Geographical Embedding of Scale-Free Networks

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Abstract

A method for embedding graphs in Euclidean space is suggested. The method connects nodes to their geographically closest neighbors and economizes on the total physical length of links. The topological and geometrical properties of scale-free networks embedded by the suggested algorithm are studied both analytically and through simulations. Our findings indicate dramatic changes in the embedded networks, in comparison to their off-lattice counterparts, and call into question the applicability of off-lattice scale-free models to realistic, everyday-life networks.

Key words: Internet, scale-free, networks, embedding, lattice
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1. Introduction

The Internet and the World Wide Web (WWW), the electricity power grid, networks of flight connections, of social contacts, and neuronal networks of the brain are few of the many examples of networks that surround us and that may be usefully described as graphs [1–5].

Graph theory is rooted in the 18th century, beginning with the work of Euler. Early efforts focused on properties of special (and usually small) graphs. In the 1960s, Paul Erdős and Alfréd Rényi [6–8] initiated the study of *random graphs*,

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known also as ER graphs. The unlimited size and randomness of ER graphs made them natural contenders for models of large networks encountered in everyday life.

In 1967 the psychologist Stanley Milgram asked himself how many acquaintances, on average, connect between a person on the East coast and another on the West coast of the USA. Following some research he finally concluded that the number of acquaintances is surprisingly small, just about six [9]. This celebrated "six degrees of separation" achieved widespread popularity. It was later realized that an average short path, typically of order $\ln N$ (N being the number of nodes, or the *size* of the network), connects between randomly selected nodes in most naturally occurring networks. Recognizing Milgram's contribution the effect came to be known as the *small world* property of networks. ER graphs possess the small world property.

However, an important ingredient was still missing. In networks of social contact the people that are connected to a certain individual (such as family, acquaintances at work, and friends) are highly likely to be connected among themselves. This high degree of *clustering* is absent in ER graphs. *Small World Networks*, introduced by Watts and Strogatz [11,10], exhibit high degrees of clustering as well as the small world property. They mark an important stage in the modeling of everyday networks and are studied in their own right.

It was recently realized that in addition to a high clustering index and the small world property real life networks exhibit yet a third important characteristic: a *scale-free* degree distribution. The *degree* of node i is the number of links, k_i, connected to the node. The likely degree of the various nodes in most observed networks follows a power-law distribution:

$$P(k) = ck^{-\lambda} , \qquad m \leq k \leq K , \tag{1}$$

where c is an appropriate normalization factor, m is the minimal degree of any given node, and the cutoff degree K depends on the size of the network, $K \sim mN^{1/(\lambda-1)}$ [12,13]. The term *scale-free* refers to the fact that the moments $\langle k^n \rangle$ for $n \geq \lfloor \lambda \rfloor$ do not exist(diverge), in the limit of $N \to \infty$. Scale-free networks possess the small world property, and it is possible to construct them with a high degree of clustering. Many important examples follow this pattern, including the Internet [14,15] and the WWW (both for incoming and outgoing links) [16,17], social networks, and virtually any large network arising in some natural context. See [3,4], for excellent reviews, and [5], for a timely anthology.

An oft-neglected aspect in the modeling of everyday-life networks is the fact that they are embedded in physical (Euclidean) space and possess a *geography*, in addition to their *topology*. The spatial location of nodes and the length of the connecting links is never a consideration in the models of networks dis-

cussed above. Nevertheless, routers of the Internet and social networks lay on the two-dimensional surface of the globe; neuronal networks in brains occupy three-dimensional space, etc. The likelihood of connections between the nodes in such networks is certainly affected by their geographical proximity, and one expects nontrivial consequences arising from this interplay between geography and topology [18]. Yet, most studies so far have focused on topological networks, where the nodes and links exist in some abstract space, devoid of metric.

In this paper we consider the embedding of networks in Euclidean space. In Section 1, we discuss general aspects of embedding and introduce a specific algorithm for the embedding of random networks of arbitrary degree distribution. This algorithm favors connections to nearest nodes and economizes on the total length of the links. We show that infinite networks can be thus embedded if and only if the degree distribution has compact support, that is, provided that there exists a sharp upper cutoff $K < \infty$ for the degree of any given node. If the distribution has no compact support, then embedding introduces an artificial cutoff and is only possible in a restricted sense. In Section 2 we apply the embedding algorithm to the widespread case of scale-free networks and study key structural properties of the resulting lattices. We conclude with a discussion of open work and a comparison to an interesting related algorithm, proposed by Warren, Sander, and Sokolov [19], in Section 3.

