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Fluctuations in Mean-Field Self-Organized Criticality

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We present two models that exhibit self-organized criticality at the mean-field level. These can be variously interpreted in epidemiological or chemical reaction terms. By studying the master equation for these models we find, however, that only in one of them does the self-organized critical behavior survive in the face of fluctuations. For this model we show the spectrum of the evolution operator to have spectral collapse, i.e., instead of a gap, as would occur in noncritical behavior, there are eigenvalues that approach zero as an inverse power of system size.

KEY WORDS: Self-organized criticality; directed percolation; epidemic models; chemical reactions; transfer matrix; spectral collapse.

1. INTRODUCTION

Self-organized criticality has been proposed[1] as a general principle underlying the common appearance of fractal structures—despite their nongenericity from the phase transition point of view—as well as other natural phenomena. Subsequent studies related some of these features to spectral properties of associated operators.[2] Fully developed models of self-organized criticality tend to require computer analysis: it is already difficult to treat critical phenomena analytically without including a time dependence of "parameters"!

In a previous publication[3] we considered a mean-field model in which the critical phenomenon was itself so simple that the time-dependent progress of the system toward criticality (under rules we gave for the full

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model) could be studied analytically. The model has been called[4] "percolitis" because it can be phrased in epidemiological terms. The time evolution of the percolitis parameters, that is, the self-organized criticality, was studied in the mean-field approximation and we presented numerical evidence that fluctuations in the system did not destroy the approach of the system to criticality. We remark, incidentally, that the work for whose didactic needs the percolitis model was invented exhibits a kind of soft self-organized criticality[5] which has yet to be fully understood.

In other studies,[6] the spectral properties of the transfer matrix for a modified version of the mean-field model were examined. One objective was to find the finite-size scaling properties of the model. A second objective was preparation for the present work.

In the current paper, our objective is to study whether mean-field self-organized critical behavior is preserved in the presence of fluctuations and also to observe whether and how self-organized criticality leads to spectral collapse in linear operators, such as the transfer matrix, associated with the model. The answer to the issue of fluctuations is not simple: we shall present two versions of a model, both of which display self-organized critical behavior at the level of mean-field theory, such that the fluctuations destroy this behavior for one variant and preserve it for the other. The starting point is the single-step directed percolation model of ref. 6, where at each time step, a sick person can spontaneously recover or can infect a healthy person with some probability. We can also think of this model as a chemical reaction for two species S and R

$$S \xrightarrow{1} R$$

$$S + R \xrightarrow{\xi} 2S$$

The total population $S + R = K$ is a conserved quantity. The symbols S and R refer both to the name of the species and to the number of its elements. The quantities above the arrows indicate the reaction rates. With either the chemical or the epidemic interpretation, the model depends on a parameter ξ and exhibits critical behavior at $\xi = 1$. The analysis of the metastable state for $\xi > 1$ and the critical approach to equilibrium were studied numerically in ref. 6 and these results were confirmed and extended by analytical methods in ref. 7.

In the present paper, we couple this system to another species called I by a third chemical reaction $2S \to S + I$ ("I" can be thought of as "immune" or "inert"). In this case K $(= S + R)$ is no longer conserved, the effective ξ parameter (the rate for $S + R \to 2S$, see below) now depends on K, and the system has self-organized critical behavior at the mean-field

level. (The precise meaning of our term "effective" is given in Section 5.) For a generic set of initial conditions, the parameter ξ adjusts itself so that it tends to its critical value 1. However, when we look in detail at the fluctuations of the birth and death process, we will see that the true process—as opposed to its mean-field estimate—does not preserve this self-organized critical behavior. Nevertheless, if we use the reaction $3S \to 2S + I$ instead of $2S \to S + I$, the birth and death process does exhibit self-organized critical behavior.

We describe these two models in Section 2, where we also write the master equations and the backward equations. In Section 3 we establish self-organized criticality at the mean-field level. Section 4 analyzes the structure of the backward equation. This analysis is used in Section 5 to describe the largest eigenvalue of the master equation by perturbation theory applied to the spectrum of the master equation of the single-step percolation model of ref. 7. In Section 6 we perform a rescaling and matched asymptotics for the equation for the generating function (much as in ref. 7) and find that the matched asymptotics can be successfully applied to the version of the model for which our other analysis shows the survival of the self-organized criticality in the face of fluctuations, but not for the version for which it does not.

2. DESCRIPTION OF THE MODELS

In this work we use a chemical picture of the models. A population of N particles is composed of three chemical species called S, R, and I. At each time step we have

$$S(t) + R(t) + I(t) = N \tag{2.1}$$

N does not vary. For convenience we introduce the number of particles of species S *or* R,

$$K = S + R \tag{2.2}$$

so that $K \geqslant S$. Because of (2.1) and (2.2), the state of the population can be given by (S, K). Let us assume that at time t, the population is in the state (S, K). At the next time step $t + \Delta t$, where Δt is a small time interval, the population varies according to the following rules:

R.1. An S particle can spontaneously become an R particle

$$S \to R$$

so that this is a transition from the state (S, K) to the state $(S-1, K)$. The probability of this transition is

$$S\,\Delta t \tag{2.3}$$

In effect this is the conditional probability

$$\Pr[\text{State at time } t+\Delta t \text{ is } (S-1, K) \mid \text{State at time } t \text{ is } (S, K)] = S\,\Delta t$$

There is no explicit parameter associated with this transition (i.e., its rate is unity), so that it is here that the time scale is established.

R.2. An R particle can meet an S particle and transform into an S particle,

$$R + S \to 2S$$

The transition is from the state (S, K) to the state $(S+1, K)$. The (conditional) probability of this transition is

$$\frac{xS(K-S)\,\Delta t}{(N-1)} \tag{2.4}$$

so that x (with appropriate scaling and state-dependent factors) is the rate of this reaction.

R.3. Two S particles meet and transform into an S particle and an I particle,

$$2S \to S + I$$

The transition is from the state (S, K) to the state $(S-1, K-1)$. The (conditional) probability of this transition is

$$\frac{yS(S-1)\,\Delta t}{(N-1)} \tag{2.5}$$

so that y (with appropriate scaling and state-dependent factors) is the rate of this reaction.

