# Velocity Distribution in a Viscous Granular Gas

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We investigate the velocity relaxation of a viscous one-dimensional granular gas in which neither energy nor momentum is conserved in a collision. Of interest is the distribution of velocities in the gas as it cools, and the time dependence of the relaxation behavior. A Boltzmann equation of instantaneous binary collisions leads to a two-peaked distribution, as do numerical simulations of grains on a line. Of particular note is that in the presence of friction there is no inelastic collapse, so there is no need to invoke additional assumptions such as the quasi-elastic limit.

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## I. INTRODUCTION

Velocity distributions in dilute granular gases are generically away from equilibrium because the collision processes in such gases are dissipative. Even in the dilute limit the velocities of different particles may be strongly correlated [1], and therefore the usual description in terms of single particle distribution functions may not be sufficient to determine all the properties of the granular gas. Nevertheless, the single particle distribution $P(r, v, t)$ contains important information, and is particularly interesting because in a granular gas it typically deviates from the Maxwell-Boltzmann distribution. Quite aside from possible spatial inhomogeneity effects such as particle clustering, even the single particle velocity distribution $P(v, t)$ in general differs from the Maxwell-Boltzmann form. These deviations have been a subject of intense interest in recent years [2–9]. They can be observed in granular gases that achieve a steady state because external forcing balances the dissipative collisions among particles [10–12], or they can be observed in unforced gases as they cool down [14–20].

Typically, gases equilibrate or achieve a steady state via collision processes defined by the conservation of energy and momentum [21]. In granular media the situation is more complex because *energy is not conserved* [22], and most work on relaxation has focused on the concomitant consequences. A particularly bothersome behavior of granular gases induced by energy non-conservation is the so-called "inelastic collapse," whereby the energy of the gas goes to zero in a finite time [14]. In the "quasi-elastic limit" the inelastic collapse is avoided, and one obtains non-trivial asymptotic velocity distributions [10, 11, 15, 16, 23] with features similar to the ones obtained in this paper.

Friction induces not only non-conservation of energy but also *non-conservation of momentum*, leading to interesting new relaxation behavior [12, 24–26]. In fact, we have recently shown [26] that in the absence of conservation laws, random linear mixing can lead to velocity distributions with algebraic or exponential tails, with nontrivial characteristic exponents. In general, conservation laws play a crucial role in the universality of the usual velocity distribution properties.

Our focus here is the effect of viscosity, and consequently, of momentum non-conservation, on the velocity distribution as a one-dimensional dilute granular gas cools down. The model consists of $N$ grains on a line (or a circle, since we use periodic boundary conditions). The grains move freely except during collisions, governed by the Hertz potential, $V(\delta_{k,k+1}) = \frac{a}{n}|\delta|^n_{k,k+1}$ when $\delta \le 0$, and $V(\delta_{k,k+1}) = 0$ when $\delta > 0$. Here $\delta_{k,k+1} \equiv y_{k+1} - y_k$ is the relative displacement of granules $k$ and $k+1$ from the positions in which they just touch each other without compression, and $a$ is a prefactor determined by Young's modulus and Poisson's ratio. The exponent $n$ is $5/2$ for spheres, it is 2 for cylinders, and in general depends on geometry. In this paper we only consider cylindrical grains, which leads to considerable simplification while still capturing important general features of the non-Maxwellian distributions. We stress that the one-sided (only repulsion) granular potential even with $n = 2$ is entirely different from a two-sided harmonic potential.

A collision between real granules always involves some momentum and energy loss [27] since some fraction of the incident momentum excites sound waves in the interior of the granules. These granule excitations relax, heating up their mass. Here we take this momentum dissipation into account through a viscosity term during the collisions. Hence, *during a collision* of the granules $k$ and $k + 1$, their equations of motion are respectively

$$\begin{aligned} \frac{\mathrm{d}^2 x_k}{\mathrm{d}t^2} &= -\gamma \frac{\mathrm{d}x_k}{\mathrm{d}t} - (x_k - x_{k+1})^{n-1}, \\ \frac{\mathrm{d}^2 x_{k+1}}{\mathrm{d}t^2} &= -\gamma \frac{\mathrm{d}x_{k+1}}{\mathrm{d}t} + (x_k - x_{k+1})^{n-1}, \end{aligned} \tag{1}$$

where $y_k = C_n x_k$, $\tau = C_n t/v_0$, $\gamma = (\tilde{\gamma}/m v_0)C_n$, $C_n \equiv (m v_0^2/a)^{1/n}$, and $\tilde{\gamma}$ is the friction coefficient. The arbitrary velocity $v_0$ sets the energy scale of the system.