1 The embedding algorithm

Our general problem is that of embedding a given infinite graph in Euclidean space. Suppose first that the graph contains no cycles (loops), i.e., it is a tree. Let the volume of a node be v. The total volume of nodes within chemical shell l (nodes up to l links away from a given node) increases as $\langle k\rangle^l v$, where $\langle k\rangle = \int kP(k)\,dk$ is the average degree of the nodes. On the other hand, the volume enclosed within a Euclidean distance r scales as r^d (in d-dimensions). Unless the links become progressively long, the exponential growth in l cannot be sustained within the much slower algebraic growth in r. If the network is to be statistically homogeneous, a lengthening of successive links is unacceptable. We conclude that infinite trees, such as the Cayley tree (Fig 1a), are not embeddable in Euclidean space [21]. Allowing for loops, the volume constraint is reduced to the extent that embedding might be possible while retaining homogeneity, and without a change in the degree distribution. An example is shown in Fig 1b.

In order to make further headway we restrict ourselves to *random* graphs, of arbitrary degree distribution $P(k)$, and the following embedding algorithm [22]. To each site i of a d-dimensional lattice assign a random connectivity k_i, taken

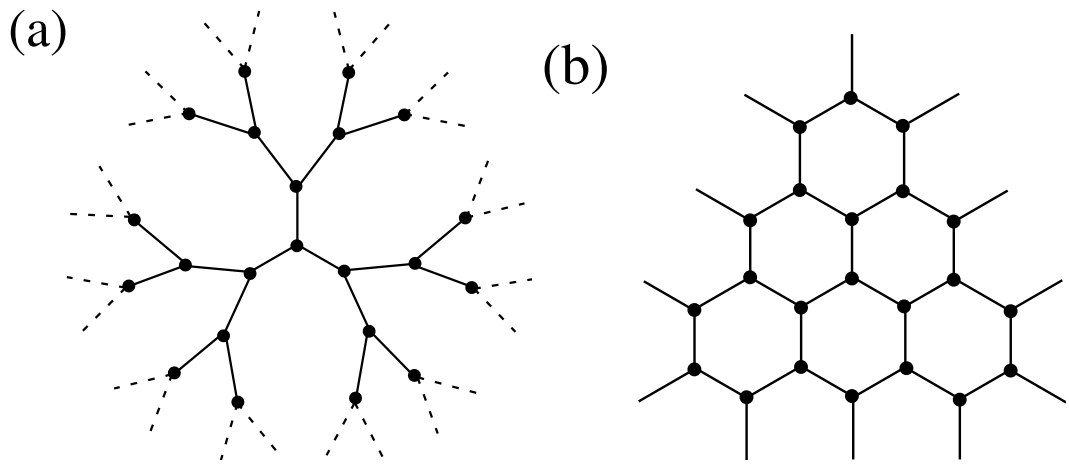

Fig. 1. Embedding problem. The Cayley tree (a) is a regular graph (all nodes have the *same* connectivity, $k = 3$) without loops. It cannot be embedded in Euclidean space, unless one allows for loops: The hexagonal lattice (b) is one example of embedding achieved in this way. Notice that the degree distribution remains unchanged.

from the degree distribution $P(k)$. Select site j at random and connect it to its closest neighbors, until its preassigned connectivity k_j is realized, or until all sites within distance

$$r(k_j) = Ak_j^{1/d} \tag{2}$$

have been explored. A link to site l is allowed provided that: (a) the site is not saturated (its connectivity has not yet reached the preassigned k_l), and (b) its distance from site j is smaller than $Ak_l^{1/d}$. Repeat this procedure for all sites. This algorithm makes sense in the context of social networks, where connections are typically confined to one's immediate neighborhood. The implied economy in the physical length of links might render the algorithm useful for the modeling of other networks of interest.