The alternative rule, mentioned above, to be used in place of R.3 is R.3′:

R.3′. Three S particles can meet and transform into two S particles and an I particle

$$3S \to 2S + I$$

The transition is from the state (S, K) to the state $(S-1, K-1)$. The (conditional) probability of this transition is

$$\frac{yS(S-1)(S-2)\,\Delta t}{(N-1)(N-2)} \tag{2.5′}$$

so that here, too, y is the rate of the reaction.

Let $P(S, K, t)$ be the probability that the system is in the state (S, K) at time t. Using the foregoing rules, we can express $P(S, K, t+\Delta t)$ in terms of $P(S, K, t)$. Taking $\partial P(S, K, t)/\partial t$ to be the limit as $\Delta t \searrow 0$ of $[P(S, K, t+\Delta t) - P(S, K, t)]/\Delta t$, we obtain the following master equations: For the rules R.1, R.2, and R.3

$$\frac{\partial P(S, K, t)}{\partial t} = (S+1)\, P(S+1, K, t) + \frac{x(S-1)(K-S+1)}{N-1} P(S-1, K, t) + \frac{yS(S+1)}{N-1} P(S+1, K+1, t) - \left(S + \frac{xS(K-S)}{N-1} + \frac{yS(S-1)}{N-1} \right) P(S, K, t) \tag{2.6}$$

For the rules R.1, R.2, and R.3′

$$\frac{\partial P(S, K, t)}{\partial t} = (S+1)\, P(S+1, K, t) + \frac{x(S-1)(K-S+1)}{N-1} P(S-1, K, t) + \frac{yS(S+1)(S+2)}{(N-1)(N-2)} P(S+1, K+1, t) - \left(S + \frac{xS(K-S)}{N-1} + \frac{yS(S-1)(S-2)}{(N-1)(N-2)} \right) P(S, K, t) \tag{2.6′}$$

Note that Eqs. (2.6) and (2.6′) differ only in the y term. Equations (2.6) and (2.6′) can be summarized in matrix form:

$$\frac{\partial P(S, K, t)}{\partial t} = \sum_{S', K'} P(S', K', t)\, U(S', K'; S, K) \tag{2.7}$$

where U is the transfer matrix. Clearly there are two U's, corresponding to the unprimed and primed equations. Their detailed form will be analyzed in Section 4.

The backward equation associated with (2.7) uses the transpose of U:

$$\frac{\partial w(S, K, t)}{\partial t} = \sum_{S', K'} U(S, K; S', K')\, w(S', K', t) \tag{2.8}$$

where $w(S, K, t)$ is the backward vector at time t. The matrix $U(S, K; S', K')$ has eigenvalues λ_α with left eigenvectors $p_\alpha(S, K)$ and right eigenvectors $w_\alpha(S, K)$. If $p_\alpha(S, K)$ and $w_\beta(S, K)$ are left and right eigenvectors associated with different eigenvalues, we have the orthogonality property

$$\sum_{S, K} p_\alpha(S, K)\, w_\beta(S, K) = \delta_{\alpha\beta} \tag{2.9}$$

For future use in Section 4, we explicitly write the backward equations associated with Eqs. (2.6) and (2.6′). As usual, in the following, unprimed and primed equations correspond to unprimed and primed rules (R.3 and R.3′, respectively):

$$\frac{\partial w(S,K,t)}{\partial t} = Sw(S-1,K,t) + \frac{xS(K-S)}{N-1} w(S+1,K,t) + \frac{yS(S-1)}{N-1} w(S-1,K-1,t) - \left(S + \frac{xS(K-S)}{N-1} + \frac{yS(S-1)}{N-1}\right) w(S,K,t) \tag{2.10}$$

$$\frac{\partial w(S,K,t)}{\partial t} = Sw(S-1,K,t) + \frac{xS(K-S)}{N-1} w(S+1,K,t) + \frac{yS(S-1)(S-2)}{(N-1)(N-2)} w(S-1,K-1,t) - \left(S + \frac{xS(K-S)}{N-1} + \frac{yS(S-1)(S-2)}{(N-1)(N-2)}\right) w(S,K,t) \tag{2.10′}$$

Note from (2.6) or (2.6′) that the eigenvectors of the forward equation with eigenvalue $\lambda = 0$ are of the form

$$P_{K_0}(S,K) = \delta(S)\,\delta(K-K_0) \tag{2.11}$$

for all possible $K_0 \geqslant 0$. One eigenvector of the backward equation with eigenvalue 0 is obviously

$$w(S,K) = 1 \qquad \text{for all} \quad S, K \tag{2.12}$$

This expresses the conservation of probability. In particular, from (2.9), any left eigenvector $P(S,K)$ with eigenvalue $\lambda \neq 0$ will satisfy

$$\sum_{S,K} P(S,K) = 0$$

3. MEAN-FIELD SELF-ORGANIZED CRITICALITY

Consider the mean-field limit for the models of the previous section. We study the quantities

$$s(t) = \frac{\langle S(t)\rangle}{N}, \qquad k(t) = \frac{\langle K(t)\rangle}{N} \tag{3.1}$$

Equations of motion for $s(t)$ and $k(t)$ are derived by multiplying (2.6) or (2.6′) by S (or respectively, by K), summing over S (or K), and dividing by N. If we neglect, in the limit $N \to \infty$, all cumulants and fluctuations we obtain the mean-field equations. For rules R.1, R.2, and R.3, they are

$$\begin{aligned} \frac{ds}{dt} &= -s + xs(k-s) - ys^2 \\ \frac{dk}{dt} &= -ys^2 \end{aligned} \tag{3.2}$$

For rules R.1, R.2, and R.3′, they are

$$\begin{aligned} \frac{ds}{dt} &= -s + xs(k-s) - ys^3 \\ \frac{dk}{dt} &= -ys^3 \end{aligned} \tag{3.2′}$$

For both systems (3.2) and (3.2)′ the stationary solutions are $s=0$, $k=k_0$ for any value k_0.

