

Pulse Inputs

The step function, shown in Figure 1a and represented as

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases} \quad (1)$$

makes an abrupt transition from 0 to 1 at time $t = 0$. Define the impulse function, $\delta(t)$, to be

$$\delta(t) = \frac{d}{dt}u(t) = \begin{cases} 0 & t < 0 \\ \text{undefined} & t = 0 \\ 0 & t > 0 \end{cases} \quad (2)$$

Since $\delta(t)$ is undefined at time 0, we consider the function $u_\varepsilon(t)$ shown in Figure 1b. This function makes the transition from 0 to 1 over the time interval from 0 to ε . Notice that

$$\lim_{\varepsilon \rightarrow 0} u_\varepsilon(t) = u(t)$$

Let

$$\delta_\varepsilon(t) = \frac{d}{dt}u_\varepsilon(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{\varepsilon} & 0 < t < \varepsilon \\ 0 & t > \varepsilon \end{cases}$$

We see that $\delta_\varepsilon(t)$ is the pulse function shown in Figure 1c. Notice that for any value of ε the area under the pulse is given by

$$\int_{-\infty}^{+\infty} \delta_\varepsilon(t) dt = \int_0^\varepsilon \frac{1}{\varepsilon} dt = 1$$

Now let

$$\delta(t) = \lim_{\varepsilon \rightarrow 0} \delta_\varepsilon(t)$$

This definition of $\delta(t)$ is consistent with the definition given in Eq. 2. We see that $\delta(t)$ is a pulse having infinite magnitude, infinitesimal duration and area equal to 1. We can't readily draw such a pulse so we represent $\delta(t)$ by an arrow as shown in Figure 1d. The height of the arrow is equal to the area of the impulse function. (The area of the impulse function is sometimes called the strength of the impulse. Also, the impulse function is sometimes called the delta function.)

An important property of the impulse function is

$$\int_{-\infty}^{+\infty} f(t) \delta(t) dt = f(0) \quad (3)$$

Letting $f(t) = 1$ gives
$$\int_{-\infty}^{+\infty} \delta(t) dt = 1$$

showing once again that the area under the impulse function is 1. More interestingly, Eq. 3 can be used to determine the Laplace transform of the impulse function

$$\mathcal{A}[\delta(t)] = \int_{0-}^{\infty} e^{-st} \delta(t) dt = e^0 = 1$$

Next, we consider some techniques that are useful for finding Laplace transforms of other pulse functions. We can delay a function $f(t)$ by time τ by replacing each occurrence of t by $t - \tau$. Consider the function

$$f(t) = mt + b$$

shown in Figure 2a. Suppose we wish to shift (delay) it to τ seconds later. This function has a single occurrence of t and so we replace it by $t - \tau$ to obtain

$$f(t - \tau) = m(t - \tau) + b = mt + (b - m\tau)$$

shown in Figure 2b. Next consider the function

$$g(t) = f(t) u(t) = (mt + b) u(t)$$

This function, shown in Figure 2c, is identical to $f(t)$ when $t > 0$ but $g(t) = 0$ when $t < 0$. Suppose we wish to delay $g(t)$ by τ seconds. The function $g(t)$ contains two occurrences of t and we must replace each occurrence of t by $t - \tau$.

$$g(t - \tau) = (m(t - \tau) + b) u(t - \tau)$$

Shown in Figure 2d, $g(t - \tau)$ is indeed a delayed copy of $g(t)$. Notice that $f(t - \tau) u(t - \tau)$ is different than both $f(t - \tau) u(t)$ and $f(t) u(t - \tau)$.

Figure 3 shows how these techniques can be used to represent pulse functions. Starting with $f(t) = 1.5 t$, a straight line that passes through the origin in Figure 3a, we multiply by a step function so that the product is 0 for time $t < 0$. The function $f(t) u(t)$ together with a delayed copy, $f(t - 10) u(t - 10)$, are shown in Figure 3b. Subtracting the delayed copy gives

$$g(t) = f(t) u(t) - f(t - 10) u(t - 10) = 1.5 t u(t) - 1.5 (t - 10) u(t - 10) \quad (4)$$

shown in Figure 3c. Subtracting an appropriately scaled and delayed step function yields the pulse shown in Figure 3d:

$$h(t) = g(t) - 15 u(t - 10) = 1.5 t u(t) - 1.5 (t - 10) u(t - 10) - 15 u(t - 10) \quad (5)$$

