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Boundaries of Weak Peak Points in Noncommutative Algebras of Lipschitz Functions

Research

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Abstract: It has been shown that any Banach algebra satisfying $\|f^2\| = \|f\|^2$ has a representation as an algebra of quaternion-valued continuous functions. Whereas some of the classical theory of algebras of continuous complex-valued functions extends immediately to algebras of quaternion-valued functions, similar work has not been done to analyze how the theory of algebras of complex-valued Lipschitz functions extends to algebras of quaternion-valued Lipschitz functions. Denote by $\mathrm{Lip}(X, \mathbb{F})$ the algebra over $\mathbb{R}$ of $\mathbb{F}$-valued Lipschitz functions on a compact metric space (X, d), where $\mathbb{F} = \mathbb{R}, \mathbb{C}$, or $\mathbb{H}$, the non-commutative division ring of quaternions. In this work, we analyze a class of subalgebras of $\mathrm{Lip}(X, \mathbb{F})$ in which the closure of the weak peak points is the Shilov boundary, and we show that algebras of functions taking values in the quaternions are the most general objects to which the theory of weak peak points extends naturally. This is done by generalizing a classical result for uniform algebras, due to Bishop, which ensures the existence of the Shilov boundary. While the result of Bishop need not hold in general algebras of quaternion-valued Lipschitz functions, we give sufficient conditions on such an algebra for it to hold and to guarantee the existence of the Shilov boundary.

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1. Introduction

Let X be a compact Hausdorff space and $C(X,\mathbb{F})$ the space (over $\mathbb{R}$ or $\mathbb{C}$) of continuous $\mathbb{F}$-valued functions on X, where $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$ – the non-commutative division ring of quaternions. Given a subalgebra $\mathcal{A} \subset C(X,\mathbb{F})$, we are interested in classifying *boundaries* for $\mathcal{A}$, subsets $B \subset X$ such that every $f \in \mathcal{A}$ attains its maximum modulus on B. In the case that $\mathbb{F} = \mathbb{C}$ and $\mathcal{A}$ is uniformly closed with complex scalars, it is well-known that there exists a minimal closed boundary with respect to inclusion, the so-called *Shilov boundary* [15]. In fact, it has been shown that the Shilov boundary exists for any point-separating subalgebra of $C(X,\mathbb{C})$ with complex scalars, regardless of whether it is uniformly complete or contains the constant functions [11, Theorem 3.3.2].

For uniform algebras – uniformly complete subalgebras of $C(X,\mathbb{C})$ that contain the constant functions and separate points in the sense that for each $x, y \in X$ there exists a function $f \in \mathcal{A}$ with $f(x) \neq f(y)$ – there are direct methods for showing the Shilov boundary exists. For example, Bear [4] provided a topological approach that can be generalized when points in the underlying domain have certain properties. A point $x_0 \in X$ is a *peak point* for $\mathcal{A} \subset C(X,\mathbb{C})$ if there exists $f \in \mathcal{A}$ such that $|f(x)| < |f(x_0)|$ for all $x \in X \setminus \{x_0\}$. Since f attains its maximum modulus exclusively at x_0, the point must be an element of every boundary of $\mathcal{A}$. The *maximizing set* of f is the set $M(f) = \{x \in X : |f(x)| = \|f\|_\infty\}$, and x_0 being a peak point is just to say that there exists $f \in \mathcal{A}$ such that $M(f) = \{x_0\}$. We are also interested in the actual values f attains on $M(f)$; the *peripheral range* of f is the set of range values of maximum modulus and is denoted by

$$\mathrm{Ran}_\pi(f) = \{\lambda \in \mathrm{Ran}(f) : |\lambda| = \|f\|_\infty\} = f(M(f)).$$

In general, uniform algebras need not have any peak points. A *weak peak point* – also called a *p-point* or a *strong boundary point* – is a point x_0 such that $\{x_0\} = \cap_{f\in S} M(f)$ for some family of functions $S \subset \mathcal{A}$. It is well-known that, for a uniform algebra $\mathcal{A}$, the collection of weak peak points is the Choquet boundary and that the closure of the Choquet boundary is the Shilov boundary. Algebras of continuous functions over $\mathbb{R}$, however, need not have a minimal closed boundary, even algebras that contain the (real) constant functions, separate points, and are uniformly complete. Since the boundary behavior for algebras of continuous functions over $\mathbb{R}$ is substantially different from the boundary theory for algebras of continuous functions over $\mathbb{C}$, it is important to find sufficient conditions that will ensure the existence of a minimal closed boundary for such algebras.

While there is an extensive literature on algebras of continuous functions over $\mathbb{C}$, there are far fewer results on algebras of continuous functions over $\mathbb{R}$ (with [12] being one of the few books on the subject), and fewer still on algebras of quaternion-valued functions.

In [9], Jarosz defines a uniform algebra $\mathcal{A}$ as any Banach algebra satisfying the property $\|f^2\| = \|f\|^2$ for all $f \in \mathcal{A}$. In the case that $\mathcal{A}$ is an algebra over $\mathbb{C}$, it can be shown that this norm condition automatically implies that $\mathcal{A}$ is commutative and has a classical representation as a uniform (function) algebra. In the case that $\mathcal{A}$ is an algebra over $\mathbb{R}$, it cannot be deduced that $\mathcal{A}$ is commutative, as the algebra of quaternions itself demonstrates.

