**Node Equations for Op Amp Circuits**

**Introduction**

The circuits in this problem each contain one or more ideal op amps. To analyze these circuits, we write and solve a set of node equations.

Ideal op amps are described in Section 6.4 of *Introduction to Electric Circuits* by R.C. Dorf and J.A. Svoboda. Section 6.5 shows how to analyze op amp circuits using node equations.

**Worked Examples**

**Example 1:**
Consider the circuit shown in Figure 1. Find the value of the voltage measured by the voltmeter.

![Figure 1](image.png)

**Figure 1** The circuit considered in Example 1.

**Solution:** Figure 2 shows the circuit from Figure 1 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. The nodes of the circuit are numbered in Figure 2. Let \( v_1, v_2, v_3 \) and \( v_4 \) denote the node voltages at nodes 1, 2, 3 and 4 respectively.

The inputs to this circuit are the voltages of the voltage sources. These inputs are related to the node voltages at the nodes of the voltage sources by

\[
0 - v_1 = 0.75 \quad \Rightarrow \quad v_1 = -0.75 \text{ V}
\]

and

\[
v_2 - 0 = 0.5 \quad \Rightarrow \quad v_2 = 0.5 \text{ V}
\]
The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

\[ v_m = v_4 - 0 = v_4 \]

**Figure 2** The circuit from Figure 1 after replacing the voltmeter by an open circuit and labeling the nodes. (Encircled numbers are node numbers)

The noninverting input of the op amp is connected to the reference node. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 3. Consequently,

\[ v_3 = 0 \text{ V} \]

Apply KCL to node 3 to get

\[
\frac{v_1 - v_3}{20000} + \frac{v_2 - v_3}{12000} = 0 + \frac{v_3 - v_4}{120000} \quad \Rightarrow \quad 6v_1 + 10v_2 - 16v_3 = v_3 - v_4
\]

Using \( v_m = v_4 \), and \( v_3 = 0 \) shows that the output voltage is related to the input voltages by

\[ v_m = -(6v_1 + 10v_2) \]

Using \( v_1 = -0.75 \text{ V} \), and \( v_2 = 0.5 \text{ V} \) gives the value of the voltage measured by the voltmeter to be

\[ v_m = -(6(-0.75) + 10(0.5)) = -0.5 \text{ V} \]
Example 2:
Consider the circuit shown in Figure 3. Find the value of the voltage measured by the voltmeter.

Solution: Figure 4 shows the circuit from Figure 3 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. The nodes of the circuit are numbered in Figure 4. Let \( v_1 \), \( v_2 \), \( v_3 \) and \( v_4 \) denote the node voltages at nodes 1, 2, 3 and 4 respectively.

The inputs to this circuit are the voltages of the voltage sources. These inputs are related to the node voltages at the nodes of the voltage sources by

\[
0 - v_1 = 0.8 \quad \Rightarrow \quad v_1 = -0.8 \text{ V}
\]

and

\[
v_2 - 0 = 1 \quad \Rightarrow \quad v_2 = 1 \text{ V}
\]
The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

\[ v_m = v_4 - 0 = v_4 \]

The noninverting input of the op amp is connected to node 2. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 3. Consequently,

\[ v_3 = v_2 = 1 \text{ V} \]

Apply KCL to node 3 to get

\[ \frac{v_1 - v_3}{2300} = 0 + \frac{v_2 - v_4}{6900} \Rightarrow 3(v_1 - v_3) = v_3 - v_4 \]

Using \( v_m = v_4 \), and \( v_3 = v_2 \) shows that the output voltage is related to the input voltages by

\[ v_m = -3v_1 + 4v_2 \]

Using \( v_1 = -0.8 \text{ V} \), and \( v_2 = 1 \text{ V} \) gives the value of the voltage measured by the voltmeter to be

\[ v_m = -3(-0.8) + 4(1) = 6.4 \text{ V} \]

Example 3:
Consider the circuit shown in Figure 5. Find the value of the voltage measured by the voltmeter.