Alternately, starting with $g(t)$ and then subtracting and adding appropriately scaled and delayed copies of $tu(t)$ yields the pulse shown in Figure 3e:

$$k(t) = g(t) - 3.0 (t - 15) u(t - 15) + 3.0 (t - 20) u(t - 20) \quad (6)$$

(Subtracting $3.0 (t - 15) u(t - 15)$ causes $k(t)$ to begin to decrease at $t = 15$ s. Adding $3.0 (t - 20) u(t - 20)$ causes $k(t)$ to level off at $t = 20$ s. Without this last term, $k(t)$ would continue to decrease.)

To obtain the transform of the time-shifted function, we use the definition of the transform to obtain

$$\mathcal{A}[f(t - \tau)u(t - \tau)] = \int_0^{\infty} f(t - \tau)u(t - \tau)e^{-st} dt = \int_{\tau}^{\infty} f(t - \tau)e^{-st} dt$$

Now let $t - \tau = x$ to obtain

$$\mathcal{A}[f(t - \tau)u(t - \tau)] = \int_0^{\infty} f(x)e^{-s(\tau+x)} dx = e^{-s\tau} \int_0^{\infty} f(x)e^{-sx} dx = e^{-s\tau} F(s)$$

This result is summarized as

$$f(t - \tau)u(t - \tau) \leftrightarrow e^{-s\tau} F(s) \quad (7)$$

EXAMPLE 1 *Laplace Transforms of Pulse Functions*

Find the Laplace transforms of $g(t)$, $h(t)$ and $k(t)$ shown in Figure 3.

Solution

After obtaining Eqs. 4, 5 and 6, the required Laplace transform are easily determine using Eq. 7:

$$G(s) = \mathcal{A}[g(t)] = \mathcal{A}[1.5 t u(t)] - \mathcal{A}[1.5 (t - 10) u(t - 10)]$$

$$= 1.5 \left(\frac{1}{s^2} \right) - e^{-10s} \left(1.5 \left(\frac{1}{s^2} \right) \right) = \frac{1.5(1 - e^{-10s})}{s^2}$$

$$H(s) = \mathcal{A}[h(t)] = \mathcal{A}[g(t)] - \mathcal{A}[15 u(t - 10)] = \frac{1.5(1 - e^{-10s})}{s^2} - e^{-10s} \left(\frac{15}{s} \right)$$

$$K(s) = \mathcal{A}[k(t)] = \mathcal{A}[g(t)] - \mathcal{A}[30 (t - 15) u(t - 15)] + \mathcal{A}[30 (t - 20) u(t - 20)]$$

$$= \frac{1.5(1 - e^{-10s})}{s^2} - e^{-15s} \left(\frac{3.0}{s^2} \right) + e^{-20s} \left(\frac{3.0}{s^2} \right) = \frac{1.5(1 - e^{-10s} - 2e^{-15s} + 2e^{-20s})}{s^2}$$