1. *Analysis of the system* (3.2). We linearize (3.2) around a stationary point $(0, k_0)$ so that

$$\frac{ds}{dt} \sim s(-1 + xk_0) \tag{3.3}$$

If $k_0 > 1/x$, then $(0, k_0)$ is repelling; if $k_0 < 1/x$, then $(0, k_0)$ is attracting. We define a new variable δ by

$$k = \frac{1}{x} + \delta \tag{3.4}$$

and the system (3.2) becomes

$$\begin{aligned} \frac{ds}{dt} &= xs\delta - (x+y)\, s^2 \\ \frac{d\delta}{dt} &= -ys^2 \end{aligned}$$

It follows that Eq. (3.2) has two particular solutions

$$s = \frac{1}{x(t+t_0)}, \qquad k = \frac{y}{x^2(t+t_0)} + \frac{1}{x} \tag{3.5}$$

$$s = \frac{1}{y(t+t_0)}, \qquad k = \frac{1}{y(t+t_0)} + \frac{1}{x} \tag{3.6}$$

In the (s, k) space these particular solutions define two lines of slope y/x and 1, respectively, converging to the point $(0, 1/x)$. We henceforth assume

$$y < x \tag{3.7}$$

It follows that all points (s, k) such that

$$s \leqslant k < \frac{1}{x} + \frac{y}{x} s$$

are attracted by the segment $s = 0$, $0 \leqslant k < 1/x$ at an exponential rate due to (3.3). On the other hand, all points (s, k) such that $k \geqslant 1/x + sy/x$ [above the critical line in (3.5)] are attracted to the point $(0, 1/x)$ and approach it with time dependence $\sim 1/t$. This behavior is self-organized critical behavior, namely an open subset of initial conditions is attracted to the critical stationary point $(0, 1/x)$ (which separates the repelling from the attracting stationary points) with a nonexponential time dependence.

2. *Analysis of the system* (3.2′). The stability analysis of (3.2′) is the same as that of (3.2), namely the stationary points are $(0, k_0)$ and they are repelling for $k_0 > 1/x$ and attracting for $k_0 < 1/x$. Let us define a new variable

$$u = k - s - 1/x \tag{3.8}$$

so that (3.2′) becomes

$$\begin{aligned} \frac{ds}{dt} &= xus - ys^3 \\ \frac{du}{dt} &= -xus \end{aligned} \tag{3.9}$$

A particular solution is $u = 0$, $s = [2y(t + t_0)]^{-1/2}$ and as a consequence, the line $u = 0$ cannot be crossed by any other trajectory. Moreover, in the (u, s) phase space, the vector field (3.9) is vertical (i.e., parallel to the u axis) on the parabola

$$u = s^2 x/y$$

so that this parabola will be crossed by all trajectories vertically from above. Near $u = 0$, the motion is approximately given by

$$\frac{ds}{dt} \sim -ys^3 \qquad \text{so that} \quad s(t) \sim [2y(t + t_0)]^{-1/2} \tag{3.10}$$

while

$$\frac{d \log u}{dt} = -xs \sim [2y(t+t_0)]^{-1/2}$$

so that

$$u(t) \sim \exp\left\{-x\left[\frac{2}{y}(t+t_0)\right]^{1/2}\right\} \tag{3.11}$$

This implies that for large t, $u(t) \ll s(t)$ and justifies the approximation $xus \ll ys^3$ for deriving (3.10). Returning to the initial variables (s, k), we see that the trajectories with initial conditions satisfying

$$k \geqslant s + 1/x$$

converge to the critical stationary point $(0, 1/x)$ and that for $t \to +\infty$, the behavior of the trajectories is critical with exponent $-1/2$,

$$s(t) \sim (2yt)^{-1/2}$$

$$k(t) \sim 1/x + (2yt)^{-1/2} + \exp\left[-x\left(\frac{2t}{y}\right)^{1/2}\right]$$

Remark. The system (3.2) can also be analyzed with the variable u, as in (3.2′). For that case, however, u does not vanish with exponential rapidity, but only as an inverse power. Whether this is a harbinger of the ultimate breakdown of the mean-field analysis for the system that obeys R.3 is unclear.

4. ANALYSIS OF THE BACKWARD MATRIX

We return to the models described in Section 2, but go beyond the mean-field analysis. The long-time behavior of the models is controlled by the eigenvalue nearest to zero of the transfer matrix U used in Eq. (2.7). Recall that the eigenvalues of (2.10) or (2.10′) are $\lambda_\alpha \leqslant 0$, so that the asymptotic behavior of the probability for the survival of any nonabsorbing state (i.e., $S > 0$) is $\exp[-(\min|\lambda_\alpha|)t]$, the minimum being taken over the nonzero eigenvalues.

The explicit forms of U given in Eqs. (2.10) and (2.10′) show that $w(S, K)$ is coupled only to $w(S', K')$ for $K' = K$ $(K-1)$ and $S' = S$ $(S \pm 1)$. It is therefore convenient to arrange the vector $[w(S, K)]_{0 \leqslant S \leqslant K \leqslant N}$ by blocks. Correspondingly, the backward matrix will also be arranged in block form. For each $K = 0, 1, \ldots, N$, we define a $(K+1)$-dimensional vector

$$(w_K(S))_{0 \leqslant S \leqslant K}$$

by

$$w_K(S) = w(S, K) \tag{4.1}$$

The backward matrix U now has a natural block structure of the following form:

$$U = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & & \cdots & 0 \\ 0 & D_1 & 0 & 0 & 0 & 0 & & \cdots & 0 \\ 0 & A_2 & D_2 & 0 & 0 & 0 & & \cdots & 0 \\ 0 & 0 & A_3 & D_3 & 0 & 0 & & \cdots & 0 \\ 0 & 0 & 0 & A_4 & D_4 & 0 & & \cdots & 0 \\ 0 & 0 & 0 & 0 & A_5 & \cdots & & \cdots & 0 \\ 0 & 0 & 0 & \cdots & \cdots & & & & \\ \vdots & \vdots & \vdots & & \ddots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & D_{N-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & A_N & D_N \end{pmatrix}$$