Although the problem might appear relatively simple because it is one-dimensional and quasi-linear, the one-sidedness of the potential leads to analytic complexities even in the dissipationless case [25, 28, 29], and even greater complexities in the presence of dissipation [24, 25]. We wish to explore the effects of friction at *low densities*, implemented via the assumption that the collisions are always *binary*, that is, that only two granules at a time are members of any collision event, and that at any moment there is at most one collision.

Our analytic starting point is the Boltzmann equation for binary collisions in a spatially uniform gas, which describes the rate of change of the probability distribution of velocities, $P(v,t)$. If $u_1$ and $u_2$ are the initial velocities of a pair of particles just before a collision and $u_1'$ and $u_2'$ their velocities just after, then the Boltzmann equation is

$$\frac{\partial}{\partial t}P(v,t) = \iint \mathrm{d}u_1\,\mathrm{d}u_2\,\theta(u_1-u_2)(u_1-u_2)P(u_1,t)P(u_2,t) \times\left[\delta(v-u_1')+\delta(v-u_2')-\delta(v-u_1)-\delta(v-u_2)\right]. \tag{2}$$

Since the problem is one-dimensional, one can keep track of the precise conditions under which a collision between two particles of given velocities will or will not occur, and how these events will change the distribution function. In Eq. (2) we have assumed that the particle on the left is that with initial velocity $u_1$. A collision takes place if and only if $u_1 > u_2$. This restriction is enforced by the Heaviside theta function $\theta(y) = 1$ for $y > 0$, $\theta(y) = 0$ for $y < 0$, and $\theta(0) = 1/2$.

At the moment of the start of the collision, which we call $t = 0$, the velocities of a pair of grains are $u_1$ and $u_2$. The collision ends at the time $\tau = 2\pi/\sqrt{8-\gamma^2}$, when the grains lose contact. It is important to notice that for $n = 2$ the collision time *is independent* of the initial condition. This is a feature that makes the cylindrical granule geometry much simpler than other shapes. The velocities $u_1'$ and $u_2'$ at the moment of separation $\tau$ are found to be [24]

$$u_{1,2}' = u_1\frac{\mu^2\mp\mu}{2} + u_2\frac{\mu^2\pm\mu}{2}, \tag{3}$$

where $\mu \equiv e^{-\gamma\tau/2}$. In our further analysis we think of the distance traveled by the granules during a collision as negligible. For small damping $\gamma$ this distance is $(u_1+u_2)\pi/\sqrt{8}$. This is to be compared to the mean distance between particles, which can be made arbitrarily large by lowering the density. We also take the collision time as instantaneous. This collision time for small damping is $\tau \approx \pi/\sqrt{2}$, to be compared with the typical mean free time of flight of a particle between collisions. With these approximations, the only role played by the viscosity is dictated by the collision rule (3). These assumptions might conceivably be problematic for the most energetic particles that may travel a relatively long distance during a collision and a relatively short distance between collisions, but explicit analysis of these extreme events is difficult and probably not important at sufficiently low densities.

In the long-time asymptotic limit we assume a scaling solution of the form

$$P(v,t) = \frac{1}{\phi(t)}F\left(\frac{v}{\phi(t)}\right). \tag{4}$$

This scaling in Eq. (2) together with the velocity dependence of the kernel of the Boltzmann equation obtained from Eq. (3) leads to $\phi(t) \sim t^{-1}$. Haff obtained this behavior (Haff's Law) in his classic paper on a hydrodynamic inelastic hard sphere model for a granular fluid [7].

We have not found an analytic solution of the Boltzmann equation. We therefore simulate the equation numerically and also implement a further approximation that leads to an analytic solution that we can compare with the numerical results.

We directly simulate the Boltzmann equation using the following algorithm: (1) Start with $N$ grains whose velocities are independently assigned accordingly to an initial distribution $P(v,0)$; (2) Choose one pair of grains with probability proportional to the modulus of their relative velocity and let them collide, using the collision rule; (3) Increment time by twice the inverse of the modulus of the pre-collisional relative velocity. The factor of 2 accounts for picked pairs that do not collide, since our algorithm forces a collision at each step; (4) Iterate these three steps many times and for many samples.