Nonetheless, it is shown in [9] that for any such algebra there is a compact Hausdorff space X' with a decomposition $X' = X_1 \cup X_2 \cup X_3$ such that $\mathcal{A}$ is isometrically isomorphic to a subalgebra of $C(X_1, \mathbb{R}) \oplus C(X_2, \mathbb{C}) \oplus C(X_3, \mathbb{H})$. In particular, if $\mathcal{A} \subset C(X, \mathbb{H})$ for some compact Hausdorff space X, then the decomposition is natural in the sense that $X' \subset X$. This shows that every Banach algebra satisfying the norm condition $\|f^2\| = \|f\|^2$ has a representation as an algebra of quaternion-valued, continuous functions – which suggests that algebras of quaternion-valued functions are an important class of algebras to understand. Such real Banach algebras have also been recently studied by Albiac and Briem [2, 3].

The first goal of this work is to provide a new proof, using the function algebra representation given in [9], that the only associative normed algebras over $\mathbb{R}$ whose norm is multiplicative are $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. The proof, which is given in Section 2, demonstrates that results on the boundaries of spaces of complex-valued functions will naturally extend, at most, to spaces of quaternion-valued continuous functions, because the construction of boundaries via weak points requires that the range of the functions have a multiplicative norm.

The second primary goal of this work is to analyze boundaries of algebras of Lipschitz functions – with specific focus on $\mathbb{H}$-valued functions. Let (X, d) be a compact metric space, and set

$$\mathrm{Lip}(X, \mathbb{F}) = \left\{ f \in C(X, \mathbb{F}) : \sup_{x,y \in X,\, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} < \infty \right\}.$$

This subalgebra of $C(X, \mathbb{F})$ is complete with respect to the norm $\|f\|_{\mathrm{Lip}(X)} = \|f\|_\infty + L(f)$, where

$$L(f) = \sup_{x,y \in X,\, x \neq y} \frac{|f(x) - f(y)|}{d(x,y)}.$$

It is also common to norm this space by the equivalent norm $\|f\| = \max\{\|f\|_\infty, L(f)\}$. The value of the "sum" form is that $\|fg\|_{\mathrm{Lip}(X)} \leq \|f\|_{\mathrm{Lip}(X)} \|g\|_{\mathrm{Lip}(X)}$ for all $f, g \in \mathrm{Lip}(X, \mathbb{F})$, which is not the case for the "max" version of the norm.

In Section 3, we introduce the background required for the study of boundaries of function algebras and prove that some of the basic results on boundaries of weak peak points for complex-valued function algebras extend to algebras of $\mathbb{H}$-valued functions. One approach to characterizing boundaries of weak peak points in uniform algebras is to use a classical result due to Bishop [5, Theorem 2.4.1], and Section 4 contains results extending this to algebras of Lipschitz functions over $\mathbb{R}$, including algebras of quaternion-valued functions. In particular, we give a sufficient condition under which a subalgebra $\mathcal{A}$ of $\mathrm{Lip}(X, \mathbb{F})$ satisfies the results of Bishop – which ensures that the algebra has a Shilov boundary – by developing a new notion of local Lipschitz number for a function around a class of maximizing sets of other functions in the algebra. If the algebra $\mathcal{A}$ has sufficiently many functions with "corner"-like structures, then the algebra must have a Shilov boundary. This is analogous to the study of algebras of continuous functions, which are often analyzed under the condition that they contain sufficiently many "peaking functions."

2. The Generality of the Quaternions as the Range of Continuous Functions

When one generalizes from analyzing functions taking values in $\mathbb{C}$ to functions taking values in a larger set, it is desirable to extend the range to as large a set as is reasonable. Many of the results characterizing boundaries of function algebras via weak peak points require that the set in which functions take their values be a normed algebra with a multiplicative norm, i.e. satisfying $\|xy\| = \|x\|\|y\|$ for all x and y. It turns out that, for associative algebras over $\mathbb{R}$, the quaternions is the most general set with these properties. This fact has been known at least since the 1960's [18, Exercise 5.5.a] (perhaps much longer), but there are few proofs available in the literature. We give a new proof here based on quite recent results [1].

Proposition 1.
Suppose that $\mathcal{A}$ is an associative, unital, normed algebra over $\mathbb{R}$ such that $\|fg\| = \|f\|\,\|g\|$ for all $f, g \in \mathcal{A}$. Then $\mathcal{A}$ is a division algebra. In particular, $\mathcal{A}$ is isometrically algebra isomorphic to $\mathbb{R}, \mathbb{C}$, or $\mathbb{H}$.