The full backward matrix U acts on a vector $w(S, K)$ by the following rules, written in block form:

$$\begin{aligned} (Uw)_{K=0} &= 0 \\ (Uw)_{K=1} &= D_1 w_1 \end{aligned} \tag{4.2}$$

and generally for $2 \leqslant K \leqslant N$

$$(Uw)_K = A_K w_{K-1} + D_K w_K \tag{4.3}$$

Here D_K is a $(K+1) \times (K+1)$ matrix and A_K is a $(K+1) \times K$ matrix. The exact definition of these matrices is implicit in Eq. (2.10) or (2.10′). Note that for both Eqs. (2.10) and (2.10′) there is a part independent of y (depending only on the rules R.1 and R.2) and a part proportional to y (corresponding to rule R.3 or R.3′). This suggests that we build D_K in the following way. First we have

$$D_1 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.4}$$

while for $2 \leqslant K \leqslant N$, D_K takes the form

$$D_K \equiv D_K(x, y) \equiv B_K(x) + y\Delta_K \tag{4.5}$$

From (2.10) or (2.10′) it follows that $B_K(X)$ is

$$B_K(x)=\begin{pmatrix}
0 & 0 & 0 & 0 & \cdots & & 0 \\
1-\left(1+\frac{x(K-1)}{N-1}\right) & \frac{x(K-1)}{N-1} & 0 & \cdots & & & 0 \\
0 & 2 & -\left(2+\frac{2x(K-2)}{N-1}\right) & \frac{2x(K-2)}{N-1} & \cdots & & 0 \\
\vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \cdots & \cdots & S & -\left(S+\frac{Sx(K-S)}{N-1}\right) & \frac{Sx(K-S)}{N-1} \cdots & 0 \\
\vdots & \vdots & \vdots & & \ddots & & \vdots \\
0 & \cdots & \cdots & & \cdots & \cdots & \frac{x(K-1)}{N-1} \\
0 & \cdots & \cdots & & \cdots & K & -K
\end{pmatrix} \tag{4.6}$$

The first row of B_K is 0. Row number S for $1 \leqslant S \leqslant K$ of B_K is

$$\left(0,\ldots,0,+S,-\left(S+\frac{xS(K-S)}{N-1}\right),\frac{xS(K-S)}{N-1},0,\ldots,0\right) \tag{4.7}$$

where the element S is in column number $S-1$ (and the first column is labeled 0). The matrix Δ_K is diagonal. Its value depends on the choice of rule R.3 or R.3′. For R.3, we have

$$\Delta_K = -\mathrm{diag}\left(\frac{S(S-1)}{N-1}\right) \tag{4.8}$$

For R.3′ we have

$$\Delta_K = -\mathrm{diag}\left(\frac{S(S-1)(S-2)}{(N-1)(N-2)}\right) \tag{4.8'}$$

In particular, the first row of Δ_K is 0, so that the first row of D_K is 0.

So far we have concentrated on the matrix D_K of Eq. (4.3). We must still write down the $(K+1)\times K$ matrix A_K which also appears in Eq. (4.3) and is proportional to y. The first column and the first two rows of A_K are 0. For rule R.3, $2 \leqslant K \leqslant N$, A_K is

$$A_K = -\frac{y}{N-1}\begin{pmatrix}
0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 \\
0 & 2\cdot 1 & 0 & \cdots & \\
0 & 0 & 3\cdot 2 & & \\
 & & & \ddots & \\
0 & & \cdots & \cdots & K\cdot(K-1)
\end{pmatrix} \tag{4.9}$$

and row number S (counting the first row as zero) of A_K is

$$\left(0,\ldots,0,-\frac{yS(S-1)}{N-1},0,\ldots,0\right)$$

with the nonzero element in column number $S-1$ (again, the leftmost column is labeled 0). For rule R.3', $2\leqslant K\leqslant N$

$$A_K=-\frac{y}{(N-1)(N-2)}\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 3\cdot 2\cdot 1 & 0 & \cdots & \\ 0 & 0 & 0 & 4\cdot 3\cdot 2 & & \\ & & & & \ddots & \\ 0 & 0 & & \cdots & \cdots & K\cdot(K-1)\cdot(K-2) \end{pmatrix} \tag{4.9'}$$

and row number S of A_K is

$$\left(0,\ldots,0,-\frac{yS(S-1)(S-2)}{(N-1)(N-2)},0,\ldots,0\right)$$

with the nonzero element in column number $S-1$.

Now, we consider an eigenvector of U of eigenvalue λ. Recalling (4.1), this is a collection of $(K+1)$ vectors $(w_K(S))_{0\leqslant S\leqslant K}$ such that

$$(Uw)_K=\lambda w_K \tag{4.10}$$

The eigenvalue $\lambda=0$ is $(N+1)$ times degenerate. The eigenvectors of eigenvalue 0 of ${}^TU$ (i.e., of the forward matrix) are given by Eq. (2.11). Let $\lambda\neq 0$ be an eigenvalue of U. We have, using (4.10) and (4.2),

$$D_1 w_1=\lambda w_1$$

so that because $\lambda\neq 0$, either λ is an eigenvalue of D_1 or $w_1=0$. Let us assume that λ is not an eigenvalue of $D_1,\ldots,D_{K-1}$. Then $w_1,\ldots,w_{K-1}$ are 0 and we have from (4.3)

$$D_K w_K=\lambda w_K$$

so that either λ is an eigenvalue of D_K or $w_K=0$. Finally we have proved that each eigenvalue of U is an eigenvalue of D_K for a certain K.

Conversely, it is easy to see that if the eigenvalues of the D_K's are all different (except the eigenvalue $\lambda = 0$ which is common to all D_K's), then every eigenvalue λ of D_K is an eigenvalue of U for the eigenvector $(w_J^{(\lambda)})_{0 \leqslant J \leqslant N}$ defined by

$$w_J^{(\lambda)} = 0 \qquad \text{if} \quad J < K$$

$$D_K w_K^{(\lambda)} = \lambda w_K^{(\lambda)}$$

$$(D_J - \lambda)\, w_J^{(\lambda)} = A_J w_{J-1}^{(\lambda)} \qquad \text{for} \quad J > K$$

If a certain eigenvalue λ is shared by several matrices D_K, we consider the largest K such that λ is an eigenvalue of D_K and define a corresponding eigenvector $(w_J^{(\lambda)})_{0 \leqslant J \leqslant N}$ as above. Then $D_J - \lambda$ is invertible for $J > K$.

We have thus shown that the spectrum of U is the union of the spectra of the matrices D_K. In Section 5 we will analyze the spectrum of $D_K = B_K(x) + y\Delta_K$, considering y to be small and the expression $y\Delta_K$ to be a perturbation. The spectrum of D_K is therefore found as a perturbation of that of B_K.

