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A Criterion for Stability in Random Boolean Cellular Automata

James F. Lynch[1]
Department of Mathematics & Computer Science
Clarkson University
Potsdam, NY 13699-5815

Abstract

Random boolean cellular automata are investigated, where each gate has two randomly chosen inputs and is randomly assigned a boolean function of its inputs. The effect of non-uniform distributions on the choice of the boolean functions is considered. The main results are that if the gates are more likely to be assigned constant functions than non-canalyzing functions, then with very high probability, the automaton will exhibit very stable behavior: most of the gates will stabilize, and the state cycles will be bounded in size.

Introduction

Boolean cellular automata are models of parallel computation that have attracted much attention from researchers in complex systems and artificial life. Computer simulation of these automata have shown that they often exhibit stable and robust behavior, even when randomly constructed. The implications of this evidence have been described in numerous articles by S. Kauffman and others (see for example [1], which includes an extensive bibliography). Only recently, however, have rigorous mathematical methods been applied to the study of boolean cellular automata. This article is a continuation of the efforts begun in Luczak and Cohen [3] and Lynch [4]. We investigate randomly constructed boolean cellular automata, where each gate has two inputs, as in most of Kauffman's simulations. However, instead of randomly assigning one of the 16 boolean functions of two arguments to each gate with equal probability, we consider the effect of non-uniform distributions, using a classification of boolean functions introduced by Kauffman (op. cit.). (We still require some mild symmetry conditions on the probabilities.)

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The boolean functions can be partitioned into the canalyzing and noncanalyzing functions. A formal definition will be given in the next section, but for now it suffices to note that among the canalyzing functions are the constant functions; i.e. the function that outputs 0 regardless of its inputs and its negation that always outputs 1. Further, among the two-argument boolean functions, there are only two non-canalyzing functions: the *equivalence* function that outputs 1 if and only if both of its inputs have the same value, and its negation the *exclusive or*.

Our main result is that if the function assigned to each gate is more likely to be constant than non-canalyzing, then with very high probability the automaton will exhibit very stable behavior. Specifically, it will have these four properties:

(1) Almost all of the gates in the automaton will stabilize, regardless of the starting state. That is, they eventually settle into a state (0 or 1) that never changes.
(2) Almost all of the gates are weak, that is, changing their state does not affect the state cycle that is entered.
(3) From any starting state, the state cycle will be entered quickly.
(4) The state cycles are bounded in size.

This shows, perhaps surprisingly, that the nonconstant canalyzing functions, which include the *or* and the *and* functions, have a neutral effect on the stability of the automaton. It is the non-canalyzing gates that seem to be the sources of instability.

Our results, and the earlier results in [3] and [4] support the belief that stability and emergent order are widespread phenomena in boolean cellular automata. In addition, they give a simple, exact condition that implies stability. Further, the distributions where the probabilities of constant and non-canalyzing gates are equal (as in Kauffman's model) appear to be thresholds between very stable and more complex behavior. This will be described in a future article. At present, it is known to the author that when the two probabilities are equal, with high probability almost all of the gates still stabilize and are weak, but the state cycles are no longer bounded in size. Most of them are larger than n^c for some $c > 0$. Very little is known about the behavior of random automata when the probability of non-canalyzing gates exceeds that of constant gates.

§2. Definitions

We will now give precise definitions of the notions that were alluded to in the previous section. Let n be a natural number. A *boolean cellular automaton* with n gates consists of a directed graph D with vertices $1, \ldots, n$ (referred to as *gates*) and a sequence $\boldsymbol{f} = (f_1, \ldots, f_n)$ of boolean functions. In this article, each gate will have indegree two, and each boolean function will have two arguments. We say that gate j is an *input* to gate i if (j, i) is