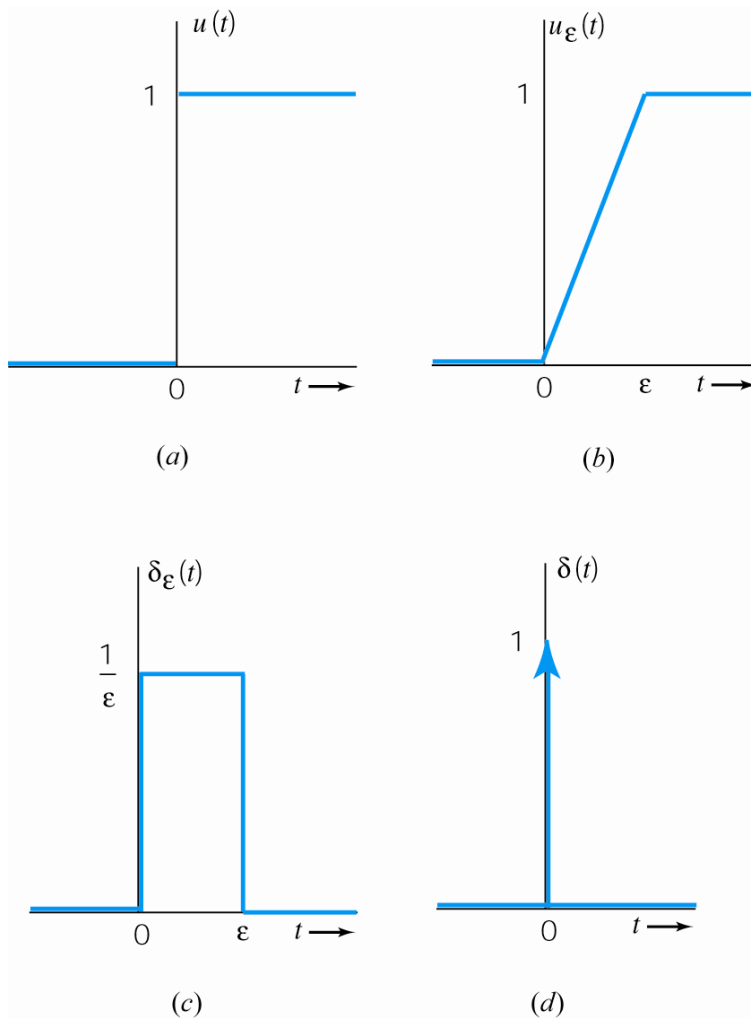


FIGURE 1 (a) A step function, (b) an approximation to the step function, (c) a pulse function and (d) the impulse function..

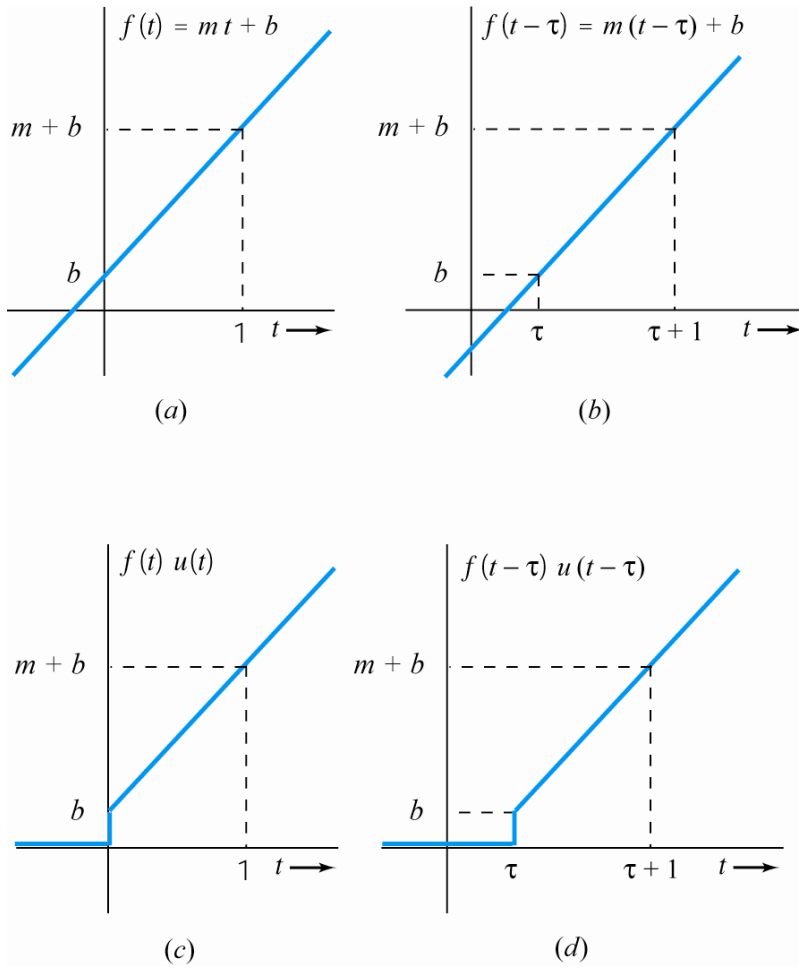


FIGURE 2 (a) A function, (b) a delayed copy of the function, (c) a new function formed by multiplying $f(t)$ by a step function and (d) a delayed copy of the new function.