In our simulations we took $N = 100$ and averaged our results over 1000 samples. In Fig. 1 we show the resulting velocity distribution for $\gamma = 0.9$ at different times. It is clear that the initial symmetric and single peaked distribution develops two distinct peaks as it starts to collapse to the ultimate equilibrium distribution, a $\delta$-function at $v = 0$. In the simulation underlying this figure the initial distribution was a double peaked Gaussian, chosen because it converges quickly. We find the same asymptotic behavior for the initially exponential distribution. The validity of the scaling solution Eq. (4) is clearly evident in the scaled rendition of the velocity distribution shown in Fig. 2 for different times.

A two-peaked velocity distribution has previously been found in a number of studies (see e.g. [13]), in particular in the context of one-dimensional momentum-conserving granular gases that exhibit inelastic collapse. It occurs when particles whose (momentum-conserving) inelastic collisions are described by a constant restitution coefficient $r$ can collide infinitely often in a finite time, which drives their energy to zero in a finite time [14]. Whether or not this occurs depends on the number of particles $N$. There is a monotonically increasing relation between $N$ and $r$ for the critical value of $N_c(r)$ above which collapse occurs, with $N_c \to \infty$ as $r \to 1$. To avoid the collapse, it is customary to introduce extraneous assumptions such as the "quasi-elastic limit" which invokes the limit $r \to 1$ and $N \to \infty$ in such a way as to always remain below the collapse threshold. In this limit, a double-peaked velocity distribution is also observed [13–16]. Another way

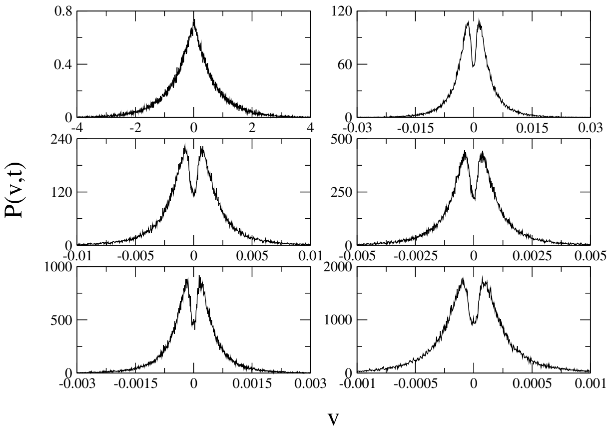


FIG. 1: Velocity distribution obtained using the simulation algorithm detailed in the text. From left to right and top to bottom, the panels correspond to time 0, 8000, 16000, 32000, 64000 and 128000 in the adimensional units used in this paper. The initial distribution is a symmetric exponential. Notice the change in the scales as time proceeds.

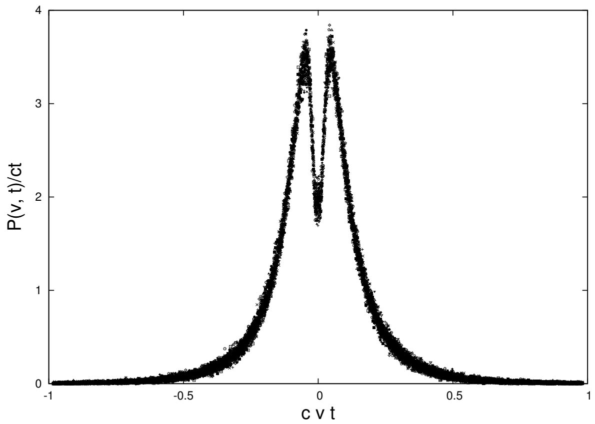


FIG. 2: Asymptotic behavior of the scaled velocity distribution. Different symbols stand for different times ($1000 \times 2^n$ with $n = 0, 1, \cdots, 13$). (1000, 2000, 4000, 8000, 16000, 32000, 64000, 128000, 256000, 512000, 1024000, 2048000, 4096000 and 8192000). The constant $c$ in the figure is arbitrary and was chosen to facilitate comparison with Fig. 3.

to avoid the collapse is to impose elasticity (or stickyness) on collisions between particles with relative velocity smaller than a given threshold (the collapse is avoided regardless of the actual value of the threshold, as long as there is one) [13].