5. SPECTRUM OF B_K AND PERTURBATION ANALYSIS OF D_K

The matrix B_K is related to a matrix that was introduced in refs. 6 and 7. There, we introduced a stochastic process, the single-step percolation process, with rules R.1 and R.2. In this process, the total population was M and among these M persons at each time there were $S(t)$ sick people. The state of the system was completely specified by S, with $0 \leqslant S \leqslant M$. For that system the transition probabilities were given by

$$\Pr[S(t + \Delta t) = S(t) - 1] = S(t)\, \Delta t$$

$$\Pr[S(t + \Delta t) = S(t) + 1] = \frac{\xi S(t)[M - S(t)]\, \Delta t}{M - 1}$$

(ξ being the rate). Going to the limit $\Delta t \searrow 0$, we obtained the master equation

$$\frac{\partial P(S, t)}{\partial t} = (S + 1)\, P(S + 1, t) + \frac{(S - 1)(M - S + 1)}{M - 1}\, \xi P(S - 1, t)$$

$$- \left[S + \frac{S(M - S)}{M - 1}\, \xi \right] P(S, t) \tag{5.1}$$

The matrix associated with the backward equation was the matrix $T_M(\xi)$ given by

$$T_M(\xi)=\begin{pmatrix} 0 & 0 & 0 & 0 & \cdots & & 0 \\ 1-\left(1+\frac{\xi(M-1)}{M-1}\right) & \frac{\xi(M-1)}{M-1} & 0 & \cdots & & & 0 \\ \vdots & \vdots & \vdots & \ddots & & & \vdots \\ 0 & \cdots & \cdots & S-\left(S+\frac{\xi S(M-S)}{M-1}\right) & \frac{\xi S(M-S)}{M-1} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \ddots & & \vdots \\ 0 & \cdots & \cdots & & \cdots & \cdots & \frac{\xi(M-1)}{M-1} \\ 0 & \cdots & \cdots & & \cdots & M & -M \end{pmatrix} \tag{5.2}$$

Note that $T_M(\xi)$ is not simply obtained from $B_K(x)$ by changing K to M and x to ξ, because in $B_K(x)$ we have the term $xS(K-S)/(N-1)$ and in $T_M(\xi)$ we have instead $\xi S(M-S)/(M-1)$. Thus in T_M, M plays the roles of both K and N. We shall return to this point.

In refs. 6 and 7, the spectrum of $T_M(\xi)$ was studied in considerable detail. We now recall those results. Apart from the trivial eigenvalue, $\mu=0$, corresponding to the extinction of the disease, $P(S)=\delta(S)$, the first excited state was analyzed and the following conclusions drawn for and from its eigenvalue μ_1:

1. $\xi=1$ is critical. Mean-field analysis of this model proves that $S=0$ is stable for $\xi<1$ and $S=1-1/\xi$ is stable for $\xi>1$.

2. For $\xi>1$, the stochastic process leads to extinction, because the only stationary state is

$$P(S)=\delta(S)$$

On the other hand, for $\xi>1$ there is a long-lived metastable state whose inverse lifetime is

$$\begin{aligned} |\mu_1| &\sim \exp\left[-(M-1)\left(1-\frac{1}{\xi}-\log\xi\right)\right] \\ &\sim \exp\left[-(M-1)\frac{(\xi-1)^2}{2}\right] \end{aligned} \tag{5.3}$$

the last form holding for ξ slightly above 1.

3. At $\xi=1$ exactly, the first eigenvalue rescales as

$$\mu_1 \sim \frac{\nu_1}{\sqrt{M}} \tag{5.4}$$

and ν_1 is the ground-state energy of a certain Schrödinger-like equation.

4. If we are near criticality, namely if

$$\xi = 1 + \frac{\alpha}{\sqrt{M}} \tag{5.5}$$

then again $\mu_1 \sim v_1/\sqrt{M}$, where $v_1 = v_1(\alpha)$ is the ground-state energy of a certain Schrödinger-like equation whose potential term is parametrized by α and where:

(a) For $\alpha \sim -\infty$, $v_1 \sim -|\alpha|$.

(b) For $\alpha \sim +\infty$, $v_1 \sim -C\alpha^2 e^{\alpha^2/2}$ and C is a constant.

We now return to the analysis of the spectrum of $B_K(x)$. As we have seen above, the only difference between $B_K(x)$ and $T_M(\xi)$ is the term $xS(K-S)/(N-1)$, which is replaced by $\xi S(M-S)/(M-1)$. We identify K with M and then define

$$\xi = \frac{xK}{N} \tag{5.6}$$

The critical value $\xi = 1$ of the $T_M(\xi)$ process corresponds exactly to the fact that $K/N \sim 1/x$, which is the self-organized critical state to which tend all trajectories starting above the line $k = s + 1/x$ in the mean-field analysis of Section 3. We sometimes refer to "ξ" as an "effective" rate for the process $S + R \to 2S$. In using this term we have in mind that we are truly in the larger model (which includes rule R.3 or R.3′), but that K is not changing rapidly, so that the system behaves like the smaller model (rules R.1 and R.2 only). As such, its transition rate for the process $S + R \to 2S$ is xK/N. On the other hand, the smaller model comes with its own parameter ξ [as in (5.1) or (5.2)], which is a fixed quantity in the context of that model alone. However, when the larger model temporarily approximates the smaller one, it does so with an effective ξ parameter, taking the value xK/N.

If $\xi = xK/N$ is well above 1, we obtain an eigenvalue like (5.3), leading to a long-lived metastable state.

Let us now consider the neighborhood of the critical value $\xi = 1$, defined as in (5.5). More precisely, let us assume that

$$K = N/x + \alpha\sqrt{N} \tag{5.7}$$

so that

$$\xi = \frac{xK}{N} \sim 1 + \frac{\alpha\sqrt{x}}{\sqrt{K}} \tag{5.8}$$

[Recall that K is in fact the M of the $T_M(\xi)$ matrix.] The largest nonzero eigenvalue of $T_M(\xi)$ or of B_K is then given by

$$\mu_1 = \frac{v_1(\alpha\sqrt{x})}{\sqrt{K}} \qquad (v_1 < 0) \tag{5.9}$$

But we need the corresponding eigenvalue λ_1 of $D_K = B_K + y\Delta_K$, so that in first-order perturbation theory (with respect to y) the eigenvalue of D_K is

$$\lambda_1 \sim \mu_1 + y\langle v_{\mu_1}| \Delta_K |w_{\mu_1}\rangle \tag{5.10}$$

where $\langle v_{\mu_1}|$ and $|w_{\mu_1}\rangle$ are the left and right eigenvectors of $T_M(\xi)$ for the eigenvalue μ_1 normalized by $\langle v_{\mu_1}|w_{\mu_1}\rangle = 1$.