an edge of D. A boolean cellular automaton $B = \langle D, \boldsymbol{f} \rangle$ defines a map from $\{0,1\}^n$ (the set of 0-1 sequences of length n) to $\{0,1\}^n$ in the following way. For each $i = 1, \ldots, n$ let $j_i < k_i$ be the inputs of i. Given $x = (x_1, \ldots, x_n) \in \{0,1\}^n$, $B(x) = (f_1(x_{j_1}, x_{k_1}), \ldots, f_n(x_{j_n}, x_{k_n}))$. B may be regarded as a finite state automaton with state set $\{0,1\}^n$ and initial state x. That is, its state at time 0 is x, and if its state at time t is $y \in \{0,1\}^n$ then its state at time $t+1$ is $B(y)$. Our first set of definitions pertains to the aspects of stability that will be studied.

Definition 2.1. Let B be a boolean cellular automaton and $x \in \{0,1\}^n$.

(1) We put $B^t(x)$ for the state of B at time t, and $f_i^t(x)$ for the value of its ith component, or gate, at time t.
(2) Since the number of states is finite, i.e. 2^n, there exist times $t_0 < t_1$ such that $B^{t_0}(x) = B^{t_1}(x)$. Let t_1 be the first time at which this occurs. Then $B^{t+t_1-t_0}(x) = B^t$ for all $t \geq t_0$. We refer to the set of states $\{B^t(x) : t \geq t_0\}$ as the *state cycle* of x in $\langle D, \boldsymbol{f} \rangle$, to distinguish it from a cycle of D in the graph-theoretic sense.
(3) The *tail* of x in $\langle D, \boldsymbol{f} \rangle$ is $\{B^t(x) : t < t_0\}$.
(4) Gate i *stabilizes* in t steps on input x if for all $t' \geq t$, $f_i^{t'}(x) = f_i^t(x)$.
(4) Gate i is *weak* if for any input x, letting $\overline{x}^i$ be identical to x except that its ith component is $1 - x_i$,

$$\exists t_0 \exists d \forall t (t \geq t_0 \Rightarrow B^t(x) = B^{t+d}(\overline{x}^i)).$$

That is, changing the state of i does not affect the state cycle that is entered.

While we are primarily interested in stability, the related notion of forcing seems to be easier to deal with combinatorially. Thus most of our results pertain to forcing in boolean cellular automata, but as will be evident, they translate directly into results about stability.

Definition 2.2. Let $f(x_1, x_2)$ be a boolean function of two arguments.

(1) We say that f *depends* on argument x_1 if for some $v \in \{0,1\}$, $f(0,v) \neq f(1,v)$. A symmetric definition applies when f depends on x_2. Similarly, if $\langle D, \boldsymbol{f} \rangle$ is a boolean cellular automaton, $f_i = f$, and the inputs of gate i are j_{i1} and j_{i2}, then for $m = 1, 2$, i depends on j_{im} if f depends on x_m.
(2) The function f is said to be *canalyzing* if there is some $m = 1$ or 2 and some values $u, v \in \{0,1\}$ such that for all $x_1, x_2 \in \{0,1\}$, if $x_m = u$ then $f(x_1, x_2) = v$. Argument x_m of f is said to be a *forcing argument* with *forcing value* u and *forced value* v. Likewise, if $\langle D, \boldsymbol{f} \rangle$ is a boolean cellular automaton and f_i is a canalyzing function with forcing argument x_m, forcing value u and forced value v, then input j_{im} is a *forcing input* of gate i. That is, if the value of j_{im} is u at time t, then the value of i is guaranteed to be v at time $t+1$.

All of these definitions generalize immediately to boolean functions of arbitrarily many arguments. In the case of two argument boolean functions, the only non-canalyzing functions are *equivalence* and *exclusive or*. The two constant functions $f(x,y) = 0$ and $f(x,y) = 1$ are trivially canalyzing, as are the four functions that depend on only one argument:

$$f(x,y) = x,$$
$$f(x,y) = \neg x,$$
$$f(x,y) = y, \text{ and}$$
$$f(x,y) = x.$$

The remaining eight boolean functions of two arguments are canalyzing, and they are all similar in the sense that both arguments are forcing with a single value, and there is one forced value. A typical example is the *or* function. Both arguments are forcing with 1, and the forced value is 1.