![Figure 5](image-url) The circuit considered in Example 3.
**Solution:** Figure 6 shows the circuit from Figure 5 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. The nodes of the circuit are numbered in Figure 6. Let $v_1$, $v_2$, $v_3$ and $v_4$ denote the node voltages at nodes 1, 2, 3 and 4 respectively.

The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

$$v_m = v_4 - 0 = v_4$$

The inputs to this circuit are the voltage of the voltage source and the currents of the current sources. The voltage of the voltage source is related to the node voltages at the nodes of the voltage sources by

$$0 - v_3 = 2.75 \Rightarrow v_3 = -2.75 \text{ V}$$

![Figure 6](image)

**Figure 6** The circuit from Figure 5 after replacing the voltmeter by an open circuit and labeling the nodes. (Encircled numbers are node numbers)

Apply KCL to node 2 to get

$$\frac{v_3 - v_2}{30000} = 0 + 60 \times 10^{-6} \Rightarrow v_3 - v_2 = 1.8 \text{ V}$$

Using $v_3 = -2.75$ V gives

$$v_2 = -4.55 \text{ V}$$

The noninverting input of the op amp is connected to node 2. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 1. Consequently,

$$v_1 = v_2 = -4.55 \text{ V}$$
Apply KCL to node 1 to get

\[ 20 \times 10^{-6} = 0 + \frac{v_1 - v_4}{40000} \quad \Rightarrow \quad v_1 - v_4 = 0.8 \text{ V} \]

Using \( v_m = v_4 \), and \( v_1 = -4.55 \text{ V} \) gives the value of the voltage measured by the voltmeter to be

\[ v_m = -4.55 - 0.8 = -5.35 \text{ V} \]

**Example 4:**
Consider the circuit shown in Figure 7. Find the value of the voltage measured by the voltmeter.

![Figure 7](image)

**Solution:** Figure 8 shows the circuit from Figure 7 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. Figure 8 shows the circuit after numbering the nodes. Let \( v_1, v_2, v_3 \) and \( v_4 \) denote the node voltages at nodes 1, 2, 3 and 4 respectively.

The input to this circuit is the voltage of the voltage source. This input is related to the node voltages at the nodes of the voltage source by

\[ 0 - v_1 = 3.35 \quad \Rightarrow \quad v_1 = -3.35 \text{ V} \]

The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

\[ v_m = v_4 - 0 = v_4 \]
The noninverting input of the op amp is connected to the reference node. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 2. Consequently,

\[ v_2 = 0 \text{ V} \]

Apply KCL to node 2 to get

\[ \frac{v_1 - v_2}{20000} + \frac{v_2 - v_3}{40000} \Rightarrow v_3 = -2v_1 + 3v_2 = -2v_1 \]

Apply KCL to node 3 to get

\[ \frac{v_2 - v_3}{40000} + \frac{v_3 - v_4}{10000} + \frac{v_3 - v_4}{8000} \Rightarrow 5v_4 = -v_2 + 10v_3 = 10v_3 \]

Combining these equations gives

\[ v_4 = 2v_3 = -4v_1 \]

Using \( v_m = v_4 \), and \( v_1 = -3.35 \text{ V} \) gives the value of the voltage measured by the voltmeter to be

\[ v_m = -4(-3.35) = 13.4 \text{ V} \]
Example 5:
Consider the circuit shown in Figure 9. Find the value of the voltage measured by the voltmeter.

![Circuit Diagram](image)

**Figure 9** The circuit considered in Example 5.

**Solution:** Figure 10 shows the circuit from Figure 9 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. The nodes of the circuit are numbered in Figure 10. Let $v_1$, $v_2$, $v_3$, $v_4$, $v_5$ and $v_6$ denote the node voltages at nodes 1, 2, 3, 4, 5 and 6 respectively.