An interesting hydrodynamic analysis based on numerical simulations that also arrives at a two-peaked velocity distribution (and that also deals with the issue of the inelastic collapse) leads to the conclusion that the usual hydrodynamic variables (density, velocity, and granular temperature) are not sufficient, and that an additional variable, the third moment of the fluctuating velocity, must also be included [16]. Benedetto *et al.* study the problem analytically (again in the quasi-elastic limit) and also arrive at the conlcusion that the velocity distribution is non-Maxwellian. A very recent study of a two-dimensional collection of disks in a channel finds that the shape of the velocity distribution depends on the coefficient of restitution and on the Knudsen number (the ratio of the channel width and the mean free path of the grains) [30]. They find a bimodal distribution when the Knudsen number is high but a unimodal distribution when it is low.

We stress again that in the presence of friction there is no inelastic collapse so that no extraneous assumptions are needed, and the double-peaked distribution occurs regardless of the parameter values as long as there is friction.

While the Boltzmann equation does not appear amenable to analytic solution, we can formulate a simpler model that may incorporate its main features. In this model we have an ensemble of rings of size $l$ each containing only two particles. In each ring, the dynamics is perfectly deterministic: the two balls keep colliding with each other, back and forth, moving toward each other with ever decreasing velocities. For each one ring, after $n$ collisions, the velocities, which we call $u_n$ and $v_n$, are obtained by repeated application of the collision rule (3),

$$\begin{aligned} u_n &= \frac{\mu^{2n} + (-\mu)^n}{2} u_0 + \frac{\mu^{2n} - (-\mu)^n}{2} v_0, \\ v_n &= \frac{\mu^{2n} - (-\mu)^n}{2} u_0 + \frac{\mu^{2n} + (-\mu)^n}{2} v_0, \end{aligned} \tag{5}$$

where $u_0$ and $v_0$ are the initial velocities. Note that each of these velocities alternates from positive to negative as the particles move in one direction and then another in the ring. Since the grains are in a ring, the distance they have to travel between two collisions is $l$. The travel time is $l/|u-v|$, where $u$ and $v$ are their current velocities between collisions. Hence, the time that has elapsed by the $n^{th}$ collision is

$$t_n = \frac{l}{|u_0 - v_0|} \sum_{k=0}^{n-1} \mu^{-k} = \frac{l}{|u_0 - v_0|} \frac{\frac{1}{\mu^n} - 1}{\frac{1}{\mu} - 1} \tag{6}$$

The sign alternation of the velocities in Eq. (5) is not important in the effort to understand the long time behavior of the distribution. We can approximate the velocities by envelope functions $A_+(t)$ and $A_-(t)$ and write $u_n = A_+(t)u_0 + A_-(t)v_0$, $v_n = A_-(t)u_0 + A_+(t)v_0$. The envelope functions can be found by solving Eq. (6) for $\mu^n$, substituting this into Eq. (5) ignoring the minus signs in the $(-\mu)^n$ factors, and setting $t_n \equiv t$. One finds

$$\begin{aligned} A_\pm(t) &= \frac{1}{2}\left[1 + \frac{|u_0 - v_0|}{l}\left(\frac{1}{\mu} - 1\right)t\right]^{-2} \\ &\pm \frac{1}{2}\left[1 + \frac{|u_0 - v_0|}{l}\left(\frac{1}{\mu} - 1\right)t\right]^{-1}. \end{aligned} \tag{7}$$

Hence, if initially the particles had a velocity distribution $P(v,0)$, then the velocity distribution as a function of time is

$$P(v,t) = \iint \delta(v - A_+(t)u_0 - A_-(t)v_0) \times P(u_0,0)P(v_0,0)du_0dv_0. \quad (8)$$

Asymptotically,

$$A_+(t) = -A_-(t) \sim \frac{1}{2}\frac{l}{|u_0-v_0|t}\left(\frac{1}{\mu}-1\right)^{-1}, \quad (9)$$

so that

$$P(v,t) = \iint \delta\left(v - \frac{1}{2}\frac{l}{t}\left(\frac{1}{\mu}-1\right)^{-1}\mathrm{sgn}(u_0-v_0)\right) \times P(u_0,0)P(v_0,0)du_0dv_0. \quad (10)$$

Therefore, $P(u,t)$ consists of two $\delta$-peaks, one at positive and one at negative velocities. If the initial distribution is symmetric about zero velocity and is properly normalized, then

$$P(v,t) \sim \frac{1}{2}\delta\left(v - \frac{l}{2t}\left(\frac{1}{\mu}-1\right)^{-1}\right) + \frac{1}{2}\delta\left(v + \frac{l}{2t}\left(\frac{1}{\mu}-1\right)^{-1}\right). \quad (11)$$

The peaks thus move toward the final velocity $v = 0$ as $1/t$. This time dependence is in agreement with the Boltzmann equation analysis.