Definition 2.3. Again, $\langle D, \boldsymbol{f} \rangle$ is a boolean cellular automaton. Using induction on t, we define what it means for gate i to be *forced to a value v in t steps*.

If f_i is the constant function $f(x_1, x_2) = v$, then i is forced to v in t steps for all $t \geq 0$.

If the inputs j_{i1} and j_{i2} of i are forced to u_1 and u_2 respectively in t steps, then i is forced to $f_i(u_1, u_2)$ in $t+1$ steps.

If f_i is a canalyzing function with forcing argument x_m, forcing value u, and forced value v, and j_{im} is forced to u in t steps, then i is forced to v in $t+1$ steps.

By induction on t it can be seen that if i is forced in t steps, then it stabilizes for all initial states x in t steps.

The following combinatorial notions will be used in characterizing forcing structures. We assume the reader is familiar with the basic concepts of graph theory (see e.g. Harary [2]). Unless otherwise stated, *path* and *cycle* shall mean directed path and cycle in the digraph D.

Definition 2.4. (1) For any gate i in $\langle D, \boldsymbol{f} \rangle$ with inputs j_{i1} and j_{i2}, let

$$N_0^-(i) = \{i\} \text{ and}$$
$$N_{d+1}^-(i) = N_d^-(j_{i1}) \cup N_d^-(j_{i2}).$$

(2) Then

$$S_d^-(i) = \bigcup_{c \leq d} N_c^-(i).$$

That is, $S_d^-(i)$ is the set of all gates that are connected to i by a path of length at most d.

(3) In a similar way we define $N_d^+(i)$ and $S_d^+(i)$, the set of all gates reachable from i by paths of length at most d.

(4) For any nonnegative integer d, a *d-unforced path* is a sequence of distinct gates $P = (i_1, \ldots, i_p)$ such that i_{r+1} depends on i_r for $1 \leq r < p$ and none of the gates are forced in d steps.
(5) A *d-unforced cycle* is the same except $i_1 = i_p$.

Note that whether i is forced in d steps is completely determined by the restriction of D and $\boldsymbol{f}$ to $S_d^-(i)$.

We will examine the asymptotic behavior of *random* boolean cellular automata. For each boolean function f of two arguments, we associate a probability $a_f \in [0,1]$, where $\sum_f a_f = 1$. The random boolean cellular automaton with n gates is the result of two random processes. First, a random digraph where every gate has indegree two is generated. Independently for each gate, its two inputs are selected from the $\binom{n}{2}$ equally likely possibilities. Next, each gate is independently assigned a boolean function of two arguments, using the probability distribution $\langle a_f : f : \{0,1\}^2 \to \{0,1\}\rangle$. We will use $\tilde{B} = \langle \tilde{D}, \tilde{\boldsymbol{f}}\rangle$ to denote a random boolean cellular automaton generated as above. For any properties $\mathcal{P}$ and $\mathcal{Q}$ pertaining to boolean cellular automata, we put $\mathrm{pr}(\mathcal{P}, n)$ for the probability that the random boolean cellular automaton on n gates has property $\mathcal{P}$ and $\mathrm{pr}(\mathcal{P}|\mathcal{Q}, n)$ for the conditional probability that $\mathcal{P}$ holds, given that $\mathcal{Q}$ holds. Usually, we will omit the n in these expressions since it will be understood.

We classify the two argument boolean functions as follows:

(1) $\mathcal{A}$ contains the two constant functions.
(2) $\mathcal{B}_1$ contains the four canalyzing functions that depend on one argument.
(3) $\mathcal{B}_2$ contains the eight canalyzing functions that depend on both arguments.
(4) $\mathcal{C}$ contains the two non-canalyzing functions.

