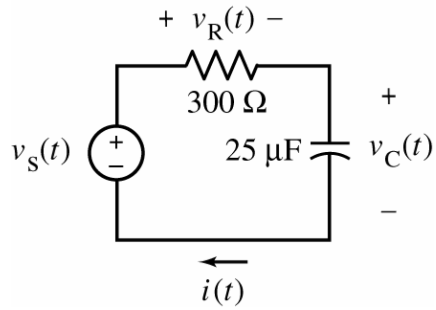
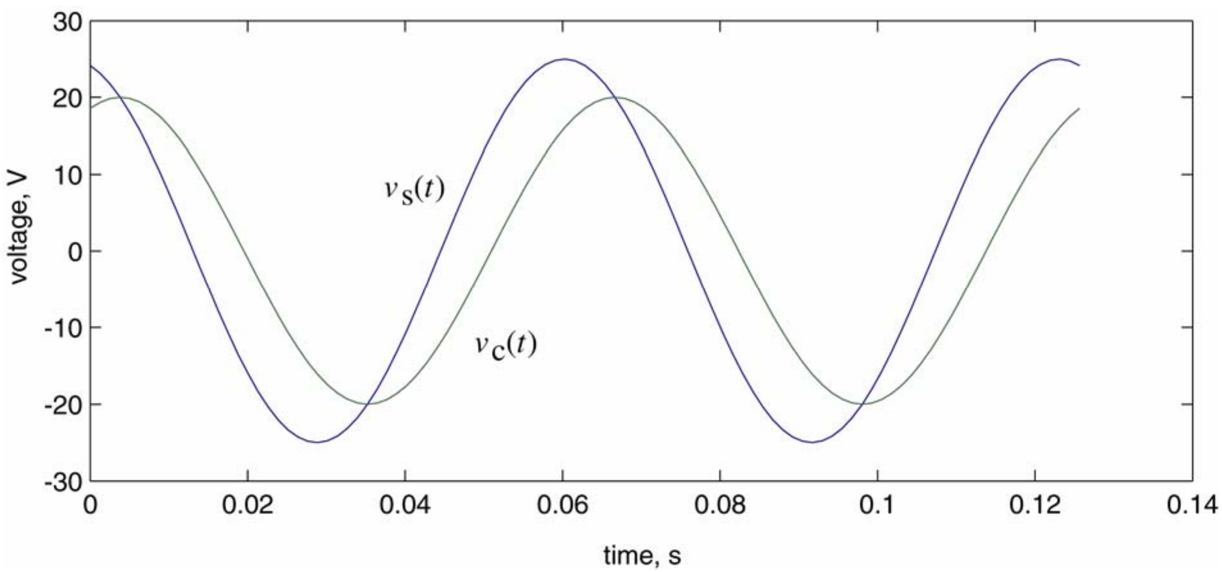
Sinusoids and Complex Numbers

The input to this circuit is the voltage source voltage, $v_s(t)$. The output is the voltage across the capacitor, $v_C(t)$. The input and output are the sinusoids shown below. We expect



$$v_s(t) = A\cos(\omega t + \theta) \quad \text{and} \quad v_C(t) = B\cos(\omega t + \phi)$$

where A, B, ω, θ and ϕ are constants to be determined.



From the plot

1) The period of both sinusoids is $T = 62.83$ ms so $\omega = \frac{2\pi}{0.06283} = 100$ rad/s .
2) The peak-to-peak amplitude of the source voltage is 50 V so $A = 25$ V.
3) The peak-to-peak amplitude of the capacitor voltage is 40 V so $B = 20$ V.
4) $v_s(0) = 24.15$ V and $v_s(t)$ is decreasing at time $t = 0$ so $\theta = \cos^{-1}\left(\frac{24.15}{25}\right) = 15°$.
5) $v_C(0) = 18.54$ V and $v_C(t)$ is increasing at time $t = 0$ so $\phi = -\cos^{-1}\left(\frac{18.54}{20}\right) = -22°$.

so

$$v_s(t) = 25\cos(100t + 15°) \text{ V} \quad \text{and} \quad v_C(t) = 20\cos(100t - 22°) \text{ V}$$

Apply KVL in the circuit to get

$$v_R(t) = v_s(t) - v_C(t) = 25\cos(100t + 15°) - 20\cos(100t - 22°)$$

First, we solve this equation using trigonometry:

Recall that $\cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

so

$$\begin{aligned}25\cos(100t + 15°) &= 25\left[\cos(100t)\cos(15°) - \sin(100t)\sin(15°)\right] \\ &= 25\cos(15°)\cos(100t) - 25\sin(15°)\sin(100t) \\ &= 24.15\cos(100t) - 6.47\sin(100t)\end{aligned}$$

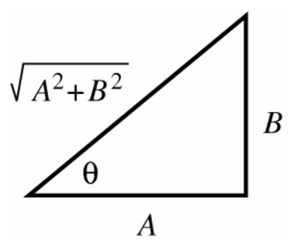
and

$$\begin{aligned}20\cos(100t - 22°) &= 20\left[\cos(100t)\cos(22°) + \sin(100t)\sin(22°)\right] \\ &= 20\cos(22°)\cos(100t) + 20\sin(22°)\sin(100t) \\ &= 18.54\cos(100t) + 7.49\sin(100t)\end{aligned}$$

The KVL equation indicates that

$$\begin{aligned}v_R(t) = v_s(t) - v_C(t) &= 25\cos(100t + 15°) - 20\cos(100t - 22°) \\ &= \left[24.15\cos(100t) - 6.47\sin(100t)\right] - \left[18.54\cos(100t) + 7.49\sin(100t)\right] \\ &= (24.15 - 18.54)\cos(100t) + (-6.47 - 7.49)\sin(100t) \\ &= 5.61\cos(100t) - 16.96\sin(100t)\end{aligned}$$

We need another trigonometric identity. Consider



$$\sin(\theta) = \frac{B}{\sqrt{A^2 + B^2}}$$

$$\cos(\theta) = \frac{A}{\sqrt{A^2 + B^2}}$$

$$\begin{aligned}&A\cos(\omega t) + B\sin(\omega t) \\ &= \sqrt{A^2 + B^2}\left[\frac{A}{\sqrt{A^2 + B^2}}\cos(\omega t) + \frac{B}{\sqrt{A^2 + B^2}}\sin(\omega t)\right] \\ &= \sqrt{A^2 + B^2}\left[\cos(\theta)\cos(\omega t) + \sin(\theta)\sin(\omega t)\right] \\ &= \sqrt{A^2 + B^2}\left[\cos(-\theta)\cos(\omega t) - \sin(-\theta)\sin(\omega t)\right] \\ &= \sqrt{A^2 + B^2}\cos(\omega t - \theta)\end{aligned}$$

where $\theta = \tan^{-1}\left(\frac{B}{A}\right)$

Consequently

$$\begin{aligned}v_R(t) = 5.61\cos(100t) - 16.96\sin(100t) &= \sqrt{5.61^2 + 16.96^2}\cos\left(100t - \tan^{-1}\left(\frac{-16.96}{5.61}\right)\right) \\ &= 15\cos(100t - (-68.1°)) \\ &= 15\cos(100t + 68.1°) \text{ V}\end{aligned}$$

Second, we will solve the KVL equation using complex numbers. To do so we associate the complex number $A\angle\theta$ with the sinusoid $A\cos(\omega t+\theta)$. That is

$$A\cos(\omega t+\theta) \quad\Leftrightarrow\quad A\angle\theta$$

Using this association:

$$\begin{aligned}25\cos(100t+15°)-20\cos(100t-22°)\Leftrightarrow 25\angle 15°-20\angle -22° &= (24.15+j\,6.47)-(18.54-j\,7.49)\\ &= (24.15-18.54)+j(6.47+7.49)\\ &= 5.61+j\,13.96=15\angle 68.1°\end{aligned}$$

Then

$$15\cos(100t+68.1°)\Leftrightarrow 15\angle 68.1°$$

We have achieved the same result with considerably less effort.

Check: $$i(t)=\frac{v_{\mathrm{R}}(t)}{300}=\frac{15\cos(100t+68.1°)}{300}=0.05\cos(100t+68.1°)$$

But also

$$\begin{aligned}i(t)=(25\times10^{-6})\frac{d\,v_{\mathrm{C}}(t)}{dt} &= (25\times10^{-6})\frac{d}{dt}(20\cos(100t-22°))\\ &= (25\times10^{-6})(20)(100)(-\sin(100t-22°))\\ &= 0.05\sin(100t+180°-22°)=0.05\sin(100t+158°)\\ &= 0.05\cos(100t+158°-90°)\\ &= 0.05\cos(100t+68°)\end{aligned}$$