Suppose that one attempts to embed an infinite network, of degree distribution $P(k)$, in an infinite lattice, by the above algorithm. Nodes with a connectivity larger than a certain cutoff $k_c(A)$ cannot be realized, because of the possible saturation of surrounding sites. Consider the number of links $n(r)$ entering a generic node from a surrounding neighborhood of radius r. Sites at distance r' are linked to the node with probability

$$Pr(k > (r'/A)^d) = \int_{(r'/A)^d}^{\infty} P(k')\, dk' \ .$$

Thus,

$$n(r) = \int_0^r dr'\, S_d r'^{d-1} \int_{(r'/A)^d}^{\infty} dk'\, P(k') \ ,$$

where S_d is the surface area of the d-dimensional unit sphere. Reversing the order of integration and carrying out the spatial integral, we obtain

$$
\begin{aligned}
n(r) &= \int_0^{(r/A)^d} dk\, P(k) \int_0^{Ak^{1/d}} dr'\, S_d r'^{d-1} \\
&\quad + \int_{(r/A)^d}^{\infty} dk\, P(k) \int_0^{r} dr'\, S_d r'^{d-1} \\
&= V_d r^d \Big\{ \Big(\frac{A}{r}\Big)^d \int_0^{(r/A)^d} kP(k)\, dk + \int_{(r/A)^d}^{\infty} P(k)\, dk \Big\} \,,
\end{aligned}
$$

where $V_d = \frac{1}{d} S_d$ is the volume of the d-dimensional unit sphere. Finally, taking the limit $r \to \infty$, we derive the cutoff

$$
k_{\rm c} = \lim_{r\to\infty} n(r) = V_d A^d \langle k \rangle \,. \tag{3}
$$

We distinguish between two cases for the degree distribution $P(k)$: with and without compact support. The distribution $P(k)$ has compact support if there exists a $K < \infty$ such that $P(k) = 0$ for all $k > K$. All finite networks have a distribution with compact support, however infinite networks might too have compact support (an example is the Cayley tree of Fig. 1a). If this is the case, then $\langle k \rangle \leq K$ is finite, and one can always select A large enough so that $k_{\rm c} > K$. In other words, *if there is compact support the network is embeddable.*

If the network is infinite and without compact support, then if $\langle k \rangle < \infty$, for any finite A the cutoff $k_{\rm c}$ is finite and the tail of the distribution for $k > k_{\rm c}$ is chopped off. (Infinite networks with diverging $\langle k \rangle$ are pathological— we are not aware of any important practical application — though they might be embeddable.) The cutoff connectivity $k_{\rm c}$ implies a cutoff length

$$
\xi = r(k_{\rm c}) = (V_d \langle k \rangle)^{1/d} A^2 \,. \tag{4}
$$

The embedded network displays the original (chopped) distribution up to length scale ξ and repeats, statistically, at length scales larger than ξ. Indeed, ξ is analogous to the correlation length in percolation theory, above criticality, where the infinite cluster is *fractal* for $r < \xi$ and is homogeneous for $r > \xi$ [23–25]. We emphasize that the cutoff $k_{\rm c}$ is of consequence even if the distribution $P(k)$ is narrow, that is, even if $\int_{k_{\rm c}}^{\infty} P(k)\, dk \ll 1$. Indeed, even in such a case the length scale ξ is finite (and controlled mainly by A).

In summary, finite networks (all practical situations) are always embeddable by our proposed algorithm. Infinite networks are strictly embeddable only if their degree distribution has compact support. Otherwise, they are embeddable in a restricted manner, with a cutoff $k_{\rm c}$ imposed by the embedding, and statistical repetition at length scales $r > \xi \sim A^2$.

2 Embedding of scale-free networks

We now apply the embedding algorithm to the widespread case of scale-free networks, Eq. (1). Generally, we consider embedding in d-dimensional lattices of size R, though in the numerical simulations shown below we limit ourselves to two-dimensional square lattices with periodic boundary conditions. Because the lattice has a finite number of sites, $N \sim R^d$, the degree distribution (1) has compact support [12,13]:

$$K \sim mN^{1/(\lambda-1)} \sim R^{d/(\lambda-1)} \ . \tag{5}$$

This, in conjunction with (2), implies a natural cutoff length

$$r_{\max} = r(K) \sim AR^{1/(\lambda-1)} \ . \tag{6}$$

The interplay between the three length scales, R, ξ, $r_{\max}$, determines the nature of the network. The embedding cutoff k_c is imposed only if $\xi < r_{\max}$, and there is statistical repetition for $r > \xi$. Otherwise ($r_{\max} < \xi$), the natural cutoff of K is attained. As long as $\min(r_{\max}, \xi) \ll R$, the finite size of the lattice imposes no serious restrictions. Otherwise ($\min(r_{\max}, \xi) \gtrsim R$), finite-size effects become important. In short, there exist 6 different regimes, characterized by the relative ordering of R, ξ, and $r_{\max}$: Regime A: $r_{\max} < R < \xi$. Natural cutoff K is attained, and no finite-size effects. Statistical repetition occurs at length scales $r > r_{\max}$, rather than ξ. Regime B: $r_{\max} < \xi < R$. Same as regime A, but statistical repetition occurs at length scales $r > \xi$. Regime C: $\xi < r_{\max} < R$. Cutoff k_c is imposed; no finite-size effects. Regime D: $\xi < R < r_{\max}$. Same as regime C. Regime E: $R < \xi < r_{\max}$. Cutoff k_c is imposed; strong finite-size effects prevent statistical repetition. Regime F: $R < r_{\max} < \xi$. Natural cutoff K is attained; strong finite-size effects. The various regimes are demarcated by the lines

$$\frac{\ln A}{\ln R} = \begin{cases} \frac{1}{2} + \frac{\lambda-3}{2(\lambda-1)}, & r_{\max} = R, \\ \frac{1}{2}, & \xi = R, \\ \frac{1}{2} - \frac{\lambda-3}{2(\lambda-1)}, & \xi = r_{\max}, \end{cases} \tag{7}$$

as shown in Fig. 2. In Fig. 3a, we present typical networks that result from our embedding method, for $\lambda = 2.5$ and $\lambda = 5$. Long range links become more noticeable as λ decreases. In Fig. 3b, we show the same networks as in Fig. 3a, where successive chemical shells are shaded differently. Chemical shell l consists of all sites that can be reached by a minimal number of l connecting

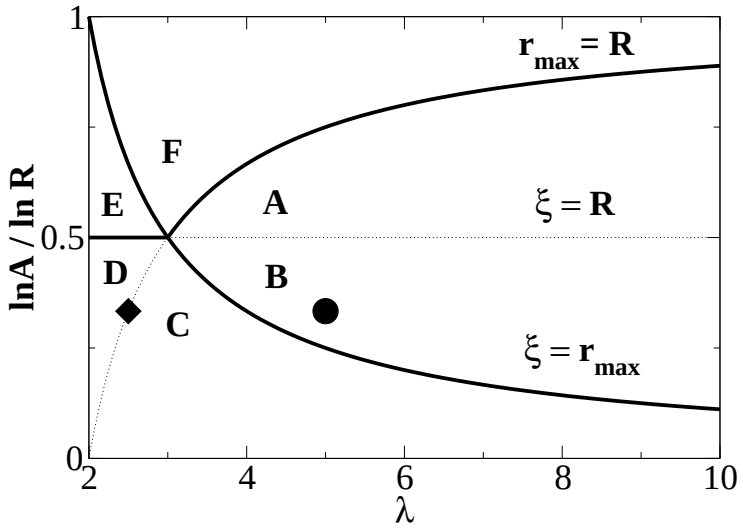

Fig. 2. Regimes of embedded scale-free networks. A: $r_{max} < R < \xi$, B: $r_{max} < \xi < R$, C: $\xi < r_{max} < R$, D: $\xi < R < r_{max}$,E: $R < \xi < r_{max}$, F: $R < r_{max} < \xi$. The diagram can be reduced into just four regimes separated by the cutoff k_c and by finite-size effects. A and B: no cutoff and no size effect; C and D: cutoff and no size effect; E: cutoff and size effect; F: no cutoff but size effect. The two symbols indicate the parameters corresponding to Fig. 3b, $\lambda = 2.5$ (diamond) and $\lambda = 5$ (circle).

links relative to a given site (the central site, in the figure). For our choice of parameters, $\lambda = 5$ falls in the region of $\xi > r_{\max}$, while for $\lambda = 2.5$, $\xi < r_{\max}$. In the latter case, we clearly see the statistical repetition beyond the length scale ξ (Fig. 3b, $\lambda = 2.5$).

The degree distribution of the embedded networks is illustrated in Fig. 4. In Fig. 4a, various values of A were selected such that $\xi < r_{\max}$, and the distribution terminates at the imposed cutoff $k_{\rm c}$. The scale-free distribution is altered slightly, for $k < k_{\rm c}$, due to saturation effects, but the overall trend is highly consistent with the original power-law. The data-collapse shown in the inset confirms the power-law, as well as the scaling of the cutoff $k_{\rm c} \sim A^d$. In Fig. 4b, $A = 10$ was fixed and values of λ were selected such that $\xi > r_{\max}$. The natural cutoff K is now attained, and the data collapse at the inset confirms the power-law distribution as well as the known relation $K \sim mR^{d/(\lambda-1)}$.

