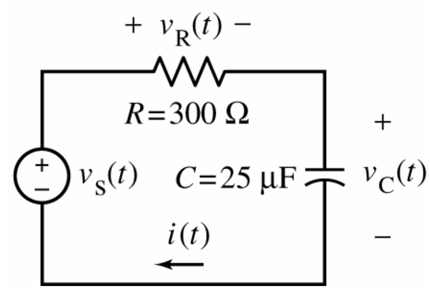
Analysis of AC Circuits



Let's represent this circuit by a differential equation. First, $i(t) = C \frac{d}{dt} v_C(t)$. Then KVL gives

$$v_s(t) = RC \frac{d}{dt} v_C(t) + v_C(t) = 0.0075 \frac{d}{dt} v_C(t) + v_C(t)$$

When the source voltage is $\quad v_s(t) = 25\cos(100t + 15°)\ \text{V}$

we expect $\quad v_C(t) = A\cos(100t + \theta)$

Substituting into the differential equation, we get

$$\begin{aligned} 25\cos(100t+15°) &= 0.0075 \frac{d}{dt}\left[A\cos(100t+\theta)\right] + A\cos(100t+\theta) \\ &= 0.0075\,(100)\,A\left[-\sin(100t+\theta)\right] + A\cos(100t+\theta) \\ &= -0.75\,A\sin(100t+\theta) + A\cos(100t+\theta) \end{aligned}$$

Solution using trigonometry:

Recall that $\quad \cos(\alpha \pm \beta) = \cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta)$

and $\quad \sin(\alpha \pm \beta) = \sin(\alpha)\cos(\beta) \pm \cos(\alpha)\sin(\beta)$

so

$$\begin{aligned} &\left[25\cos(15°)\right]\cos(100t) - \left[25\sin(15°)\right]\sin(100t) \\ &\quad = -0.75\,A\left[\sin(\theta)\cos(100t) + \cos(\theta)\sin(100t)\right] + A\left[\cos(\theta)\cos(100t) + \sin(\theta)\sin(100t)\right] \\ &\quad = A\left[\left(-0.75\sin(\theta) + \cos(\theta)\right)\cos(100t) - \left(-0.75\cos(\theta) + \sin(\theta)\right)\cos(100t)\right] \end{aligned}$$

Equating the coefficients of $\cos(100t)$ and $\sin(100t)$ gives

$$\left.\begin{aligned} 25\cos(15°) &= A\left(-0.75\sin(\theta) + \cos(\theta)\right) \\ -25\sin(15°) &= -A\left(0.75\cos(\theta) + \sin(\theta)\right) \end{aligned}\right\} \Rightarrow \quad A = 20\ \text{V} \quad \text{and} \quad \theta = -22°$$

That is $\quad v_C(t) = 20\cos(100t - 22°)\ \text{V}$

Solution using Euler's identity:

Euler's identity is: $e^{j\theta} = \cos\theta + j\sin\theta \quad \Rightarrow \quad \cos\theta = \mathrm{Re}\{e^{j\theta}\}$

Using $-\sin(\theta) = \cos(\theta + 90°)$, we can write the differential equation as

$$25\cos(100t + 15°) = 0.75\,A\left[\cos(100t + \theta + 90°)\right] + A\cos(100t + \theta)$$

Using Euler's identity

$$\mathrm{Re}\{25e^{j100t+15°}\} = 0.75A\,\mathrm{Re}\{e^{j100t+\theta+90°}\} + A\,\mathrm{Re}\{e^{j100t+\theta}\}$$

$$\begin{aligned}\mathrm{Re}\{25e^{j100t}\,e^{j15°}\} &= 0.75A\,\mathrm{Re}\{e^{j100t}\,e^{j\theta}\,e^{j90°}\} + A\,\mathrm{Re}\{e^{j100t}\,e^{j\theta}\} \\ &= \mathrm{Re}\{0.75\,Ae^{j100t}\,e^{j\theta}\,e^{j90°} + Ae^{j100t}\,e^{j\theta}\}\end{aligned}$$

Notice that $e^{j90°} = \cos 90° + j\sin 90° = 0 + j = j$. Consequently

$$\begin{aligned}\mathrm{Re}\{25e^{j15°}\,e^{j100t}\} &= \mathrm{Re}\{j\,0.75e^{j\theta}\,Ae^{j100t} + e^{j\theta}\,Ae^{j100t}\} \\ &= \mathrm{Re}\{(j\,0.75e^{j\theta} + e^{j\theta})\,Ae^{j100t}\} \\ &= \mathrm{Re}\{(1 + j\,0.75)\,Ae^{j\theta}e^{j100t}\}\end{aligned}$$

In order for this to be true at all times, it is necessary and sufficient that

$$25e^{j15°} = (1 + j\,0.75)\,Ae^{j\theta}$$

That is

$$Ae^{j\theta} = \frac{25e^{j15°}}{1 + j\,0.75} = \frac{25e^{j15°}}{1.25e^{j37°}} = 20e^{-j22°}\ \mathrm{V}$$