Proof. By hypothesis, $\|f^2\| = \|f\|^2$ for all $f \in \mathcal{A}$, so, by Abel and Jarosz [1], there exists a compact Hausdorff space X such that $\mathcal{A}$ is isometrically isomorphic to a subalgebra $\hat{\mathcal{A}}$ of $C(X, \mathbb{H})$ (under the uniform norm). Therefore we identify $\mathcal{A}$ with its function algebra representation $\hat{\mathcal{A}}$ and $f \in \mathcal{A}$ with $\hat{f} \in \hat{\mathcal{A}}$. The norm is understood as the uniform norm on $\hat{\mathcal{A}}$.

Fix $f \in \mathcal{A}$, then $f(x) = a(x) + b(x)i + c(x)j + d(x)k$, with $a, b, c, d \in C(X, \mathbb{R})$. For any $r > 0$, note that $\|(f+r)(f-r)\|_\infty = \|f+r\|_\infty \|f-r\|_\infty$ implies that there exists $y_r \in M(f+r) \cap M(f-r)$. Given any $x_1, x_2 \in X$, we may assume that $a(x_1) \leq a(x_2)$, which gives

$$\begin{aligned}
0 \leq a(x_2) - a(x_1) &= \frac{(a(x_2)+r)^2 + (a(x_1)-r)^2 - 2r^2 - a(x_2)^2 - a(x_1)^2}{2r} \\
&\leq \frac{\|f+r\|^2 + \|f-r\|^2 - 2r^2 - a(x_2)^2 - a(x_1)^2}{2r} \leq \frac{\|f+r\|^2 + \|f-r\|^2 - 2r^2}{2r} \\
&= \frac{(a(y_r)+r)^2 + (a(y_r)-r)^2 + 2b(y_r)^2 + 2c(y_r)^2 + 2d(y_r)^2 - 2r^2}{2r} = \frac{|f(y_r)|^2}{r} \leq \frac{\|f\|^2}{r}.
\end{aligned}$$

Since this holds for all $r > 0$ and every $x_1, x_2 \in X$, it must be that $a(x_1) = a(x_2)$ for every $x_1, x_2 \in X$, i.e. the real part of f is constant.

Since $\mathcal{A}$ is unital, it contains the real constant functions. Thus the fact that $\operatorname{Re}(f)$ is constant implies $\operatorname{Re}(f) \in \mathcal{A}$. Therefore $\overline{f} = 2\operatorname{Re}(f) - f \in \mathcal{A}$, which implies $f\overline{f} = |f|^2 \in \mathcal{A}$. Since $f\overline{f}$ is real and $\mathcal{A}$ contains only functions with constant real part, $|f|^2$ is constant. If $f \not\equiv 0$, then $f^{-1} = \frac{1}{|f|^2}\overline{f} \in \mathcal{A}$, meaning $\mathcal{A}$ is a division algebra.

Fröbenius proved [6] that every finite-dimensional, associative, unital, normed division algebra over $\mathbb{R}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$. This was improved by Hurwitz [8], and the general theorem – which does not depend on the assumption of finite dimensions – can be found in Rickart [14, Theorem 1.7.6] and in Żelazko [18, Theorem 5.5]. Since $\mathcal{A}$ is a unital, normed, associative, division algebra, $\mathcal{A}$ is isomorphic to $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, completing the proof. □

Since, as will be seen in the sections that follow, the study of boundaries via weak peak points is only natural when the range of the functions being analyzed is a unital algebra over $\mathbb{R}$ with a multiplicative norm, this result shows that $\mathbb{H}$ is the most general set to which the results apply.

The primary algebras of interest in this work are algebras of Lipschitz functions. While the results in [9] demonstrating that Banach algebras satisfying $\|f^2\| = \|f\|^2$ are function algebras give the most comprehensive analysis of algebras of quaternion-valued functions, these results do not apply to algebras of Lipschitz functions, since the Lipschitz norm does not satisfy $\|f\|^2 = \|f^2\|$, except in trivial cases.

3. Boundaries of Function Algebras

Following arguments detailed in [13], it is possible to construct boundaries for an algebra $\mathcal{A} \subset C(X, \mathbb{F})$ from the maximizing sets of the functions in $\mathcal{A}$. An *m-set* is a nonempty set $E \subset X$ such that $E = \cap_{f\in S} M(f)$ for some family of functions $S \subset \mathcal{A}$, and a straightforward application of Zorn's Lemma – along with the compactness of the underlying domain – gives the existence of minimal m-sets with respect to inclusion. The collection of all minimal m-sets is denoted by $\mathcal{E}^0$, and we set $\delta\mathcal{A} = \bigcup_{E\in\mathcal{E}^0} E$. If a minimal m-set E is a singleton $\{x_0\}$, then $\{x_0\}$ is a weak peak point. If all the minimal m-sets are singletons, then $\delta\mathcal{A}$ is a boundary for $\mathcal{A}$ consisting exactly of weak peak points. The next result generalizes [13, Lemma 3.4], from whose proof it is immediate, and we note that it holds for any family of continuous $\mathbb{F}$-valued functions, regardless of algebraic structure.

Lemma 3.1.
Let $\mathcal{A} \subset C(X, \mathbb{F})$ be any family of functions. Then $\delta\mathcal{A}$ is a boundary for $\mathcal{A}$.