We now turn to the relation between the Euclidean metric and chemical length in our embedded scale-free networks. The chemical distance l between two sites is the minimal number of links l connecting the two. Thus, if the Euclidean distance between the sites is r, then

$$l \sim r^{d_{\rm min}} \tag{8}$$

defines the minimal (chemical) length exponent $d_{\rm min}$. In order to compute $d_{\rm min}$

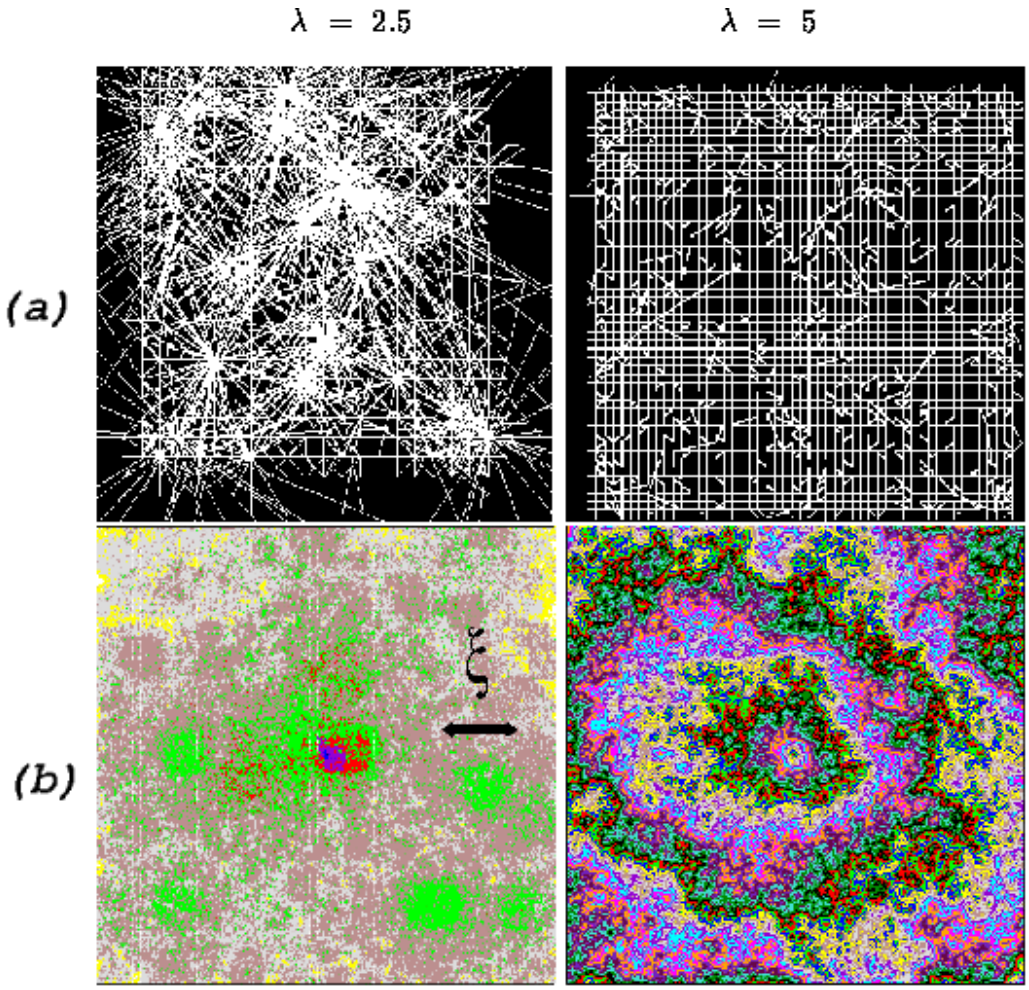

Fig. 3. Embedded scale-free networks. (a) Actual networks (links are highlighted) for graphs with $\lambda = 2.5$ and 5, embedded in 50×50-sites square lattices. (b) Chemical shells, for the same parameters as in (a), but for lattices of size 300×300. Notice the statistical repetition at length scales greater thanξ, for $\lambda = 2.5$. In this figure and throughout this article the lower cutoff was taken to be $m = 4$. This guarantees that for large values of λ the network becomes a regular square lattice.

