Homework 4

2.3 :

2. There are exactly two elements in the universe.

3. (a) $\exists x \exists y \exists z (\neg x = y \wedge \neg x = z \wedge \neg y = z \wedge \forall w (w = x \vee w = y \vee w = z))$.

(b) $\exists x \exists y \exists z \forall w (w = x \vee w = y \vee w = z)$

(c) No such sentence exists. Predicate logic can't express the concept of "finite number of elements."

4. (a)

1.	$\phi \to q_1 \wedge q_2$	premise
2.	ϕ	assumption
3.	$q_1 \wedge q_2$	MP 1,2
4.	q_1	$\wedge$e 3
5.	$\phi \to q_1$	$\to$i 2–4
6.	ϕ	assumption
7.	$q_1 \wedge q_2$	MP 1,6
8.	q_2	$\wedge$e 7
9.	$\phi \to q_2$	$\to$i 6–8
10.	$(\phi \to q_1) \wedge (\phi \to q_2)$	$\wedge$i 5,9

(b)

1.	$\phi \to \forall x Q(x)$	premise
2.	x_0	
3.	ϕ	assumption
4.	$\forall x Q(x)$	MP 1,3
5.	$Q(x_0)$	$\forall x$e 4
6.	$\phi \to Q(x_0)$	$\to$i 3–5
7.	$\forall x (\phi \to Q(x))$	$\forall x$i 2–6

(c)

1.	$\forall x(P(x) \to Q(x))$	premise
2.	$\forall x P(x)$	assumption
3.	$x_0 \quad P(x_0)$	$\forall x$e 2
4.	$P(x_0) \to Q(x_0)$	$\forall x$e 1
5.	$Q(x_0)$	MP 3,4
6.	$\forall x Q(x)$	$\forall x$i 3–5
7.	$\forall x P(x) \to \forall x Q(x)$	$\to$ i 2–6

9. (e)

1.	$\forall x(P(x) \vee Q(x))$	premise
2.	$\exists x Q(x) \vee \neg \exists x Q(x)$	LEM
3.	$\exists x Q(x)$	assumption
4.	$\forall x P(x) \vee \exists x Q(x)$	$\vee$i 3
5.	$\neg \exists x Q(x)$	assumption
6.	$x_0 \quad P(x_0) \vee Q(x_0)$	$\forall x$e 1
7.	$P(x_0)$	assumption
8.	$Q(x_0)$	assumption
9.	$\exists x Q(x)$	$\exists x$i 8
10.	$\perp$	$\neg$e 5,9
11.	$P(x_0)$	$\perp$ e 10
12.	$P(x_0)$	$\vee$e 6, 7, 8–11
13.	$\forall x P(x)$	$\forall x$i 6–12
14.	$\forall x P(x) \vee \exists x Q(x)$	$\vee$i 13
15.	$\forall x P(x) \vee \exists x Q(x)$	$\vee$e 2, 3–4, 5–14

10. (a)

1.	$p \vee q$	premise
2.	$\neg p \wedge \neg q$	assumption
3.	p	assumption
4.	$\neg p$	$\wedge$e 2
5.	$\bot$	$\neg$e 3,4
6.	q	assumption
7.	$\neg q$	$\wedge$e 2
8.	$\bot$	$\neg$e 6,7
9.	$\bot$	$\vee$e 1, 3–5, 6–8
10.	$\neg(\neg p \wedge \neg q)$	$\neg$i 2–9

(b)

1.	$\exists x P(x)$	premise
2.	$\forall x \neg P(x)$	assumption
3.	$x_0 \quad P(x_0)$	assumption
4.	$\neg P(x_0)$	$\forall x$e 2
5.	$\bot$	$\neg$e 3,4
6.	$\bot$	$\exists x$e 1, 3–5
7.	$\neg \forall x \neg P(x)$	$\neg$i 2–6

(c)

1.	$\neg(\neg p \wedge \neg q)$	premise
2.	$\neg(p \vee q)$	assumption
3.	p	assumption
4.	$p \vee q$	$\vee$i 3
5.	$\bot$	$\neg$e 2,4
6.	$\neg p$	$\neg$i 3–5
7.	q	assumption
8.	$p \vee q$	$\vee$i 7
9.	$\bot$	$\neg$e 2,8
10.	$\neg q$	$\neg$i 7–9
11.	$\neg p \wedge \neg q$	$\wedge$i 6,10
12.	$\bot$	$\neg$e 1,11
13.	$\neg\neg(p \vee q)$	$\neg$i 2–12
14.	$p \vee q$	$\neg\neg$e 13

(d) $\neg\forall x \neg P(x) \vdash \exists P(x)$, by analogy to parts (a) and (b).

1.	$\neg\forall x \neg P(x)$	premise
2.	$\neg\exists x P(x)$	assumption
3.	x_0	
4.	$P(x_0)$	assumption
5.	$\exists x P(x)$	$\exists x$i 4
6.	$\bot$	$\neg$e 2,4
7.	$\neg P(x_0)$	$\neg$i 4–6
8.	$\forall x \neg P(x)$	$\forall x$i 3–7
9.	$\bot$	$\neg$e 1,8
10.	$\neg\neg\exists x P(x)$	$\neg$i 2–9
11.	$\exists x P(x)$	$\neg\neg$e 10

11. (d)

1.	$\forall x(P(x) \leftrightarrow x = b)$	premise
2.	$P(b) \leftrightarrow b = b$	$\forall x$e 1
3.	$b = b \to P(b)$	$\wedge$e 2
4.	$b = b$	$=$ i
5.	$P(b)$	MP 3.4
6.	$x_0 \quad P(x_0) \leftrightarrow x_0 = b$	$\forall x$e 1
7.	$y_0 \quad P(y_0) \leftrightarrow y_0 = b$	$\forall x$e 1
8.	$P(x_0) \wedge P(y_0)$	assumption
9.	$P(x_0)$	$\wedge$e 8
10.	$P(x_0) \to x_0 = b$	$\wedge$e 6
11.	$x_0 = b$	MP 9,10
12.	$P(y_0)$	$\wedge$e 8
13.	$P(y_0) \to y_0 = b$	$\wedge$e 7
14.	$y_0 = b$	MP 12,13
15.	$x_0 = y_0$	symmetry (2.6), transitivity (2.7) 11,14
16.	$P(x_0) \wedge P(y_0) \to x_0 = y_0$	$\to$ i 8–15
17.	$\forall y(P(x_0) \wedge P(y) \to x_0 = y)$	$\forall y$i 7–16
18.	$\forall x \forall y(P(x) \wedge P(y) \to x = y)$	$\forall x$i 6–17
19.	$P(b) \wedge \forall x \forall y(P(x) \wedge P(y) \to x = y)$	$\wedge$i 5,18