If $\mathcal{A}$ is a point-separating, uniformly closed subalgebra of $C(X, \mathbb{C})$ over $\mathbb{C}$ – where X is a compact *metric space* – then $\delta\mathcal{A}$ is the minimal boundary [7], but there are few other sufficient conditions on an algebra $\mathcal{A}$ ensuring that $\delta\mathcal{A}$ is minimal in any sense. Nonetheless, there are conditions on the m-sets of $\mathcal{A}$ guaranteeing that $\delta\mathcal{A}$ is contained in every closed boundary, which implies that the intersection of all closed boundaries is a boundary.

For algebras of complex-valued continuous functions over $\mathbb{C}$, it is not necessary to study weak peak points in order to guarantee that there exists a minimal closed boundary. It can be shown that every subalgebra of $C(X, \mathbb{C})$ over $\mathbb{C}$ that separates points has a minimal closed boundary [11, Theorem 3.3.2]. As the next example shows, this is not the case for algebras of continuous functions over $\mathbb{R}$.

Example 3.1.
Let $\mathbb{D}$ be the open unit disk in the complex plane, $\mathbb{D}' = \{z + 3 : z \in \mathbb{D}\}$, and $X = \overline{\mathbb{D}} \cup \overline{\mathbb{D}'}$. The disk algebra, $A(\mathbb{D})$ is the collection of functions that are continuous on $\overline{\mathbb{D}}$ and analytic on $\mathbb{D}$. We extend each $f \in A(\mathbb{D})$ to $\overline{\mathbb{D}'}$ by conjugation, i.e. given $f \in A(\mathbb{D})$, we set

$$\hat{f}(z) = \begin{cases} f(z) & z \in \overline{\mathbb{D}} \\ \overline{f(z-3)} & z \in \overline{\mathbb{D}'} \end{cases}, \qquad \text{and define} \qquad \mathcal{A} = \left\{ \hat{f} : f \in A(\mathbb{D}) \right\}.$$

It is clear that $\mathcal{A}$ is closed under addition and pointwise multiplication, since $A(\mathbb{D})$ is and conjugation is both additive and multiplicative. Moreover, $\mathcal{A}$ is closed under *real* scalar multiplication and contains the real constant functions. It does not contain the complex constant functions, nor is it closed under complex scalar multiplication. It is not difficult to show that $\mathcal{A}$ is uniformly complete, so $\mathcal{A}$ is a uniformly complete, point-separating algebra of continuous functions over $\mathbb{R}$.

Since each $\hat{f} \in \mathcal{A}$ is analytic on $\mathbb{D}$, the maximum modulus principle gives that the maximum modulus of $\hat{f}$ is attained on the boundary of $\mathbb{D}$, and it is clear that this same maximum modulus is attained on the boundary of $\mathbb{D}'$. Let $z \in \overline{\mathbb{D}}$ with $|z| = 1$, then $|\hat{f}(z)| = |\hat{f}(z+3)|$ for all $f \in \mathcal{A}$. The minimal m-sets of $\mathcal{A}$ therefore cannot be singletons; each contains at least the two points $\{z, z+3\}$. In fact, the minimal m-sets are exactly the pairs $\{z, z+3\}$, where $|z| = 1$. Since the unit circle centered at 0 and the unit circle centered at 3 are two disjoint closed boundaries for $\mathcal{A}$, the intersection of all closed boundaries is empty and thus clearly not a boundary.

This is the canonical example of a *real function algebra* in the sense of Kulkarni and Limaye [12], and a real uniform algebra in the sense of Jarosz [9].

This example shows that the boundary theory for algebras over $\mathbb{R}$ is significantly different from the boundary theory for algebras over $\mathbb{C}$. Given that our interest in this work is algebras of quaternion-valued functions – which are algebras over $\mathbb{R}$ – it is necessary to provide sufficient conditions under which the minimal closed boundary does exist, and the minimal m-sets can provide such a condition. The next theorem is proven in [13, Theorem 3.11] for algebras of complex-valued functions. The proof here applies to algebras of functions taking values in $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, and $\mathbb{H}$. We begin with two lemmas, which generalize and whose proofs follow from results in [13].

Lemma 3.2.
Suppose that $\mathcal{A} \subset C(X, \mathbb{F})$ is an algebra and that E is a minimal m-set for $\mathcal{A}$. If $f \in \mathcal{A}$ attains its global maximum modulus on E, then f is constant on E.

Lemma 3.3.
Suppose that $\mathcal{A} \subset C(X, \mathbb{F})$ is an algebra and that E is a minimal m-set for $\mathcal{A}$. Then for every open set U containing E there exists $h \in \mathcal{A}$ such that h is constant on E and $E \subset M(h) \subset U$.

We now proceed with the main result of this section.

Theorem 1.
Suppose that X is a compact Hausdorff space and that $\mathcal{A} \subset C(X, \mathbb{F})$ is a subalgebra whose minimal m-sets are singletons, then $\delta\mathcal{A}$ is contained in every closed boundary. In particular, the intersection of all closed boundaries is a closed boundary.