The inputs to this circuit are the voltages of the voltage sources. These inputs are related to the node voltages at the nodes of the voltage sources by

\[
0 - v_1 = 0.15 \quad \Rightarrow \quad v_1 = -0.15 \text{ V}
\]

and

\[
v_2 - 0 = 3.5 \quad \Rightarrow \quad v_2 = 3.5 \text{ V}
\]

The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

\[
v_m = v_3 - v_6
\]
The noninverting input of the top op amp is connected to node 1. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 4. Consequently,

\[ v_4 = v_1 = -0.15 \text{ V} \]

The noninverting input of the bottom op amp is connected to node 2. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 5. Consequently,

\[ v_5 = v_2 = 0.35 \text{ V} \]

Apply KCL to node 4 to get

\[ \frac{v_3 - v_4}{140000} = 0 + \frac{v_4 - v_5}{40000} \Rightarrow v_5 = \frac{9}{2}v_4 - \frac{7}{2}v_5 \]

Apply KCL to node 4 to get

\[ \frac{v_4 - v_5}{40000} = 0 + \frac{v_5 - v_6}{100000} \Rightarrow v_6 = -\frac{5}{2}v_4 + \frac{7}{2}v_5 \]

Using \( v_m = v_3 - v_6 \) shows that the output voltage is related to the input voltages by
\[ v_m = \left( \frac{9}{2} v_4 + \frac{7}{2} v_5 \right) - \left( -\frac{5}{2} v_1 + \frac{7}{2} v_2 \right) = 7 (v_1 - v_2) \]

Using \( v_1 = -0.15 \) V, and \( v_2 = 0.35 \) V gives the value of the voltage measured by the voltmeter to be

\[ v_m = 7 \left( (-0.15) - (0.35) \right) = 7 (-0.5) = -3.5 \text{ V} \]

Example 6:
Consider the circuit shown in Figure 11. Find the value of the voltage measured by the voltmeter.

Figure 11 The circuit considered in Example 6.

Solution: Figure 12 shows the circuit from Figure 11 after replacing the voltmeter by an equivalent open circuit and labeling the voltage measured by the voltmeter. We will analyze this circuit by writing and solving node equations. The nodes of the circuit are numbered in Figure 12. Let \( v_1, v_2, v_3 \) and \( v_4 \) denote the node voltages at nodes 1, 2, 3 and 4 respectively.

The input to this circuit is the voltage of the voltage source. This input is related to the node voltages at the nodes of the voltage sources by

\[ v_1 - 0 = 1.0 \quad \Rightarrow \quad v_1 = 1.0 \text{ V} \]

The output of this circuit is the voltage measured by the voltmeter. The output voltage is related to the node voltages by

\[ v_m = v_4 - 0 = v_4 \]
Figure 12. The circuit from Figure 11 after replacing the voltmeter by an open circuit and labeling the nodes. (Encircled numbers are node numbers)

The noninverting input of the op amp is connected to the reference node. The node voltage at the inverting input of an ideal op amp is equal to the node voltage at the noninverting input. The inverting input of the op amp is connected to node 3. Consequently,

\[ v_3 = 0 \text{ V} \]

Apply KCL to node 2 to get

\[ \frac{v_1 - v_2}{6000} + \frac{v_2 - v_3}{9000} + \frac{v_2 - 0}{18000} \Rightarrow 3v_1 = 6v_2 - v_3 \]

Using \( v_1 = 1 \text{ V}, \) and \( v_3 = 0 \) gives \( v_2 = 0.5 \text{ V}. \) Next, apply KCL to node 3 to get

\[ \frac{v_2 - v_3}{18000} + \frac{v_3 - v_4}{54000} \Rightarrow 3(v_2 - v_3) = v_3 - v_4 \]

Using \( v_m = v_4, v_3 = 0 \) and \( v_2 = 0.5 \text{ V} \) gives the value of the voltage measured by the voltmeter to be

\[ v_m = -3v_2 = -1.5 \text{ V} \]