In Appendix A, based on a numerical study of the eigenvalues, we evaluate the perturbation term in (5.10) for large K (or, what is the same, for large N, because we are near criticality, so that $K \sim N/x$). The results are the following:

1. For rule R.3, Eq. (A.14) shows that

$$y\langle v_{\mu_1}| \Delta_K |w_{\mu_1}\rangle = O(1) \qquad \text{and} \qquad \langle v_{\mu_1}| \Delta_K |w_{\mu_1}\rangle < 0 \tag{5.11}$$

(for N or K tending to infinity). In particular in this case the largest nontrivial eigenvalue of D_K stays a positive distance from 0 when $N \to \infty$. We have an exponentially rapid approach to the critical state $K/N = 1/x$ and the self-organized critical behavior is destroyed.

2. For rule R.3′, Eq. (A.14′) shows that

$$y\langle v_{\mu_1}| \Delta_K |w_{\mu_1}\rangle \sim \frac{C}{\sqrt{K}} \tag{5.12}$$

In particular, in this case the largest eigenvalue of D_K stays a distance $1/\sqrt{K}$ away from zero when $N \to \infty$, which is the indication of the slow decay to the critical state. This is the same scale of spectral gap as occurs *at* criticality for percolitis and which yields the $1/t$ relaxation.

Remark 5.1. Since the major conclusions of this article depend on perturbation theory, we also made numerical checks of the validity of perturbation theory for these matrices. The perturbation result is obtained from the numerically determined eigenfunctions of T_M. The nonperturbative result is gotten by simply diagonalizing the entire matrix (including the y-dependent portion) numerically. The results compared well for small y.

Remark 5.2. Had we taken a rule R.3″ with four S's meeting to give three S's and one I, the correction would have been $O(1/K)$. Thus a system with this rule, whose mean-field theory can be derived as in Section 3, also exhibits self-organized criticality.

Remark 5.3. Although not immediately germane to our purposes, it can be seen that near criticality the eigenvalues of the matrix B_K have the following property:

$$\{\text{Number of eigenvalues} \leqslant E\} \sim E^{2/3}$$

We show this in Appendix B. Although we only prove the property asymptotically, it is found numerically to hold with extraordinary accuracy.

6. RESCALING AND PERTURBATION THEORY FOR THE EIGENVALUES

Consider the matrix D_K given by Eq. (4.5). Recall that D_K is to be considered a backward matrix. We are interested in its largest eigenvalue λ of D_K or, what is the same, of the transposed matrix ${}^T D_K$. Denote by $(p(S))_{0 \leqslant S \leqslant K}$ an eigenvector of eigenvalue λ of ${}^T D_K$, so that

$$\lambda p(S) = (S+1)\, p(S+1) + (S-1)(K-S+1)\frac{x}{N-1}\, p(S-1)$$
$$-\left(S + \frac{S(K-S)}{N-1}\, x\right) p(S) + y(\Delta_K)_S\, p(S) \tag{6.1}$$

where $(\Delta_K)_S$ is the diagonal element in row $n = S$ of Δ_K, as in (4.8) or (4.8′). Define the generating function of $p(S)$

$$f(u) = \sum_{S=0}^{K} u^S p(S)$$

It is easy to see that

$$\lambda f = (1-u)\left[\frac{\partial f}{\partial u} - \frac{xu(K-1)}{N-1}\frac{\partial f}{\partial u} + \frac{xu^2}{N-1}\frac{\partial^2 f}{\partial u^2}\right] - yRf \tag{6.2}$$

where Rf is the differential operator defined by

$$Rf = \begin{cases} \dfrac{u^2}{N-1}\dfrac{\partial^2 f}{\partial u^2} & \text{for rule R.3} \\ \dfrac{u^3}{(N-1)(N-2)}\dfrac{\partial^3 f}{\partial u^3} & \text{for rule R.3′} \end{cases} \tag{6.3}$$

In (6.2) the term $(1-u)[\cdots]$ is exactly that obtained in Section 2 of ref. 7, formula (2.6), except that the x of ref. 7 has here become $x_{\text{eff}} = x(K-1)/(N-1)$ (exactly as in Section 5 of the present article). Let us now rescale parameters as in ref. 7, defining

$$x_{\text{eff}} \equiv x\frac{K-1}{N-1} = 1 + \frac{\alpha}{N^{\beta}}, \qquad \sigma = N^{\gamma}(1-u) \tag{6.4}$$

Then the first term on the right-hand side of Eq. (6.2) becomes

$$\begin{aligned}(1-u)&\left[\frac{\partial f}{\partial u} - x_{\text{eff}} u \frac{\partial f}{\partial u} + \frac{x}{N-1} u^2 \frac{\partial^2 f}{\partial u^2}\right] \\ &\approx \sigma\left[\left(\frac{\alpha}{N^{\beta}} - \frac{\sigma}{N^{\gamma}}\right)\frac{\partial f}{\partial \sigma} + N^{\gamma-1}\frac{\partial^2 f}{\partial \sigma^2}\right]\end{aligned} \tag{6.5}$$

Moreover, the terms Rf become

$$Rf \sim \begin{cases} N^{2\gamma-1}\dfrac{\partial^2 f}{\partial \sigma^2} & \text{for rule R.3} \\ N^{3\gamma-2}\dfrac{\partial^3 f}{\partial \sigma^3} & \text{for rule R.3}' \end{cases} \tag{6.6}$$

We see that in the case of rule R.3, it is impossible to choose β and γ so that all terms in (6.5) and (6.6) rescale in the same way [the best we can do is $\gamma = \beta = 1/3$ and $\lambda = \mu/N^{1/3}$, but then the term $N^{\gamma-1}(\partial^2 f/\partial\sigma^2)$ in Eq. (6.5) does not contribute]. If we choose rule R.3′, then we can take $\gamma = \beta = 1/2$ and $\lambda \sim N^{-1/2}$, and we obtain a perfect rescaling where all terms contribute. Moreover, this rescaling is the same as that of ref. 7 (see Sections 4 and 5). This means that the eigenvalue of the matrix $D_K(x, y)$ should be a small perturbation of the one of $B_K(x)$ in the case of rule R.3′. On the other hand, in the case of rule R.3, the impossibility of matching the scaling of the terms coming from $B_K(x)$ and the term coming from $y\Delta_K$ suggests that perturbation theory will demonstrate a failure of spectral collapse (hence an absence of self-organized criticality) as we have in fact seen in Section 5. The mechanism of the failure in that case was the fact that perturbation theory gave an order of change larger, in terms of powers of N, than the "vanishing" (as a function of N) gap.