Proof. Let B be a closed subset of X, and suppose that there exists a minimal m-set E such that $E \cap B = \varnothing$. Then $E \subset X \setminus B$, which is open, and by Lemma 3.3 there exists $f \in \mathcal{A}$ such that $E \subset M(f) \subset X \setminus B$. Therefore $M(f) \cap B = \varnothing$, proving B is not a boundary for $\mathcal{A}$. Thus, if B is a closed boundary for $\mathcal{A}$, then $B \cap E \neq \varnothing$ for every minimal m-set E. Since the minimal m-sets are singletons, by assumption, we must have that $E \subset B$ for every minimal m-set E, i.e. $\delta\mathcal{A} \subset B$. Since $\delta\mathcal{A}$ is a boundary – by Lemma 3.1 – that is contained in every closed boundary, the closure of $\delta\mathcal{A}$ is the minimal closed boundary. $\square$

Therefore, for an algebra $\mathcal{A} \subset C(X, \mathbb{F})$, it is sufficient to show that the minimal m-sets are singletons in order that the Shilov boundary exist.

4. Boundaries for Algebras of Lipschitz Functions over $\mathbb{R}$

Constructing boundaries via m-sets is only useful if there is a systematic method for showing that the minimal m-sets are singletons. One technique for doing this in uniform algebras is to use a classical result, due to Bishop [5, Theorem 2.4.1]. The version given here is a generalization due to Yates [17]. Recall that $\mathrm{Ran}_\pi(f)$ is the peripheral range of f, i.e. the set of range values of maximum modulus.

Theorem 2 ([17], Lemma 3.4.3).
Let $\mathcal{A} \subset C(X, \mathbb{C})$ be a uniform algebra, $x_0 \in \delta\mathcal{A}$, and $f \in \mathcal{A}$ with $f(x_0) \neq 0$. Then there exists $h \in \mathcal{A}$ such that $x_0 \in M(h) \cap M(fh)$, $\mathrm{Ran}_\pi(h) = \{1\}$, and $\mathrm{Ran}_\pi(fh) = \{f(x_0)\}$.

The proof given by Bishop – and subsequently adapted and generalized in several directions – is unique to algebras complete with respect to the uniform norm. Nonetheless, the result carries in certain other algebras. Moreover, it is not necessary that the range be $\mathbb{C}$; the result also carries when $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{H}$. The particular algebras that we are most interested in are the algebras of Lipschitz functions on compact metric spaces.

Lemma 4.1.
If (X, d) is a compact metric space, $x_0 \in X$, and $f \in \mathrm{Lip}(X, \mathbb{F})$ with $f(x_0) \neq 0$, then there exists $h \in \mathrm{Lip}(X, \mathbb{F})$ such that $x_0 \in M(h) \cap M(fh)$, $\mathrm{Ran}_\pi(h) = \{1\}$, and $\mathrm{Ran}_\pi(fh) = \{f(x_0)\}$.

A proof of this result for complex-valued Lipschitz functions can be found in [10], and that proof applies to this case as well. Unfortunately, the result does not hold in subalgebras of $\mathrm{Lip}(X, \mathbb{F})$, even norm-complete subalgebras.

Example 4.1.
Let $\mathcal{A} = C^1([0,1], \mathbb{R})$ be the algebra of real-valued continuous functions on $[0,1]$ with continuous first derivative. This is a complete subalgebra of $\mathrm{Lip}([0,1], \mathbb{R})$ that contains the constant functions and separates points (cf. [16]). A straightforward application of the product rule shows that Lemma 4.1 does not hold in this algebra. Suppose that $x_0 \in M(h) \cap M(fh) \cap (0,1)$ for some nonzero h. Then $h'(x_0) = 0$ and $(fh)'(x_0) = 0$, so

$$0 = (fh)'(x_0) = f(x_0)h'(x_0) + f'(x_0)h(x_0) = f'(x_0)h(x_0).$$

Since $h(x_0) \neq 0$ (as $x_0 \in M(h)$), it must be that $f'(x_0) = 0$. Thus for $x_0 \in (0,1)$ we can only find an (nonzero) h such that h and fh are both maximized when $f'(x_0) = 0$.
To see that $\delta\mathcal{A} = [0,1]$, note that for any $x_0 \in [0,1]$ the function $f_{x_0}(x) = 1 - (x - x_0)^2$ attains its maximum modulus exclusively at x_0, which is sufficient to ensure that $\{x_0\}$ is a minimal m-set.

The problem in $C^1([0,1], \mathbb{R})$ is that it is, in general, not possible to find h such that both h and fh are maximized at a specified point x_0. The smoothness of the differentiable functions is what causes the result to fail, which means that, in subalgebras of $\mathrm{Lip}(X, \mathbb{F})$, we will require sufficiently many non-smooth functions in order to ensure that Lemma 4.1 holds. In order to show that each point is a minimal m-set, we only need to find a function h such that fh is maximized exclusively at x_0,

The theorem that follows gives a criterion on subalgebras of $C(X, \mathbb{F})$ that will ensure that the minimal m-sets are singletons.

Theorem 3.
Let X be a compact Hausdorff space, $\mathcal{A} \subset C(X, \mathbb{F})$ a point-separating algebra, and $E \subset X$ a minimal m-set for $\mathcal{A}$. If for every $f \in \mathcal{A}$ with $f|_E \not\equiv 0$ there exists $h \in \mathcal{A}$ with $M(h) \cap E \neq \varnothing$, $\mathrm{Ran}_\pi(h) = \{1\}$, and $M(fh) \cap E \neq \varnothing$, then E is a singleton.

