functions. This measure is given by the largest eigenvalue of ψ†ψ, considered as an operator. The definition is basis independent.

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Entanglement was considered by Schrödinger to be the essence of quantum behavior. It is the central concept in the EPR paradox and Bell inequalities, and now—because of its effective destruction of interference—looms as a significant factor in the construction of systems intended to retain quantum coherence. For such systems especially, the distinction between decoherence and error is important. Decohered pieces of wave function—pieces that have become entangled with environmental degrees of freedom—can never provide the useful interference on which the quantum computer depends. Moreover, when one combines the effect of several small entanglement episodes, it is in the amplitude that the damage occurs, meaning that this is particularly effective at the destruction of interference. Error, on the other hand, leaves one within the original Hilbert space. This allows corrective measures to be applied more easily. Also, under some circumstances, if the errors occur within a small number of degrees of freedom (say a phase), N such errors will only cause a $\sqrt{N}$ defect. It is thus important to have sharp criteria for distinguishing decoherence from error. Our measure of entanglement provides such a criterion.

If ψ(x, y) is a function of two variables, x and y, it is said to be entangled if it cannot be written as a product of two separate functions of x and y. It is clear that some functions are “more” entangled than others: there should be a sense in which the state $(|+\rangle - |-\rangle)/\sqrt{2}$ is highly entangled, while a spin in a quantum computer that has weakly interacted with a phonon is only slightly entangled (and therefore may continue to function successfully in the computer). We next provide a precise measure of entanglement. The mathematical basis of our definition is straightforward, so that we will only give general indications of proofs.

Having defined our measure of entanglement we also apply it to two extreme situations: a characterization of maximally entangled states and a criterion for the
analysis of only slightly entangled states. For the latter case, we mention results on
the possible entanglement of a confined system with its walls, in which our criterion
allows a precise measure of the damage done (which interestingly may sometimes
be zero).

Two criteria for the degree of entanglement come to mind. Let
\[ J \equiv \min_{\rho,\sigma} \int dxdy |\psi(x, y) - \rho(x)\sigma(y)|^2 \] (1)

and
\[ K \equiv \max_{f,g} \left| \int dxdy \psi^*(x, y) f(x)g(y) \right| \] (2)

with, in the second definition, normalized \( f \) and \( g \) (but not in the first definition).
In fact, both definitions produce essentially the same functions, as we show below.
Our measure of entanglement will be \( J \). Clearly a function that is not entangled
will have \( J = 0 \). It also turns out that \( J = 1 - K^2 \).

Note that this definition is independent of basis and only depends on the choice
of coordinate system for the configuration space of the degrees of freedom.

The mathematical result is most easily stated in matrix form. Let \( A \) be an
arbitrary \( n \times n \) matrix and as above (except for notational changes)
\[ J \equiv \min_{u,v} \sum_{ij} |A_{ij} - u_i v_j^*|^2 \] (3)

Then if \( u \) and \( v \) minimize \( J \), they satisfy the following
\[ A^\dagger u = v\|u\|^2, \quad Av = u\|v\|^2 \] (4)

from which follow
\[ AA^\dagger u = \lambda u, \quad A^\dagger Av = \lambda v \] (5)

and
\[ \lambda = \|u\|^2\|v\|^2 = \text{maximum eigenvalue of } A^\dagger A \] (6)

Defining \( \tilde{u} = u/\|u\|, \tilde{v} = v/\|v\|, \) and \( S_2(B) \equiv \text{Tr } B^\dagger B \) for a matrix \( B \), we can write
\[ J = S_2 \left( A - \sqrt{\lambda} |\tilde{u}\rangle\langle \tilde{v}| \right) \]. It is also clear that \( \langle \tilde{u}|A|\tilde{v}\rangle = \sqrt{\lambda} \), which is real, and finally
\[ J = 1 - \lambda \].