Then the probabilities that a gate is assigned a function in each of the categories are:

$$a = \sum_{f \in \mathcal{A}} a_f$$
$$b_1 = \sum_{f \in \mathcal{B}_1} a_f$$
$$b_2 = \sum_{f \in \mathcal{B}_2} a_f$$
$$c = \sum_{f \in \mathcal{C}} a_f$$

Lastly, we put $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ and $b = b_1 + b_2$, the probability that a gate is assigned a nonconstant canalyzing function. Throughout the rest of the article, we assume the following symmetry conditions on our distributions:

$$a_f = a_{\neg f} \text{ for all } f \in \mathcal{A} \cup \mathcal{B}_2,$$
$$a_{f(x,y)} = a_{f(y,x)} \text{ for all } f \in \mathcal{B}_1.$$

Also, log shall always mean $\log_2$.

§3. Stable Gates

As previously mentioned, a gate is stable if it is forced. Thus, a lower bound on the probability that a gate is forced also holds for the probability of stability.

Lemma 3.1. *For $d \geq 0$ and $v = 0, 1$ let*

$$\begin{aligned} p_d(v) = \mathrm{pr}(&\textit{gate } i \textit{ is forced to } v \textit{ in } d \textit{ steps} \\ &|S_d^-(i) \textit{ induces an acyclic subgraph of } \tilde{D}) \textit{ and} \\ p_d = p_d(0) + p_d(1).& \end{aligned}$$

Then

$$p_d(0) = p_d(1) \tag{3.2}$$

and p_d satisfies the following recurrence:

$$\begin{aligned} p_0 &= a, \\ p_{d+1} &= a + bp_d + cp_d^2. \end{aligned}$$

Proof. We prove Equation (3.2) by induction on d. The recurrence relation will be a byproduct. For $d = 0$, it is clear since $p_0(0) = p_0(1) = a/2$.

Next we prove Equation (3.2) for $d+1$, assuming it is true for d. Let j and k be the two inputs of i. Since $S_{d+1}^-(i)$ induces an acyclic subgraph, so do $S_d^-(j)$ and $S_d^-(k)$. The possible ways that i can be forced to v are:

(1) It is assigned the constant function $f(x, y) = v$.
(2) It is assigned some function $f \in \mathcal{B}_1$, and the input on which i depends is forced in d steps to the value that forces f to v.
(3) It is assigned some function $f \in \mathcal{B}_2$, v is the forced value of f and at least one of its inputs is forced in d steps to the value that forces f to v, or v is not the forced value of f but j and k are forced to values u and w such that $f(u, w) = v$.
(4) It is assigned some function $f \in \mathcal{C}$, and j and k are forced to values u and w such that $f(u, w) = v$.

We will derive expressions for the probability of each of the four cases, and show they are the same for $v = 0$ and 1. The probability of Case (1) is $a/2$.

If $f \in \mathcal{B}_1$, say $f(x, y) = x$, the probability that i is forced to v in $d + 1$ steps is $p_d(v)$. The other choices for $f \in \mathcal{B}_1$ are symmetric, and by the induction assumption the probability of Case (2) is $b_1 p_d(0)$.

In Case (3), it can be observed that the eight functions in $\mathcal{B}_2$ may be partitioned into four pairs, each of the form $\{f, \neg f\}$. Take a typical $f \in \mathcal{B}_2$,

say the *or* function. Then 0 is not the forced value of f, but it is for $\neg f$. The probability that f is forced to 0 is

$$p_d(0)^2,$$

and the probability that $\neg f$ is forced to 0 is

$$(1-p_d(1))p_d(1)+p_d(1)(1-p_d(1))+p_d(1)^2.$$

Summing these two probabilities and using symmetry and the induction hypothesis, we get the probability that the function assigned to i is f or $\neg f$ and i is forced to 0 in $d+1$ steps:

$$a_f \times p_d(0)^2 + a_{\neg f} \times (2p_d(1)-p_d(1)^2) = (a_f + a_{\neg f}) \times p_d(0).$$

Summing over all four pairs of functions, the probability of Case (3) is $b_2 p_d(0)$. The argument when $v=1$ is symmetric.