Proof. Suppose that the hypotheses of the theorem all hold, and fix $f \in \mathcal{A}$ with $f|_E \not\equiv 0$. Also, choose $x_0 \in E$, and set $\lambda = f(x_0)$. By hypothesis there exists $h \in \mathcal{A}$ such that $\mathrm{Ran}_\pi(h) = \{1\}$, $M(h) \cap E \neq \varnothing$, and $M(fh) \cap E \neq \varnothing$. Since $E \cap M(h)$ is an m-set – and E is minimal – it must be that $E \subset M(h)$ and $E \subset M(fh)$. Since fh attains its maximum modulus on E, Lemma 3.2 gives that fh is constant on E. Since $h(x) = 1$ for all $x \in E$, it must be that $(fh)(x) = f(x) = \lambda$ for all $x \in E$. Thus every $f \in \mathcal{A}$ that is not identically 0 on E is constant on E. Clearly every function $f \in \mathcal{A}$ that is identically 0 on E is also constant on E, so every $f \in \mathcal{A}$ is constant on E. Since $\mathcal{A}$ separates points, it must be that E is a singleton. $\square$

In particular, if this theorem holds for all $E \in \mathcal{E}^0$, then all minimal m-sets are singletons and Theorem 1 provides the existence of a minimal closed boundary.

4.1. The Shilov Boundary in Subalgebras of $\mathrm{Lip}(X, \mathbb{F})$

In this section we generalize to a specific class of algebras of Lipschitz functions and prove the existence of the Shilov boundary via weak peak points for these subalgebras of $\mathrm{Lip}(X, \mathbb{F})$, where throughout this section X is assumed to be a compact metric space. Constructing boundaries via the minimal m-sets for such algebras is an appropriate technique whenever one can guarantee the existence of certain functions that isolate the points of the minimal m-sets.

Suppose that E is a minimal m-set and that h is a function such that $E \subset M(h)$ and $h|_E \equiv \gamma$ for some $\gamma \in \mathbb{F}$. We say that h has a "corner" on E if

$$\ell_E(h) := \sup_{\varepsilon > 0} \left\{ \inf_{0 < d(x,E) < \varepsilon} \frac{\big|\, |h(x)| - |\gamma| \,\big|}{d(x, E)} \right\} > 0.$$

It is possible that there exists a value of ε_0 such that the set $\{x \in X : 0 < d(x, E) < \varepsilon\}$ is empty for all $\varepsilon \leq \varepsilon_0$. If that is the case, then we define $\ell_E(h) = +\infty$. Note that the hypothesis that $h|_E \equiv \gamma$ is not really an assumption, since $E \subset M(h)$ automatically implies that h is constant on E, by Lemma 3.2. Furthermore, in the case that h has a corner on E, then E and $M(h) \setminus E$ must be separated, and we may assume without loss of generality that $M(h) = E$.

Lemma 4.2.
Let $E \subset X$ be a minimal m-set for an algebra $\mathcal{A} \subset \mathrm{Lip}(X, \mathbb{F})$ and $h \in \mathcal{A}$ be such that $E \subset M(h)$. If $\ell_E(h) > 0$, then there exists $k \in \mathcal{A}$ such that $\ell_E(k) > 0$ and $M(k) = E$.

Proof. Suppose that $\inf \{d(x, E) : x \in M(h) \setminus E\} = 0$. Since h attains its maximum modulus on E, h is constant on E, and we denote by γ the value h takes on E. Since $\inf \{d(x, E) : x \in M(h) \setminus E\} = 0$, for any $\varepsilon > 0$ there exists $x_\varepsilon \in M(h) \setminus E$ such that $d(x_\varepsilon, E) < \varepsilon$. Moreover, as $x_\varepsilon \in M(h)$, $|h(x_\varepsilon)| - |\gamma| = 0$, which implies that $\ell_E(h) = 0$.

Thus $\ell_E(h) > 0$ implies $\inf \{d(x, E) : x \in M(h) \setminus E\} = \delta > 0$, and $U = \{x \in X : d(x, E) < \delta/2\}$ is an open set such that $E \subset U$ and $(M(h) \setminus E) \cap U = \varnothing$. By Lemma 3.3, there exists a function $g \in \mathcal{A}$ such that $E \subset M(g) \subset U$. Since $E \subset M(h)$, it is also true that $E \subset M(hg)$, but $U \cap (M(h) \setminus E) = \varnothing$ implies that $|h(x)g(x)| < \|hg\|_\infty$ on $X \setminus E$. Set $k = hg$, then $M(k) = E$. Finally, g is also constant on E, and we denote the value of g on E by η. Then

$$\frac{||h(x)g(x)| - |\gamma||\eta||}{d(x, E)} = \frac{|\gamma||\eta| - |h(x)g(x)|}{d(x, E)} = |h(x)| \frac{|\eta| - |g(x)|}{d(x, E)} + |\eta| \frac{|\gamma| - |h(x)|}{d(x, E)},$$

which implies that $\ell_E(k) = \ell_E(hg) \geq |\eta| \ell_E(h) > 0$. □

This leads to an immediate corollary.