The two $\delta$-peaks here reflect the fact that the magnitude of the velocity difference between the colliding particles eventually becomes independent of the magnitude of the initial velocity difference. Clearly, in the Boltzmann equation this is not quite the case and the peaks have a finite width as they converge. However, this width decreases in time as $1/t$, approaching the behavior of the two-particle ring model asymptotically.

In order to assess the validity of the Boltzmann equation for the viscous granule problem, and to get a sense of the possible effects of the spatial distribution of granules ignored in that formulation, we have carried out numerical simulations of a full chain of 10 000 viscous particles. Our collision rules are as indicated in Eq. (3), and, as before, we assume collisions to be instantaneous, but now we actually place the granules on a line and keep track of their positions so that spatial inhomogeneities can occur if the system is so inclined. The particle density is $10^{-3}$ with the particles initially distributed uniformly. The initial velocity distribution is taken to be a symmetric exponential, and $\gamma = 0.9$. As time proceeds, the initial single peak splits into two peaks which move inward and become narrower. Both the inward motion of the peaks and the width of the peaks change as $1/t$, as in the Boltzmann approximation. We tested the scaling hypothesis Eq. (4) on our simulations, and in Fig. 3 we show the results averaged over 100 simulations. Clearly, the scaling works quite well.

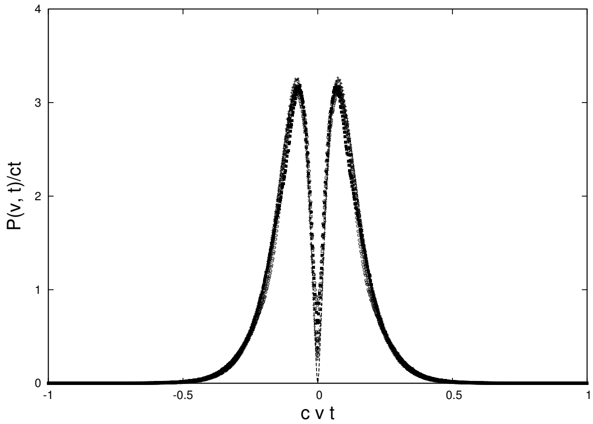


FIG. 3: Asymptotic behavior of the scaled velocity distribution. Different symbols stand for different times ($1000 \times 2^n$ with $n = 7, 8, \cdots, 13$). The constant $c$ in the figure is arbitrary.

Comparing the velocity distribution for the Boltzmann approximation (Fig. 1) and the simulations in a line, we find that the Boltzmann model converges to the scaling distribution much more rapidly. Initially the simulation results exhibit the same behavior as the Boltzmann model, but for longer times the behavior of the two distributions for small velocities begin to differ. In the Boltzmann case, the probability of finding a slow granule is much greater than in the simulation.

We conclude from these results that the Boltzmann problem in which spatial dependences are disregarded captures many of the essential features of the velocity relaxation in a viscous granular chain, the most important being the appearance of two peaks in the velocity distribution. The Boltzmann problem and even the simpler two-particle simplification of the problem also capture the time dependence of convergence of the two peaks into a single one at zero velocity. The slow $(1/t)$ convergence is due to the fact that the collision rate slows down as the gas cools. The Boltzmann approximation does not correctly capture the late time distribution of the slowest particles. This is probably due to spatial correlations that have been ignored in this approximation and that are currently under investigation [33].

As mentioned earlier, one-dimensional momentum-conserving granular gases may exhibit "inelastic collapse," whereby the energy of the gas goes to zero in a finite time [14–16]. To avoid this, the "quasi-elastic limit" or other ad hoc assumptions are frequently invoked, and in this limit, a double-peaked velocity distribution is also observed. In the presence of friction there is no inelastic collapse, and we always observe a double-peaked distribution. A comparison between those results and ours requires an understanding of spatial correlations [33].

Finally, we point out again that even in our most complete simulations we have approximated the collision events as instantaneous. While we do not believe this to be a perceptible source of error, it would be interesting (but extremely time consuming) to carry our full simulations of the model Eq. (1) with no further approximations.

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