Once you know that you want to prove the above assertions, verification is
straightforward. (It is a simple variational problem.) The vector, \( v \), was introduced
as a complex conjugate to make the matrix form more transparent. However, be-cause the original object, \( \psi(x, y) \), is a Hilbert space vector, rather than operator,
this may not always be the best notation.
We next show that the functions $\tilde{u}$ and $\tilde{v}$ that minimize $J$ also maximize $K$. For arbitrary, normalized, $w$ and $x$, let $\tilde{J} \equiv S_2(A - \gamma|w\rangle\langle x|)$, where $\gamma$ is an arbitrary complex constant. If we adjust $|x\rangle$ to make $\langle x|A|w\rangle$ real, then

$$\tilde{J} = S_2(A) + |\gamma|^2 - \langle x|A|w\rangle(\gamma + \gamma^*)$$  

(7)

Taking $\gamma$ to be real obviously can only reduce $J$. Since Eq. (7) holds for any real $\gamma$, it is seen that maximizing $\langle x|A|w\rangle$ is the same as minimizing $J$. It then follows that $\lambda = \gamma^2$, etc.

Using the measure of this paper, the state $(|+\rangle - |-\rangle)/\sqrt{2}$ gives rise to a matrix $A = i\sigma_y/\sqrt{2}$, and the maximal eigenvalue of $A^\dagger A$ is $1/2$. This is the maximal degree of entanglement. Actually it is easy to see that for $n$-state systems wave functions having the matrix $A_{jk} = \delta(k, \pi_j) \exp(i\phi_{jk})/\sqrt{n}$ are maximally entangled, where $\pi_j$ is any permutation on $n$ objects and $\phi_{jk}$ is any real matrix. The corresponding eigenvalue is $1/n$, so that $J = 1 - 1/n$. To see that this is maximal, note that for any $A$ all eigenvalues of $A^\dagger A$ must be real and non-negative ($\langle \phi|A^\dagger A|\phi\rangle = \|A\phi\|^2$). Hence minimizing the maximal eigenvalue is achieved by having all eigenvalues equal. Moreover, for $A$ arising from a normalized wave function, $S_2(A) = 1 = \text{sum of the eigenvalues}.$

In another article, 7 the entanglement measure defined here will be applied to a state that is at worst only slightly entangled. (The entanglement there arises from the inevitable interaction of a confined system—the putative quantum computer—with the “walls” keeping it in place.) Using the operator-formalism test developed here, it is shown that in fact no decoherence need arise, despite an apparent intertwining of coordinates. An advantage of our present technique is that it is not necessary to find the optimum states explicitly—only to solve the eigenvalue problem. The same is true when there is entanglement: the operator method developed here measures the entanglement, without requiring knowledge of the optimal factored state (although in that application enough is known of the operators to produce the eigenfunctions, should that be desired). We remark that for this application the index of “$A$” is continuous, but that this fact presents no complication.

The generalization of the entanglement criterion to three or more degrees of freedom is straightforward. From the minimization problem for

$$J_3 \equiv \min \int |\psi(x, y, z) - \gamma e(x)f(y)g(z)|^2dxdydz$$

(with $e$, $f$ and $g$ normalized) one obtains

$$\gamma e(x)^* = \int \psi(x, y, z)^*f(y)g(z)dydz$$  

(8)

and cyclic permutations of $(e, f, g)$, with $1 - \gamma^2$ again the minimum of $J_3$ (and $\gamma$ real). Eq. (8) and its permutations (and its obvious $N$-particle generalizations) form
a non-linear mapping among the Hilbert subspaces. Although a simple solution as in Eqs. (4–6) is not available, iterative mapping techniques can be applied.

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References

[1] E. Schrödinger, Proc. Camb. Phil. Soc. 31, 555 (1935). It is interesting to provide a full quotation: “When two systems, of which we know the states by their respective representation, enter into a temporary physical interaction due to known forces between them and when after a time of mutual influence the systems separate again, then they can no longer be described as before, viz., by endowing each of them with a representative of its own. I would not call that one but rather the characteristic trait of quantum mechanics.” For further discussion, see also M. Horne, A. Shimony and A. Zeilinger, in Quantum Coherence, J. S. Anandan, ed., World Scientific, Singapore (1990).