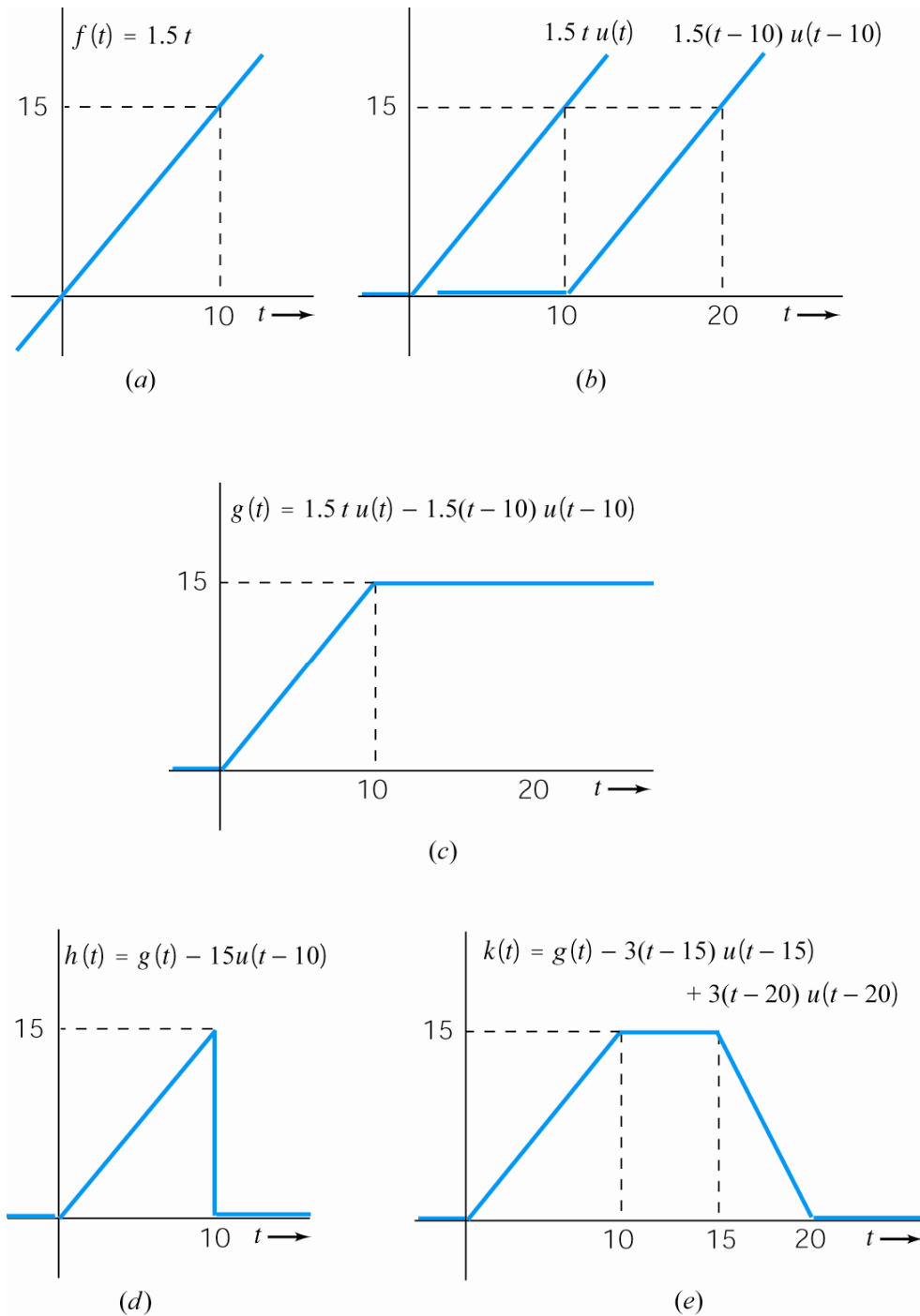


FIGURE 3 (a) A function, (b) a ramp function and a delayed copy of the ramp function, (c) a new function formed by subtracting the delayed ramp from the ramp, (d) a triangular pulse, and (e) a trapezoidal pulse.