7. CONCLUSION

In this section, we recapitulate and briefly discuss the preceding results. We started from the one-step percolation model of refs. 6 and 7,

with a fixed population K, a parameter ξ, and a variable S. This model is critical at $\xi = 1$. Now, we allow fluctuation of the population K by a coupling to a larger reservoir so that the total size is N. To do that, we allow new chemical reactions R.3 or R.3′ so that K is no longer a conserved quantity. As a result the parameter ξ becomes K dependent, namely

$$\xi = xK/N \qquad \text{[see Eq. (5.6)]}$$

where x is the rate parameter of the new model and the new model now has two variables S and K.

We then show that the mean-field theory of this new model exhibits self-organized critical behavior, namely, for generic initial data the dynamical system for the mean-field theory adjusts itself in such a way that the effective parameter ξ tends to its critical value 1 and the system tends slowly to its corresponding stationary states.

When we come back to the stochastic dynamics of these models, we see that:

(i) In rule R.3, fluctuations destroy the critical approach to equilibrium. Namely a perturbation analysis shows that the leading eigenvalue stays away from 0 when $N \to \infty$, indicating an exponentially fast approach to equilibrium.

(ii) In rule R.3′, fluctuations do not destroy the critical approach. The leading eigenvalue tends to 0 as $1/\sqrt{N}$ for large N.

We have thus exhibited two models, one of which has true self-organized criticality. The not-entirely-reliable mean-field theory suggests self-organized criticality for both; however, more comprehensive analysis of the stochastic process shows that in fact only the rules R.1, R.2, and R.3′ have this property. Finally, the model R.1, R.2, and R.3′ shows the desired spectral properties: a collapse of the eigenvalue spectrum of the time evolution operator, so that "zero modes" are available in the overall dynamical process.

APPENDIX A

In this appendix, we derive the estimates (5.11) and (5.12). First, we know that μ_1 is the first nonzero eigenvalue of the matrix $T_K(\xi)$ defined by (5.2) with the value of the parameter ξ defined by (5.8)

$$\xi = 1 + \frac{\alpha\sqrt{x}}{\sqrt{K}} \tag{A.1}$$

The first excited state of the master equation is the left eigenvector $\langle v_{\mu_1}|$ of $T_K(\xi)$. Now the generating function of the first excited state

$$f_1(u) = \sum_{S=0}^{K} u^S \langle v_{\mu_1} | S \rangle \tag{A.2}$$

satisfies the generating function equation associated with the master equation (5.1)

$$\mu_1 f_1 = (1-u)\left[\frac{\xi u^2}{K-1}\frac{\partial^2 f_1}{\partial u^2} + (1-\xi u)\frac{\partial f_1}{\partial u}\right] \tag{A.3}$$

[see Eq. (3.12) of ref. 7 with the changes $s \to u$, $N \to K$, and $x \to \xi$]. Next we apply the results of ref. 7, Section 5. We define new variables

$$v = (1-u)\sqrt{K} \tag{A.4}$$

$$\eta = (2v)^{1/2} \tag{A.5}$$

and rescale

$$\mu_1 = \frac{\nu_1}{\sqrt{K}}$$

This yields the equation

$$\nu_1 f_1 = \left(\frac{1}{2}\frac{\partial^2}{\partial \eta^2} - \left(\frac{1}{2\eta} + \frac{\eta^3}{4} - \frac{\alpha\sqrt{x}\,\eta}{2}\right)\frac{\partial}{\partial \eta}\right) f_1 \tag{A.6}$$

which is Eq. (5.3) of ref. 7 with the replacement $\alpha \to \alpha\sqrt{x}$. Let us define

$$G' = \frac{1}{\eta} + \frac{\eta^3}{2} - \alpha\sqrt{x}\,\eta$$

Then Eq. (A.6) is the backward equation associated with the Fokker–Planck operator

$$L = \frac{\partial^2}{\partial \eta^2} + \frac{\partial}{\partial \eta} G'$$

The ground state of L is $\exp[-G(\eta)]$ (at the least this is a formal ground state). To estimate the first excited state of L, we go to the Schrödinger picture, defining a Hamiltonian by $H = -\exp(G/2)\, L \exp(-G/2)$. At the level of H we showed in ref. 7 that the excited state f_1 corresponds to

$$\psi_{\eta_0}(\eta) = \exp[-G(\eta)/2] \int_0^{\min(\eta,\eta_0)} \exp[G(y)]\, dy$$

where η_0 has to be chosen as $[2\alpha\sqrt{x}]^{1/2}$ and the corresponding f_1 is then

$$f_1 = \int_0^{\min(\eta,\eta_0)} \exp[G(y)]\,dy = \int_0^{\min(\eta,\eta_0)} y \exp\left(\frac{y^4}{8} - \frac{\alpha\sqrt{x}\,y^2}{2}\right) dy \tag{A.7}$$

It remains to normalize f_1. First,

$$f_1|_{u=1} = f_1|_{\eta=0} = 0$$

which means that $\sum_{S=0}^{K} \langle v_{\mu_1}|S\rangle = 0$, so that $\langle v_{\mu_1}|w_0\rangle = 0$, where $w_0(S) = 1$ is the trivial eigenvector of $T_K(\xi)$ with eigenvalue 0. Second, we have to normalize $\langle v_{\mu_1}|$ so that $\langle v_{\mu_1}|w_{\mu_1}\rangle = 1$, where $|w_{\mu_1}\rangle$ is the right eigenvector of $T_K(\xi)$ with eigenvalue μ_1.