Lastly, let $f \in \mathcal{C}$, say f is *exclusive or*. The probability that i is forced to 0 in $d+1$ steps is $p_d(0)^2 + p_d(1)^2$, and the probability that i is forced to 1 in $d+1$ steps is $2p_d(0)p_d(1)$. By the induction assumption, these probabilities are equal. Similar reasoning applies when f is *equivalence*. Thus the probability of Case (4) is $2cp_d(0)^2$ regardless of whether v is 0 or 1.

We have shown that for $v=0$ or 1,

$$p_{d+1}(v) = \frac{a}{2} + bp_d(0) + 2cp_d(0)^2,$$

proving Equation (3.2). Furthermore

$$p_{d+1} = a + bp_d + cp_d^2. \quad \square$$

Corollary 3.3. *We have* $p_d > 1-(1-a+c)^{d+1}$.

Proof. Let $q_d = 1-p_d$. By the Lemma,

$$\begin{aligned} q_0 &= 1-a < 1-a+c \text{ and} \\ 1-q_{d+1} &= a + b(1-q_d) + c(1-q_d)^2. \end{aligned}$$

Since $a+b+c=1$,

$$\begin{aligned} q_{d+1} &= (1-a+c)q_d - cq_d^2 \\ &\le (1-a+c)q_d \end{aligned}$$

and the result follows by induction on d. $\square$

Theorem 3.4. *Let $a > c$. For any positive $\alpha < 1/2$ there is a constant $\beta > 0$ such that*

$$\lim_{n\to\infty} \mathrm{pr}(\tilde{B} \text{ has at least } n(1-n^{-\beta}) \text{ gates that stabilize in } \alpha \log n \text{ steps}) = 1.$$

Proof. We use the following slight modification of a Fact from [3].

Fact. For any positive α and function $\omega(n)$ that increases to infinity,

$$\lim_{n\to\infty} \mathrm{pr}(\tilde{B} \text{ has at most } \omega(n)n^{2\alpha} \text{ gates belonging to cycles of length } < \alpha \log n) = 1.$$

Let $\mathbf{Y}$ be the random variable that counts the number of gates in $\tilde{B}$ that do not belong to a cycle of length $< \alpha \log n$ and do not stabilize in $\alpha \log n$ steps. Let $\mathbf{E}(\mathbf{Y})$ be its expectation. By Corollary 3.3,

$$\begin{aligned}\mathbf{E}(\mathbf{Y}) &\le n(1-a+c)^{\alpha \log n} \\ &= n^{1-\gamma} \text{ for some } \gamma > 0.\end{aligned}$$

By Markov's inequality,

$$\begin{aligned}\mathrm{pr}(\mathbf{Y} \ge n^{1-\gamma/2}) &\le n^{-\gamma/2} \\ &\to 0 \text{ as } n \to \infty.\end{aligned}$$

This implies, together with the Fact, that with probability asymptotic to 1, there are at most

$$n^{1-\gamma/2} + \omega(n)n^{2\alpha}$$

gates that do not stabilize in $\alpha \log n$ steps, for any function ω that increases to infinity. Taking $\beta = \min(\gamma/2, 1-2\alpha)$ and $\omega = \log$, the Theorem follows. □

§4. Unstable Structures

We now study the sizes and shapes of the unstable components of $\tilde{B}$. Actually, we will be looking at unforced components. Since a collection of gates is unforced if it is unstable, showing that the unforced structures have certain restrictions implies the same for unstable structures. The next lemma is central to all these results.