Hence $A = 20\ \mathrm{V} \quad \text{and} \quad \theta = -22°$

As before $v_C(t) = 20\cos(100t - 22°)\ \mathrm{V}$

Solution using phasors:
Associate the complex number $A\angle\theta$ with the sinusoid $A\cos(\omega t+\theta)$. That is

$$A\cos(\omega t+\theta) \quad \Leftrightarrow \quad A\angle\theta$$

We say that $A\angle\theta$ is the phasor corresponding to $A\cos(\omega t+\theta)$. Using phasors we can transform the differential equation

$$25\cos(100t+15°) = 0.75\,A\left[\cos(100t+\theta+90°)\right] + A\cos(100t+\theta)$$

into
$$25\angle 15° = j\,0.75\,A\angle(\theta+90°) + A\angle\theta$$

Using $1\angle 90° = j$

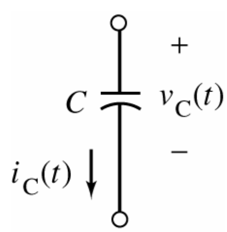
$$25\angle 15° = (1+j\,0.75)\,A\angle\theta \quad \Rightarrow \quad A\angle\theta = \frac{25\angle 15°}{1+j\,0.75} = \frac{25\angle 15°}{1.25\angle 37°} = 20\angle -22° \text{ V}$$

Hence
$$A = 20 \text{ V} \quad \text{and} \quad \theta = -22°$$

As before
$$v_C(t) = 20\cos(100t-22°) \text{ V}$$

Solution using phasors and impedances:

Consider the capacitor. When the voltage is



$$v_C(t) = A\cos(\omega t+\theta)$$

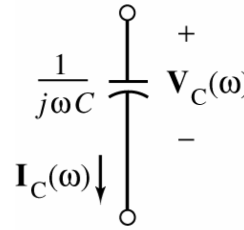
The current is

$$i_C(t) = C\frac{d}{dt}v_C(t) = -C\,\omega\,A\sin(\omega t+\theta) = C\,\omega\,A\cos(\omega t+\theta+90°)$$

Denote the corresponding phasors as

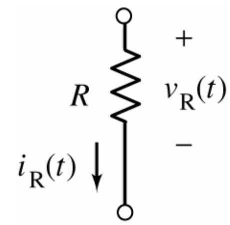
$$\mathbf{V}_C(\omega) = A\angle\theta \text{ V} \quad \text{and} \quad \mathbf{I}_C(\omega) = C\,\omega\,A\angle(\theta+90°) = j\,\omega C\,A\angle\theta \text{ A}$$

Define the impedance of the element to be the ratio of the voltage phasor to the current phasor:



$$\mathbf{Z}_C(\omega) = \frac{\mathbf{V}_C(\omega)}{\mathbf{I}_C(\omega)} = \frac{A\angle\theta}{j\,\omega C\,A\angle\theta} = \frac{1}{j\,\omega C} \quad \Omega$$

Consider the resistor. When the voltage is

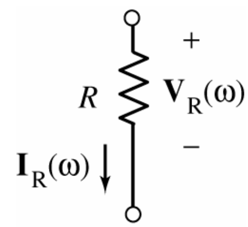


$$v_R(t) = A\cos(\omega t + \theta)$$

The current is

$$i_R(t) = \frac{v_R(t)}{R} = \frac{A}{R}\cos(\omega t + \theta)$$

The impedance of the resistor is the ratio of the voltage phasor to the current phasor:

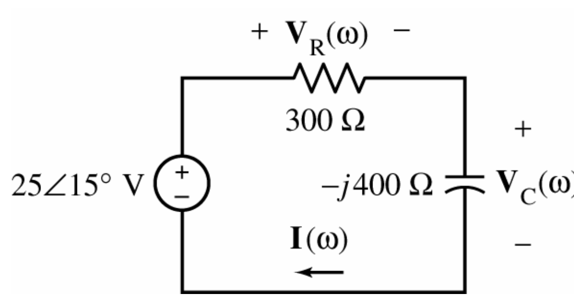


$$\mathbf{Z}_R(\omega) = \frac{\mathbf{V}_R(\omega)}{\mathbf{I}_R(\omega)} = \frac{A\angle\theta}{\frac{A}{R}\angle\theta} = R \;\; \Omega$$

(The impedance is numerically equal to the resistance.)

In the present case, $\mathbf{Z}_R(\omega) = 300 \;\; \Omega$ and $\mathbf{Z}_C(\omega) = \frac{1}{j\omega C} = \frac{1}{j100\left(25\times10^{-6}\right)} = -j400 \;\; \Omega$.

Consider this circuit in which the source is labeled using the phasor and the resistor and capacitor are labeled using impedances.



Notice that $\mathbf{V}_R(\omega) = \mathbf{Z}_R(\omega)\,\mathbf{I}(\omega)$ and $\mathbf{V}_C(\omega) = \mathbf{Z}_C(\omega)\,\mathbf{I}(\omega)$ and apply KVL to get

$$25\angle15° = 300\,\mathbf{I}(\omega) - 400\,\mathbf{I}(\omega)$$

Solve for the phasor current:

$$\mathbf{I}(\omega) = \frac{25\angle15°}{300 - j400} = \frac{25\angle15°}{500\angle-53°} = 0.05\angle68° \text{ A}$$

Using $\mathbf{V}_C(\omega) = \mathbf{Z}_C(\omega)\,\mathbf{I}(\omega)$ gives:

$$\mathbf{V}_C(\omega) = -j400\,(0.05\angle68°) = (400\angle-90°)(0.05\angle68°) = 20\angle-22° \text{ V}$$

The sinusoid corresponding to this phasor is

$$v_C(t) = 20\cos(100t - 22°) \text{ V}$$

as before.