Corollary 1.
Suppose that $\{x_0\}$ is a minimal m-set for an algebra $\mathcal{A} \subset \mathrm{Lip}(X, \mathbb{F})$ and that $h \in \mathcal{A}$ is such that $\{x_0\} \subset M(h)$ and $\ell_{\{x_0\}}(h) > 0$. Then there exists $k \in \mathcal{A}$ such that $M(k) = \{x_0\}$; in particular every weak peak point is a strong peak point.

We need one final lemma before we proceed with the main result.

Lemma 4.3.
Suppose that E is a minimal m-set for $\mathcal{A}$, $h \in \mathcal{A}$, $E \subset M(h)$, $\|h\|_\infty = 1$, and $0 < \ell_E(h) < +\infty$. Then $\ell_E(h^n) = n\ell_E(h)$. If $\ell_E(h) = +\infty$, then $\ell_E(h^n) = +\infty$.

Proof. In the case that $\ell_E(h) = +\infty$, then there exists $\varepsilon > 0$ such that $\{x \in X : 0 < d(x, E) < \varepsilon\} = \varnothing$, which is independent of h. Thus $\ell_E(h^n) = +\infty$ for all $n \in \mathbb{N}$.

If $0 < \ell_E(h) < +\infty$, then note that

$$1 - |h^n(x)| = (1 - |h(x)|) \left(1 + |h(x)| + |h^2(x)| + \ldots + |h^{n-1}(x)|\right) \leq (1 - |h(x)|)n,$$

which implies that $\ell_E(h^n) \leq n\ell_E(h)$. Since $E \subset M(h)$ and $\|h\|_\infty = 1$, Lemma 3.2 gives that $|h(x)| = 1$ for all $x \in E$, which means that $1 + |h(x)| + |h^2(x)| + \ldots + |h^{n-1}(x)| = n$ for all $x \in E$. Since h is continuous, given $\eta > 0$ there exists $\varepsilon > 0$ such that $1 + |h(x)| + |h^2(x)| + \ldots + |h^{n-1}(x)| \geq n - \eta$ whenever $0 < d(x, E) < \varepsilon$. Therefore $1 - |h^n(x)| \geq (1 - |h(x)|)(n - \eta)$ whenever $0 < d(x, E) < \varepsilon$. Thus $\ell_E(h^n) \geq \ell_E(h)(n - \eta)$, which, by the liberty of choice of η, gives that $\ell_E(h^n) \geq n\ell_E(h)$. □

We can characterize a class of algebras over $\mathbb{R}$ of $\mathbb{F}$-valued Lipschitz functions for which there exists a minimal closed boundary.

Theorem 4.
Let (X, d) be a compact metric space and $\mathcal{A}$ a subalgebra of $\mathrm{Lip}(X, \mathbb{F})$. If for every minimal m-set $E \subset X$ there exists $h \in \mathcal{A}$ such that $E \subset M(h)$ and $\ell_E(h) > 0$, then for every $f \in \mathcal{A}$ with $f|_E \not\equiv 0$ there exists $k \in \mathcal{A}$ such that $E \subset M(k)$ and $M(fk) = E$.

Proof. Without loss of generality we may assume that $\|h\|_\infty = 1$, so by Lemma 4.2 we may also assume that $M(h) = E$. Fix $f \in \mathcal{A}$, then it is further without loss of generality that we assume $\max\limits_{x \in E} |f(x)| = 1$, since f and $\frac{1}{\max\limits_{x\in E}|f(x)|} f$ have the same maximizing set.
Firstly, suppose that $L(f) = 0$. Then f is constant on X, so $|f(x)| = \|f\|_\infty$ for all $x \in X$. Therefore $|h(x)f(x)| < |f(x)| = \|f\|_\infty$ for all $x \notin E$ and $|h(x)f(x)| = |f(x)| = \|f\|_\infty$ for all $x \in E$, so $M(fh) = E$.
On the other hand, suppose that $L(f) \neq 0$. If $\ell_E(h) = +\infty$, then E is open, which implies that there exists $n \in \mathbb{N}$ such that $|h^n(x)| < \frac{1}{\|f\|_\infty}$ for all $x \in X \setminus E$, so $|h^n(x)f(x)| < \frac{1}{\|f\|_\infty}|f(x)| \leq 1$, i.e. $M(fh^n) = E$.
If $0 < \ell_E(h) < +\infty$, then by Lemma 4.3 there exists $n \in \mathbb{N}$ such that $\ell_E(h^n) > 2L(f)$, so there is also an $\varepsilon > 0$ such that