Lemma 4.1. *For any nonnegative integers d and l such that $l \le (\log n)^2$,*

$$\mathrm{pr}(\tilde{B} \text{ has a } d\text{-unforced path of length } l) \le n[2(1-a+c)^d + b + 2c + o(1)]^l.$$

Proof. We select a chain of $l+1$ gates as follows. Begin with i_{l+1}. Then select the two element set I_{l+1} of inputs to i_{l+1}. From I_{l+1}, select the gate which is the predecessor of i_{l+1} in the chain, and call it i_l. Call the other gate j_l. Repeat this selection process with i_l, and so on, ending with i_1 and j_1. The number of possible sequences $(i_1, \ldots, i_{l+1})$ and $(j_1, \ldots, j_l)$ that can be selected this way is bounded by

$$n \times \left[\binom{n}{2} \times 2\right]^l.$$

Also,

$$\mathrm{pr}\left(\bigwedge_{r=1}^{l} ((i_r, i_{r+1}) \in \tilde{D} \wedge (j_r, i_{r+1}) \in \tilde{D})\right) = \binom{n}{2}^{-l}.$$

For $r = 0, \ldots, l+1$ let $\mathcal{P}_r$ be the event that $(i_1, \ldots, i_r)$ is a d-unforced path. We will finish the proof by showing that

$$\mathrm{pr}(\mathcal{P}_r) \leq [(1-a+b)^d + (b+2c)/2 + o(1)]^r.$$

This will follow from

$$\mathrm{pr}(\mathcal{P}_r | \mathcal{P}_{r-1}) \leq (1-a+b)^d + (b+2c)/2 + o(1).$$

To prove this, let $\mathcal{Q}_r$ be the event that $N^-_{d-1}(j_r)$ does not induce a tree and $N^-_{d-1}(j_r) \cap \bigcup_{s=1}^{r-1} N^-_d(i_s) = \emptyset$. First, we show that

$$\mathrm{pr}(\neg \mathcal{Q}_r) = o(1). \tag{4.2}$$

Now $\mathcal{Q}_r$ fails only if there exists a path P of length $p \leq d$ beginning at some gate k and ending at i_r and another path Q of length q, $1 \leq q \leq d$, beginning at k, disjoint from P except at k and possibly its other endpoint, which must be in P or $\{i_1, \ldots, i_r\}$. There are no more than n^p ways of choosing P and no more than $n^{q-1} \times (p+r)$ ways of choosing Q. The probability of any such choice is bounded above by $(2/n)^{p+q}$. Therefore the probability that P and Q exist is bounded above by

$$\sum_{p=0}^{d} \sum_{q=1}^{d} 2^{p+q}(p+r)n^{-1} = O((\log n)^2 n^{-1}),$$

proving Equation (4.2).

Now we examine the conditional probability of $\mathcal{P}_r$, given that $\mathcal{P}_{r-1} \wedge \mathcal{Q}_r$. One possibility is that j_{r-1} is not forced in $d-1$ steps. Since $\mathcal{Q}_r$ holds, this event is independent of $\mathcal{P}_{r-1}$, and by Corollary 3.3 this has probability $\leq (1-a+c)^d$. The other possibility is that j_{r-1} is forced in $d-1$ steps, but i_r is not forced in d steps. There are three cases to consider:

(1) $f_{i_r} \in \mathcal{B}_1$
(2) $f_{i_r} \in \mathcal{B}_2$
(3) $f_{i_r} \in \mathcal{C}$

In Case (1), the input on which i_r depends must be i_{r-1} and not j_{r-1}. Given that $f_{i_r} \in \mathcal{B}_1$, the probability that i_r depends on i_{r-1} is $1/2$ because of the symmetry condition $a_{f(x,y)} = a_{f(y,x)}$. Thus the probability of Case (1) is $b_1/2$.

In Case (2), f_{i_r} can be forced by a single value on either input. Since j_{r-1} is forced, it must be forced to the value v that does not force f_{i_r}. Given that j_{r-1} is forced, by Lemma 3.1, the conditional probability that it is forced to v is $1/2$. Therefore the probability of Case (2) is $b_2/2$.