we regard the chemical shells as being roughly smooth, at least in the regime $\xi > r_{\max}$, as suggested by Fig. 3b($\lambda = 5$). Let the width of shell l be $\Delta r(l)$, then

$$l = \int dl = \int \frac{dr}{\Delta r(l)} \sim r^{d_{\min}} \ , \tag{9}$$

since $\Delta l = 1$. The number of sites in shell l, $N(l)$, is, on the one hand, $N(l) \sim r(l)^{d-1}\Delta r(l)$. On the other hand, since the maximal connectivity in shell l is $K(l) \sim N(l)^{1/(\lambda-1)}$, the thickness of shell $(l+1)$ is $\Delta r(l+1) \sim r(K(l)) \sim AK(l)^{1/d}$. Assuming (for large l) that $\Delta r(l+1) \sim \Delta r(l)$, we obtain

$$\Delta r(l) \sim r^{(d-1)/[d(\lambda-1)-1]} \ . \tag{10}$$

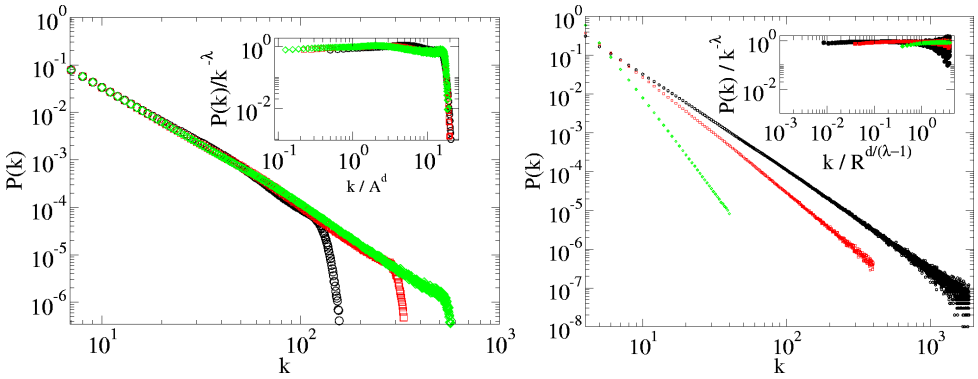

Fig. 4. Degree distribution of embedded scale-free networks, (a) when the cutoff k_c is imposed, and (b) when the natural cutoff K is achieved. For (a) the lattice size is $R = 400$, the distribution exponent is $\lambda = 2.5$, and $A = 2$ (circles), 3 (squares), 4 (diamonds). The inset confirms the scaling of Eq. (3). For (b) $R = 100$, $A = 10$, and $\lambda = 2.5$ (circles), 3.0 (squares),5.0 (diamonds). The inset confirms the scaling of Eq. (5).

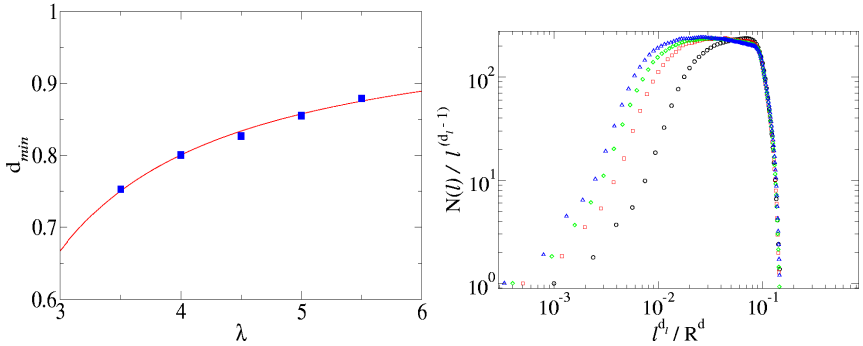

Fig. 5. Minimal length exponent $d_{\min}$. (a) $d_{\min}$ vs. λ. The analytic result of Eq. (11) (curve) matches nicely the results measured from simulations(squares). (b) Data collapse for the scaling function $\Phi(l^{d_l}/R^d)$, for $\lambda = 4$ and lattice sizes $R = 1000$ (circles), 2000 (squares), 2500 (diamonds) and 3000 (triangles).