$$\inf_{0<d(x,E)<\varepsilon} \frac{1 - |h^n(x)|}{d(x, E)} > 2L(f).$$

Choose $\varepsilon_1 > 0$ so that $\varepsilon_1 \leq \min\left\{\varepsilon, \frac{1}{2L(f)}\right\}$, then $\frac{1-|h^n(x)|}{d(x,E)} > 2L(f)$ for all $x \in V := \{x \in X : 0 < d(x, E) < \varepsilon_1\}$. Set $X_1 = \{x \in X : |f(x)| \geq 1\}$, then clearly $|h^n(x)f(x)| < 1$ for each $x \in X \setminus X_1$, so let $x \in V \cap X_1$ and choose $x_e \in E$ such that $d(x, E) = d(x, x_e)$. Then $|f(x)| \geq 1$ implies

$$1 - |f(x_e)| \leq |f(x)| - |f(x_e)| \leq |f(x) - f(x_e)| \leq L(f)d(x, x_e) < L(f)\varepsilon_1 \leq \frac{1}{2},$$

which gives $|f(x_e)| > \frac{1}{2}$. As $x_e \in E$ and $\max\limits_{y\in E} |f(y)| = 1$, it follows that $1 \leq \frac{1}{|f(x_e)|} < 2$. Since $x \in V$,

$$\begin{aligned}\frac{1}{|f(x_e)|} - |h^n(x)| \geq 1 - |h^n(x)| > 2L(f)d(x, x_e) > \frac{1}{|f(x_e)|}L(f)d(x, x_e) \geq \frac{1}{|f(x_e)f(x)|}L(f)d(x, x_e)\\ \geq \frac{|f(x) - f(x_e)|}{|f(x_e)f(x)|} = \left|\frac{1}{f(x_e)} - \frac{1}{f(x)}\right|.\end{aligned}$$

Therefore

$$|h^n(x)| < \left|\frac{1}{f(x_e)}\right| - \left|\frac{1}{f(x_e)} - \frac{1}{f(x)}\right| \leq \left|\frac{1}{f(x_e)} - \left(\frac{1}{f(x_e)} - \frac{1}{f(x)}\right)\right| = \frac{1}{|f(x)|},$$

which proves $|h^n(x)f(x)| < 1$. Thus $|h^n(x)f(x)| < 1$ for all $x \in V$, so we set $U = V \cup E = \{x \in X : d(x, E) < \varepsilon_1\}$, which is an open neighborhood of E such that $|h^n(x)f(x)| \leq 1$ for all $x \in U$ and $|h^n(x)f(x)| = 1$ implies $x \in E$. By Lemma 3.3, there exists a function $g \in \mathcal{A}$ such that $E \subset M(g) \subset U$, and it is without loss of generality that we assume $\|g\|_\infty = 1$. There also exists $m \in \mathbb{N}$ such that $\max\limits_{x\in X\setminus U} |g(x)|^m < \frac{1}{\|f\|_\infty}$. Thus, (i) for $x \in E$,

$|h^n(x)g^m(x)f(x)| \le (1)(1)(1) = 1$, (ii) for $x \in V$, $|h^n(x)g^m(x)f(x)| \le |h^n(x)f(x)| < 1$, and (iii) for $x \in X \setminus U$, $|h^n(x)g^m(x)f(x)| \le |g^m(x)f(x)| < \frac{1}{\|f\|_\infty}\|f\|_\infty = 1$. As $\max_{x\in E}|h^n(x)g^m(x)f(x)| = 1$, it follows that $k = h^n g^m$ implies $M(fk) \subset E$. Since E is a minimal m-set, it must actually be that $E = M(fk)$, which is what was to be shown. □

Remark: As was previously noted, these kinds of arguments are made in uniform algebras by assuming the completeness of the algebra. In the setting of Lipschitz algebras, the hypothesis that there exist functions $h \in \mathcal{A}$ with $\ell_E(h) > 0$) for every minimal m-set E should be viewed as a replacement for the assumption of completeness.

This leads to the final result concerning the Shilov boundary for algebras of Lipschitz functions that satisfy the hypotheses of Theorem 4.

Corollary 2.
If $\mathcal{A}$ is a point-separating subalgebra of $\mathrm{Lip}(X, \mathbb{F})$ such that for every minimal m-set E there exists $h \in \mathcal{A}$ with $E \subset M(h)$ and $\ell_E(h) > 0$, then the intersection of all closed boundaries is a closed boundary.

Proof. Let E be a minimal m-set for $\mathcal{A}$, and let $f \in \mathcal{A}$ be any function. If f is identically 0 on E, then f is clearly constant on E, so assume that f is not identically 0 on E.

By Theorem 4, there exists $k \in \mathcal{A}$ such that $E \subset M(k)$ and $M(fk) = E$. By Lemma 3.2, every function that attains its maximum modulus on E is constant on E, so k and fk are both constant on E. Since both k and fk are constant on E – and k is not 0 on E – it must be that f is also constant on E.

By assumption, the algebra $\mathcal{A}$ separates the points of X, so the fact that every function $f \in \mathcal{A}$ is constant on E implies that E is a singleton. Since all minimal m-sets are singletons, Theorem 1 gives that the closure of $\delta\mathcal{A}$ is the minimal closed boundary. □

This corollary shows that the Shilov boundary exists for all subalgebras of $\mathrm{Lip}(X, \mathbb{F})$ that satisfy the hypotheses of Theorem 4, regardless of whether the range of the functions is $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$.

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