The probability of Case (3) is c. Altogether,

$$\mathrm{pr}(\mathcal{P}_r | \mathcal{P}_{r-1}) \leq \mathrm{pr}(\neg \mathcal{Q}_r) + (1 - a + c)^d + \frac{b}{2} + c,$$

and the Lemma follows. □

Lemma 4.3. *For any nonnegative integers d and l such that $l \leq (\log n)^2$,*

$$\mathrm{pr}(\tilde{B} \textit{ has a } d\textit{-unforced cycle of length } l) \leq [2(1 - a + c)^d + b + 2c + o(1)]^l.$$

Proof. The same proof as in Lemma 4.1 applies, except the factor n disappears because $i_1 = i_{l+1}$. □

Lemma 4.4. *For any nonnegative integers d and l such that $l \leq (\log n)^2$, for any gate i in $\tilde{B}$,*

$$\mathrm{pr}(\tilde{B} \textit{ has a } d\textit{-unforced path of length } l \textit{ beginning at } i) \leq [2(1 - a + c)^d + b + 2c + o(1)]^l.$$

Proof. The same proof applies here because $i_1 = i$ is already chosen.

Theorem 4.5. *If $a > c$ then there exists $\alpha > 0$ such that*

$$\lim_{n \to \infty} \mathrm{pr}(\tilde{B} \textit{ has at least } n(1 - n^{-\alpha}) \textit{ weak gates }) = 1.$$

Proof. Choose d so that $2(1-a+c)^d + b + 2c < 1$. This is possible because $a > c$ and $a + b + c = 1$. Take $l = \log n/3$ and let $\mathbf{Y}$ be the random variable that counts the number of gates in $\tilde{B}$ that do not belong to a cycle of length $< l$ and are not weak. Consider any such gate i. If all d-unforced chains starting at i are of length $< l$, then the gates in $N_l^+(i)$ are not affected by the state of i. Thus the state of i does not affect the state cycle that $\tilde{B}$

enters, and i would be weak. Therefore there must be a d-unforced path beginning at i of length l. By Lemma 4.4,

$$\mathbf{E}(\mathbf{Y}) \leq n^{1-\gamma} \text{ for some } \gamma > 0.$$

The rest of the proof proceeds as in the proof of Theorem 3.4. □

Lemma 4.6. *Let $a > c$. For sufficiently large d there exists a constant β such that*

$$\lim_{n\to\infty} \text{pr}(\tilde{B} \textit{ has a } d\textit{-unforced path of length } \beta \log n) = 0.$$

Proof. Since $a > c$ and $a + b + c = 1$, for sufficiently large d and n, there is $\alpha < 1$ such that

$$2(1 - a + c)^d + b + 2c + o(1) \leq \alpha.$$

Then for any $\beta > 0$, $\alpha^{\beta \log n} = n^{\beta \log \alpha}$, and by Lemma 4.1, the result follows. □

Lemma 4.7. *Let $a > c$. For sufficiently large d and every $\epsilon > 0$, there is k such that*

$$\text{pr}(\tilde{B} \textit{ has a } d\textit{-unforced cycle of length} \geq k) < \epsilon.$$

Proof. Take α and β as in the last Lemma. Then

$$\begin{aligned}\text{pr}(\tilde{B} \text{ has a } d\text{-unforced cycle of length} > \beta \log n) \\ \leq \text{pr}(\tilde{B} \text{ has a } d\text{-unforced path of length } \beta \log n) \\ = o(1).\end{aligned}$$