Using this expression in (9) yields

$$d_{\min} = \frac{\lambda - 2}{\lambda - 1 - 1/d} \ . \tag{11}$$

Thus, for $d > 1$, the dimension $d_{\min} < 1$. This result is opposite to all known disordered media, where $d_{\min} \geq 1$. (A particularly simple example is provided by a polymer chain, or its self-avoiding-walk model, where the end-to-end distance is well-approximated by the Flory relation: $r \sim l^{3/(d+2)}$, i.e.,$d_{\min} = (d+2)/3$.) We have also computed the fractal dimension of the networks and found that the interior of the l-clusters is compact, with $d_f = d$ [22]. (Fig. 3 suggests that the hull of the clusters is fractal.) This result was confirmed by simulations. It then follows that the fractal dimension of the network in chemical space is also anomalous: $d_l = d_f/d_{\min} = d/d_{\min} > d$ (for $d > 1$).

In Fig. 5a, we plot $d_{\min}$ as measured from simulations, and compared with the analytical result of Eq. (11). The scaling suggested in Fig. 5b, $N(l) \sim l^{d_l-1}\Phi(l^{d_l}/R^d)$, is valid only for $\xi > r_{\max}$. For $R \to \infty$, we expect that the network is scale-free up to length scale ξ and the analogous scaling would be $N(l) \sim l^{d_l-1}\Psi(l^{d_l}/\xi^d)$, where $\Psi(x \gg 1) \sim x^{(d-d_l)/d_l}$. It remains an open challenge to conceive of a general scaling relation that would encompass all of the regimes A – F.

3 Discussion

In summary, we propose an algorithm for the embedding of networks in Euclidean space. Finite networks can always be embedded in this way. Infinite networks are strictly embeddable only if the degree distribution has compact support. If not, an artificial cutoff k_c results from our embedding technique, and the network repeats itself, in a statistical sense, at length scales greater than $\xi(k_c)$. We have applied our embedding algorithm to scale-free graphs, and studied the resulting networks.

Concurrently with our work (and independently), Warren, Sander and Sokolov [19] proposed an embedding algorithm very similar to ours. The main difference in their approach is that connecting to saturated sites is allowed, and a node is connected to as many of its closest neighbors as necessary, until its target connectivity is fulfilled. In this case, our computation of k_c in Section 1 is still valid, only that now it is a *lower* cutoff for the connectivity of the sites. On the other hand, there is no restriction, in this version of embedding, on the upper cutoff of the distribution. We anticipate that ξ here plays the role of a 'geometrical' length scale, similar to the lattice spacing for a critical percolation cluster grown on a lattice. (The cluster is self-similar only at length scales larger than the lattice spacing.)

Embedding scale-free networks in Euclidean space results in some dramatic changes from the original graphs. Consider, for example, the small world property, common to off-lattice scale-free networks grown by any of the known techniques. The small world property disappears at length scales greater than ξ: two sites separated by a physical distance $r = n\xi$, $n \gg 1$, would be connected by at least n intervening links, due to the statistical repetition. The number of sites within radius r is $N \sim n^d\xi^d$, so $l \sim N^{1/d}$. Not surprisingly, this is the same scaling as for Euclidean d-dimensional lattices. For distances $r < \xi$, we have $l \sim r^{d_{\min}} \sim N^{d_{\min}/d}$, which represents small-world behavior, since it improves on the Euclidean $l \sim N^{1/d}$, but a far cry from the off-lattice random graph $l \sim \ln N$ or from the scale free with $\lambda < 3$ where $l \sim loglogN$ [20]. Another example of a dramatic change is the qualitative difference between percolation in off-lattice and embedded networks. Indeed, Warren et al., [19]

find that the percolation transition takes place in embedded lattices even for $\lambda < 3$, in contrast to off-lattice scale-free networks [3,12]. In view of the fact that many everyday-life networks are embedded in physical space, we ought to reconsider how the topological properties of scale-free graphs — properties that we normally attribute to the physical networks — are affected by the embedding.

Acknowledgments

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[21] An amusing example is provided by star-burst dendrimers (SBD) —star-shaped polymers that grow in the pattern of the Cayley tree of Fig. 1a. The SBDs typically cannot grow beyond chemical shell $l = 6$, due to lack of space. See, for example, D. ben-Avraham, L. S. Schulman, S. H. Bossmann, C. Turro, and N. J. Turro, *J. Phys. Chem. B* **102**, 5088 (1998).

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