Now, a numerical study of $|w_{\mu_1}\rangle$ shows that $\langle S|w_{\mu_1}\rangle$ is approximately 1 everywhere except at $S=0$, where it is 0; therefore we should have

$$1 = \langle v_{\mu_1}|w_{\mu_1}\rangle = \langle v_{\mu_1}|w_{\mu_0}\rangle - \langle v_{\mu_1}|S=0\rangle = -\langle v_{\mu_1}|S=0\rangle = -f_1|_{u=0}$$

[actually $|w_{\mu_1}\rangle$ must be 0 at $S=0$ because it is orthogonal to the trivial stationary state $\delta(S)$ of the master equation (5.1)]. This means that we must take the normalized generating function of the normalized state $\langle v_{\mu_1}|$ to be

$$\tilde{f}_1(u) = -\frac{f_1(u)}{f_1|_{u=0}} \tag{A.8}$$

Now we can prove (5.11) for the rule R.3. We observe that, using (A.4),

$$\left.\frac{d^2 f_1}{du^2}\right|_{u=1} = K \left.\frac{d^2 f_1}{dv^2}\right|_{v=0} = K(v - \alpha\sqrt{x}) \exp\left(\frac{1}{2}v^2 - \alpha\sqrt{x}\,v\right)\Big|_{v=0} \tag{A.9}$$

Moreover, using (A.2),

$$\frac{1}{K}\left.\frac{d^2\tilde{f}_1}{du^2}\right|_{u=1} = \sum_{S=0}^{K} \frac{S(S-1)}{K} \langle v_{\mu_1}|S\rangle \tag{A.10}$$

(with the normalized eigenstate $\langle v_{\mu_1}|$). The second member of (A.10) can be also rewritten

$$\langle v_{\mu_1}| \frac{S(S-1)}{K} |w_0\rangle = -\langle v_{\mu_1}| \Delta_K |w_0\rangle \frac{N}{K} \tag{A.11}$$

because of (4.8). On the other hand, $|w_{\mu_1}\rangle$ is approximately the same as $|w_0\rangle$, except for $S=0$, where it is 0, so that

$$\langle v_{\mu_1}| \Delta_K |w_0\rangle \sim \langle v_{\mu_1}| \Delta_K |w_{\mu_1}\rangle$$

and finally we obtain from (A.10)

$$\frac{1}{K}\frac{d^2\tilde{f}_1}{du^2}\bigg|_{u=1} \sim -\frac{N}{K}\langle v_{\mu_1}|\, \Delta_K\, |w_0\rangle \tag{A.12}$$

From (A.11) and (A.9) we have

$$\langle v_{\mu_1}|\, \Delta_K\, |w_0\rangle \sim \frac{K}{N}\frac{\alpha\sqrt{x}}{f_1|_{u=0}} \tag{A.13}$$

But $u=0$ corresponds to $\eta = +\infty$, so that

$$-f_1|_{u=0} = -\int_0^{\eta_0} y \exp\left(\frac{y^4}{8} - \frac{\alpha\sqrt{x}\, y^2}{2}\right) dy$$

where $\eta_0 = (2\alpha\sqrt{x})^{1/2}$. This integral can be easily done, namely

$$-f_1|_{u=0} = -\int_0^{\alpha\sqrt{x}} dv \exp\left(\frac{1}{2}v^2 - \alpha\sqrt{x}\, v\right) \sim \frac{-1}{\alpha\sqrt{x}}$$

Now in (A.12), $K/N \sim 1/x$, so that finally

$$\langle v_{\mu_1}|\, \Delta_K\, |w_{\mu_1}\rangle \sim -\alpha^2 = O(1) \tag{A.14}$$

The same kind of computation can be done for the rule R.3′. Instead of (A.11) we have

$$-\langle v_{\mu_1}|\, \Delta_K\, |w_0\rangle \left(\frac{N}{K}\right)^2 = \langle v_{\mu_1}|\, \frac{S(S-1)(S-2)}{(K-1)(K-2)}\, |w_0\rangle \tag{A.11′}$$

Similarly, in place of (A.9) we have

$$\begin{aligned}\frac{d^3 f_1}{du^3}\bigg|_{u=1} &= -K^{3/2}\frac{d^3 f_1}{dv^3}\bigg|_{v=0} \\ &= -K^{3/2}\frac{d}{dv}\left[(v-\alpha\sqrt{x})\exp\left(\frac{1}{2}v^2 - \alpha\sqrt{x}\, v\right)\right]\bigg|_{v=0} \\ &\sim -K^{3/2}(1+\alpha^2 x)\end{aligned} \tag{A.9′}$$

and instead of (A.10), we have

$$\begin{aligned}\frac{1}{K^2}\frac{d^3\tilde{f}_1}{du^3}\bigg|_{u=1} &= \sum_{S=0}^{K} \frac{S(S-1)(S-2)}{(K-1)(K-2)}\langle v_{\mu_1}|\, S\rangle \\ &\sim \langle v_{\mu_1}|\, \frac{S(S-1)(S-2)}{(K-1)(K-2)}\, |w_0\rangle\end{aligned} \tag{A.10′}$$

so that finally

$$\langle v_{\mu_1} | \Delta_K | w_{\mu_1} \rangle \sim -\left(\frac{K}{N}\right)^2 \frac{1}{K^2 f_1|_{u=0}} (K^{3/2})(1+\alpha^2 x)$$

$$\sim -\frac{1}{x^2}[\alpha\sqrt{x} + (\alpha\sqrt{x})^3]\frac{1}{K^{1/2}} \qquad \text{(A.14')}$$

APPENDIX B

In this appendix we show that near criticality the eigenvalues of the matrix B_K have the following property:

$$\{\text{Number of eigenvalues} \leqslant E\} \sim E^{2/3}$$

Recalling Eq. (A.6), we observe that this equation holds not only for the ground state, but for the entire class of eigenstates of B_K obtained by the rescaling of variables. As in ref. 7, Eq. (A.6) can be transformed to a Schrödinger-like equation ("H" $= -\frac{1}{2}\partial^2/\partial\eta^2 + V$) with potential

$$V(\eta) = \frac{3}{8\eta^3} + \eta^2\left(-\frac{1}{4} + \frac{\alpha^2}{8}\right) - \frac{\alpha\eta^4}{8} + \frac{\eta^6}{32}$$

By a standard Tauberian theorem (which is the correspondence principle for large eigenvalues)

$$\{\text{Number of eigenvalues} \leqslant E\} = \int_{-E^{1/6}}^{E^{1/6}} [2(E-\eta^6/32)]^{1/2} \sim E^{2/3}$$

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