For any $l \leq \beta \log n$, by Lemma 4.3,

$$\begin{aligned}\text{pr}(\tilde{B} \text{ has a } d\text{-unforced cycle of length } l) &\leq \alpha^l \text{ and} \\ \text{pr}(\tilde{B} \text{ has a } d\text{-unforced cycle of length} \geq k) &\leq \sum_{l=k}^{\beta \log n} \alpha^l + o(1) \\ &\leq \frac{\alpha^k}{1-\alpha} + o(1) \\ &< \epsilon\end{aligned}$$

if $k > \log((1 - \alpha)\epsilon)/\log \alpha$. □

Lemma 4.8. *Let $a > c$. For sufficiently large d*

$$\lim_{n\to\infty} \mathrm{pr}(\tilde{B} \textit{ has } d\textit{-unforced cycles connected by a } d\textit{-unforced path }) = 0.$$

Proof. Let $\epsilon > 0$. By Lemmas 4.6 and 4.7, with probability greater than $1 - \epsilon/2$, we can assume there are no d-unforced cycles larger than k and no d-unforced paths longer than $\beta \log n$. Let $k_1, k_2 \leq k$ and $l \leq \beta \log n$. Taking α as in Lemma 4.6 and using the same argument, the probability that there exist d-unforced cycles of length k_1 and k_2 connected by a path of length l is bounded above by $2n^{-1}\alpha^{k_1+k_2+l}$. Summing over all $k_1, k_2 \leq k$ and $l \leq \beta \log n$, the probability that there exist d-unforced cycles connected by a d-unforced path is $\epsilon/2 + o(1)$. □

Theorem 4.9. *If $a > c$ then there is a constant β such that*

$$\lim_{n\to\infty} \mathrm{pr}(\tilde{B} \textit{ has a tail longer than } \beta \log n) = 0.$$

Proof. We take d as first used in Lemma 4.6. After d steps, the only gates that are not yet stable are those in d-unforced paths and cycles. We may assume that all these cycles and paths are disjoint except possibly at the endpoints of the paths. By Lemma 4.8, with probability $1 - o(1)$, no path begins and ends at a cycle.

Let l be the length of the longest d-unforced path in $\langle \tilde{D}, \tilde{\boldsymbol{f}} \rangle$ and m be the size of its largest d-unforced cycle. After l more steps, the only gates that are not yet stable are those in d-unforced cycles and paths beginning at an unforced cycle.

Now consider the state of the gates in these cycles, i.e. the projection of the state of $\langle \tilde{D}, \tilde{\boldsymbol{f}} \rangle$, where we look only at the values of the gates in the d-unforced cycles. This state will reach its state cycle in at most $2m$ steps. Then $\langle \tilde{D}, \tilde{\boldsymbol{f}} \rangle$ will reach its state cycle in at most l more steps. The Theorem then follows from Lemma 4.6. □

Theorem 4.10. *If $a > c$ then for every $\epsilon > 0$ there is s such that*

$$\mathrm{pr}(\tilde{B} \textit{ has a state cycle larger than } s) < \epsilon.$$

Proof. Any path (or cycle) of unstable gates is also a d-unforced path (or cycle). By Lemma 4.8, no d-unforced cycle is connected by a d-unforced path to a d-unforced cycle. Therefore no cycle of unstable gates is connected by a path of unstable gates to a cycle of unstable gates. In other words, the unstable gates are partitioned into disjoint sets, each set inducing a subgraph of $\tilde{D}$ with one cycle and possibly some paths that begin on the cycle but are pairwise disjoint off the cycle. Take any input x, and consider

the restriction $\overline{x}$ of x to the gates in any one of these partitions. Let m be the size of the cycle induced by this partition. The state cycle entered by $\overline{x}$ has size t or $2t$ for some factor t of m. By Lemma 4.7, with probability $1-\epsilon/2$, $m < k$. The size of the state cycle entered by x is the least common multiple of the sizes of the state cycles of the partitions. This is bounded above by $s = 2\text{lcm}(1, \ldots, k-1)$. $\square$

Corollary 4.11. *Let $a > c$ and $\omega(n)$ be any unbounded increasing function. Then*

$$\lim_{n\to\infty} \text{pr}(\tilde{B} \textit{ has a state cycle larger than } \omega(n)) = 0.$$

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