Homework 5

2.4 :

2. **(a)** $\mathcal{M} \models \phi$: For any x, take $y = x + 1$ and $z = x$. Then $x < y$, $z < y$, and $x \not< z$, so ϕ is true.

(b) $\mathcal{M}' \models \phi$: For any x, take $y = 2x$ and $z = x$. Then $2x = y$ and $2z = y$. If $x = 0$ then $2x = z$ and $2z = x$. If $x \neq 0$, $2x \neq z$; in either case, $2x = z \to 2z = x$, so ϕ is true.

(c) $\mathcal{M}'' \models \phi$: For any x, take $y = x$ and $z = x$. Then $x < y + 1$, $z < y + 1$, $x < z + 1$, and $z < x + 1$, so ϕ is true.

12. **(c)** Valid:

1.	$\forall x(P(x) \to \exists y Q(y))$	assumption
2.	$x_0 \quad P(x_0) \to \exists y Q(y)$	$\forall x$ e 1
3.	$P(x_0) \vee \neg P(x_0)$	LEM
4.	$P(x_0)$	assumption
5.	$\exists y Q(y)$	MP 2,4
6.	$y_0 \quad Q(y_0)$	assumption
7.	$P(x_0)$	assumption
8.	$Q(y_0)$	copy 6
9.	$P(x_0) \to Q(y_0)$	$\to$ i 7–8
10.	$\exists y(P(x_0) \to Q(y))$	$\exists y$ i 9
11.	$\exists y(P(x_0) \to Q(y))$	$\exists y$ e 5, 6–10
12.	$\neg P(x_0)$	assumption
13.	$P(x_0)$	assumption
14.	$\bot$	$\neg$ e 12, 13
15.	$Q(y)$	$\bot$ e 14
16.	$P(x_0) \to Q(y)$	$\to$ i 13–15
17.	$\exists y(P(x_0) \to Q(y))$	$\exists y$ i 16
18.	$\exists y(P(x_0) \to Q(y))$	$\vee$ e 3, 4–11, 12–17
19.	$\forall x \exists y(P(x) \to Q(y))$	$\forall x$ i 2–18
20.	$(\forall x(P(x) \to \exists y Q(y))) \to \forall x \exists y(P(x) \to Q(y))$	$\to$ i 1–19

(d)

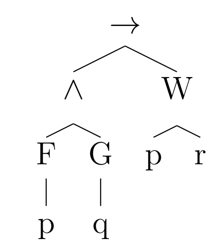
1.	$\forall x \exists y (P(x) \to Q(y))$	assumption
2.	x_0 $\exists y (P(x_0) \to Q(y))$	$\forall x$ e 1
3.	y_0 $P(x_0) \to Q(y_0)$	assumption
4.	$P(x_0)$	assumption
5.	$Q(y_0)$	MP 3,4
6.	$\exists y Q(y)$	$\exists y$ i 5
7.	$P(x_0) \to \exists y Q(y)$	$\to$ i 4–6
8.	$P(x_0) \to \exists y Q(y)$	$\exists y$ e 2, 3–7
9.	$\forall x (P(x) \to \exists y Q(y))$	$\forall x$ i 2–8
10.	$\forall x \exists y (P(x) \to Q(y)) \to \forall x (P(x) \to \exists y Q(y))$	$\to$ i 1–9

2.5:

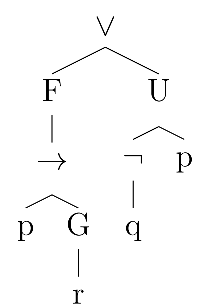
1. (e) Let $\mathcal{M}$ be the model with universe $\{0, 1\}$ where $P^{\mathcal{M}} = \{0\}$ and $Q^{\mathcal{M}} = \{1\}$. Then $\mathcal{M} \models \exists x P(x) \wedge \exists y Q(y)$ but $\mathcal{M} \not\models \exists z (P(z) \wedge Q(z))$. So if there were a proof of $\exists x P(x), \exists y Q(y) \vdash \exists z (P(z) \wedge Q(z))$, by soundness, $\exists x P(x), \exists y Q(y) \models \exists z (P(z) \wedge Q(z))$, which we have just shown is false.

3.2:

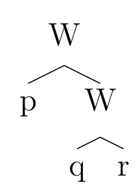
1. (a)



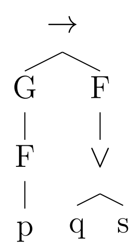
(b)



(c)



(d)



2. (b) (i) Any path starting with $q_3, q_4, \ldots$

(ii) $\mathcal{M}, q_3 \not\models \phi$ because no path starting with $q_3, q_1, \ldots$ satisfies ϕ.

(d) (i) $q_3, q_2, q_2, \ldots$.

(ii) $\mathcal{M}, q_3 \not\models \phi$ because no path starting with $q_3, q_4, \ldots$ satisfies ϕ.

(e) (i) Any path starting with $q_3, q_4, q_3, q_1, \ldots$.

(ii) $\mathcal{M}, q_3 \not\models \phi$ because no path starting with $q_3, q_1, \ldots$ satisfies ϕ.