CS458/558 EXAM II March 5, 2015

Solutions

Credit will be given for partially correct answers, so it is to your advantage to explain your answers *clearly* and *concisely*. You may answer the questions in any order you choose. Just remember to number each problem clearly and write your name on the top of each sheet you turn in.

1. (20 points)

   (a) Show that the Horn formula

   $$(s \to t)\wedge(s\wedge v\wedge r \to p)\wedge(\top \to s)\wedge(w\wedge s \to\bot)\wedge(r\wedge t\wedge u \to w)\wedge(r\wedge t \to s)\wedge(s\wedge t \to r)$$

   is satisfiable by showing a truth assignment to its propositional atoms that makes the formula $\top$.

   Using the marking algorithm, these atoms get marked: r, s, t, and the algorithm terminates without a clause where all the atoms to the left of the $\to$ are marked, but with $\bot$ to the right of the $\to$. So a satisfying truth assignment is:

   $$\begin{aligned} p &= \bot \\ r &= \top \\ s &= \top \\ t &= \top \\ u &= \bot \\ v &= \bot \\ w &= \bot \end{aligned}$$

   (b) Write a Horn clause that, if it is added to the above formula, it makes the formula unsatisfiable.

   There are many choices, e.g., $(s \to\bot)$.

2. (20 points) You may find it helpful to use the results of some of the homework problems here.

   (a) Show that $\{\leftrightarrow, \bot\}$ is not an adequate set of connectives for propositional logic.
   $\bot\equiv \neg(p \leftrightarrow p)$, and since we know from the homework (1.5: 3. (c)) that $\{\leftrightarrow, \neg\}$ is not adequate, $\{\leftrightarrow, \bot\}$ is not adequate.

   (b) Show that $\{\leftrightarrow, \bot, \vee\}$ is an adequate set of connectives for propositional logic.
   $\neg p \equiv p \leftrightarrow\bot$, and since we know from the homework (1.5: 3.) that $\{\vee, \neg\}$ is adequate, $\{\leftrightarrow, \bot, \vee\}$ is adequate.

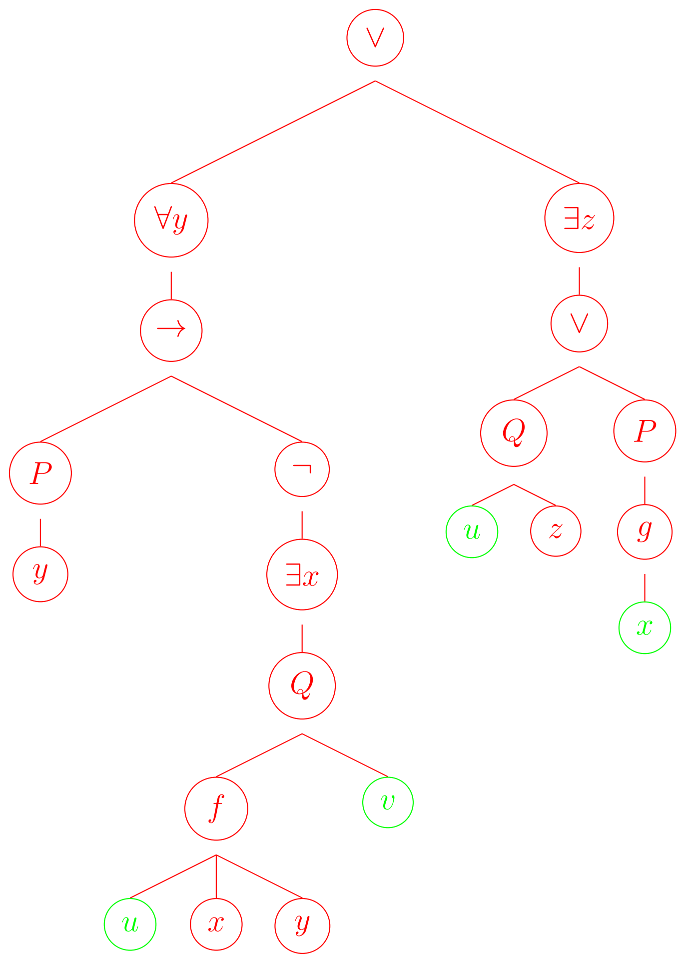
3. (30 points) Let ϕ be the formula

$$\forall y(P(y) \to \neg\exists x Q(f(u,x,y),v)) \lor \exists z(Q(u,z) \lor P(g(x)))$$

where P is a unary predicate symbol, Q is a binary predicate symbol, f is a ternary function symbol, and g is a unary function symbol. Let t be the term

$$f(u, g(y), u)$$

(a) Sketch the parse tree of ϕ.



(b) Circle all the occurrences of free variables in the parse tree of part (a).

Indicated with green.

(c) Write $\phi[t/u]$. Is t free for u in ϕ?

$$\forall y(P(y) \to \neg\exists x Q(f(f(u,g(y),u),x,y),v)) \lor \exists z(Q(f(u,g(y),u),z) \lor P(g(x)))$$

t is not free for u in ϕ.

(d) Write $\phi[t/x]$. Is t free for x in ϕ?

$$\forall y(P(y) \to \neg\exists x Q(f(u,x,y),v)) \lor \exists z(Q(u,z) \lor P(g(f(u,g(y),u))))$$

t is free for x in ϕ.

(e) Write $\phi[t/z]$. Is t free for z in ϕ?

It's the same as ϕ, and t is free for z in ϕ.

4. (30 points) Show that the following sequent is valid by writing a proof using the rules of natural deduction (p. 27 and pp. 107–117):

$$\forall x B(x,c), \forall x(\neg P(x) \to \neg \exists y B(x,y)), \forall x P(x) \to Q(c) \vdash \exists y Q(y)$$

where B is a binary predicate symbol, P is a unary predicate symbol, and c is a constant symbol.

1.	$\forall x B(x,c)$	premise
2.	$\forall x(\neg P(x) \to \neg \exists y B(x,y))$	premise
3.	$\forall x P(x) \to Q(c)$	premise
4.	$x_0 \quad B(x_0,c)$	$\forall x$ e 1
5.	$\exists y B(x_0,y)$	$\exists y$ i 4
6.	$\neg P(x_0) \to \neg \exists y B(x_0,y)$	$\forall x$ e 2
7.	$\neg\neg \exists y B(x_0,y)$	$\neg\neg$ i 5
8.	$\neg\neg P(x_0)$	MT 6,7
9.	$P(x_0)$	$\neg\neg$ e 8
10.	$\forall x P(x)$	$\forall x$ i 4–9
11.	$Q(c)$	MP 3,10
12.	$\exists y Q(y)$	$\forall y$ i 11