On Totally Shattered and Hyperuniversal Graphs*

(preliminary draft)

MATTHEW DAY
University of Texas at Austin

SARAH GOOD
University of Washington

HESTER GRAVES
Wellesley College

AMARDA SHEHU
Clarkson University

CHRISTINO TAMON
Clarkson University

CATHERINE VANDERWAART
Swarthmore College

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Abstract

We study the class of totally shattered graphs and their connection to strong universal graphs. The class of totally shattered graphs is a graph-theoretic analogue of (n, k)-universal set systems. We also consider the testing dimension of graphs, which is a natural extension of the Vapnik-Chervonenkis dimension for graphs, and prove some extremal and structural properties of graphs with specific testing dimensions. Finally, we explore a conjecture that totally shattered graphs are quasirandom.

1 Introduction

Let $G = (V, E)$ be a finite, simple, undirected graph. For a vertex $v \in V$, the open (resp. closed) neighborhood of v is defined as $N(v) = \{u \mid (u, v) \in E\}$ (resp. $\overline{N}(v) = N(v) \cup \{v\}$). A set of vertices S is *shattered* (resp. *externally* shattered) by G if for every subset A of S, there is a vertex $v_A \in V$ such that $A = \overline{N}(v_A) \cap S$ (resp. $A = N(v_A) \cap S$). So shatterings use closed neighborhoods whereas external shatterings allow only open neighborhoods. In this case, we also say that v_A *shatters* A in S or that v_A *yields* the shattering of A within S. Unless otherwise stated, we refer to non-external shatterings as simply shatterings. An easy but important observation is that, if $A, B \subseteq S$, $A \neq B$, v_A shatters A in S and v_B shatters B in S, then $v_A \neq v_B$. The *Vapnik-Chervonenkis (VC) dimension* of a graph G is the size of the largest shattered subset of $V(G)$. The *testing dimension* of a graph G is the largest k so that *every* subset $S \subseteq V$ of k vertices in G is shattered. The notation $VCdim(G)$ and $tdim(G)$ are used to denote the VC and testing dimensions of G, respectively. We use $xtdim(G)$ to denote the *external* testing dimension of G. The term shattering means non-external shattering; when external shattering is meant, it will be explicitly mentioned.

The notion of shatterings for hypergraphs is well-studied in statistical learning theory and combinatorial geometry Total shatterings in hypergraphs or set-systems was studied by Seroussi and Bshouty [16] under the name of (n, k)-universal sets. Graph shatterings were considered earlier by Anthony, Brightwell, and Cooper [2] and by Kranakis et al. [10]. The latter paper also contained several nice generalizations and complexity results on these problems. Total shattering in graphs

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was explored in [9] in response to a remark made in [2] about this natural extension of the Vapnik-Chervonenkis dimension of graphs. In fact, the external k-shattering condition is equivalent to the k-extendible property in finite model theory [15].

Interestingly, there is a connection between totally shattered graphs and *universal* graphs, i.e., graphs which contain all small graphs as (induced) subgraphs. This connection was mentioned in [9] but was used earlier in works on explicit construction of universal graphs [6]. Universal graphs have been studied in the 1960s by Moon [11] and Chung and Graham [7] and others. These graphs are important for applications in areas such as circuit design and testing, data representation, and fault-tolerance in networks. In this work, we describe a stronger notion of universality for graphs that gives a tight characterization of totally shattered graphs. Although this notion, which we call *hyperuniversality*, has been studied as a property of random graphs (see Bollobas [4]), our interest is to use it in characterizing totally shattered graphs.

Another main problem studied in this work is the determination of the function $t(k)$ which is defined as the minimum n so that there is a graph on n vertices with testing dimension k. In [9], it was observed that $2^k+k-1 \leq t(k) \leq k^2 2^k(1+o(1))$. The lower bound is obtained by looking at how many vertices are required to shatter a single set of k vertices. The upper bound is obtained by a simple probabilistic argument on random graphs. Here, we improve the lower bound by exploiting the fact that the minimum dominating set in a graph with testing dimension k must be of size at least k. Using this, we manage to show that $t(k) = \Omega(k2^k)$. It is an open question to determine the exact asymptotics of $t(k)$.

We describe some notation that we used throughout this note. We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. For a graph $G = (V, E)$ and for a subset $U \subseteq V$ of vertices in G, let $G[U]$ denote the induced subgraph with vertex set U in G. The degree of a vertex v is denoted $deg(v)$. We denote $\delta(G)$ to be the minimum degree of G (smallest degree of a vertex) and $\Delta(G)$ to be the maximum degree of G (largest degree of a vertex). If G is a connected graph, then the vertex-connectivity of G, i.e., the minimum number of vertices of G whose removal disconnects G, is denoted $\kappa(G)$. A graph G is called k-connected if $\kappa(G) \geq k$. A dominating set of G is a set D of vertices where each vertex of G is either in D or adjacent to some vertex in D. The smallest dominating set of G is denoted $\gamma(G)$.

2 Total Shatterings of Graphs

The smallest graph with a testing dimension equals 1 is $\overline{K}_2$. In what follows we will deal only with graphs of testing dimension of at least 2. We briefly review some observations made in [9] about totally shattered graphs. For some of the claims, we have included some proof sketches.

Fact 1 *If a graph $G = (V, E)$ has a k-dominating set, then $tdim(G) < k$.*

Proof: There is no empty shattering for the set of dominating vertices. □

Proposition 2 *Let $G = (V, E)$ be a graph with $tdim(G) = k \geq 2$. Then* **(i)** *$diam(G) = 2$.* **(ii)** *For each $v \in V$, $2^{k-1} - 1 \leq deg(v) \leq |V| - 2^{k-1} + 1$.* **(iii)** $2^{k-2} \leq \kappa(G) \leq |V| - 2^{k-1} + 1$.

A graph G is called *universal* for a family of graphs $\mathcal{H}$ if G contains all graphs $H \in \mathcal{H}$ as induced subgraphs. The notion of universal graphs was studied by Moon [11] (see also Chung and Graham [7]) and has been studied by others in relation to sparse graphs and small trees. We say that a graph is k-universal if it is universal for all graphs with at most k vertices.

Proposition 3 *Let $k \geq 1$. If $G = (V, E)$ is a graph with tdim$(G) = k$, then G is k-universal.*

Recall that $t(k)$ is the minimum n such that there is a graph on n vertices with a testing dimension of k. In [9], it was found that $t(2) = 8$. The problem of determining exact values for $t(3)$ and beyond is still an open question. Nevertheless, some information on the asymptotic behavior of $t(k)$ is available.

Proposition 4 *For $k \geq 2$, $2^k + k - 1 \leq t(k) \leq k^2 2^k(1 + o(1))$.*

Proof: (sketch) To see the lower bound, take a k-clique (which exists since G is k-universal). There must exist $2^k - 1$ external vertices that yield all the shatterings (except for the full set) for the k-clique. For the upper bound, use the probabilistic method and consider the probability that a random graph $G(n, 1/2)$ is not externally k-shattered. This is at most $\binom{n}{k} 2^k (1 - 2^{-k})^{n-k}$; forcing this last expression to be strictly less than 1 and solving for n, we get the upper bound of $k^2 2^k(1 + o(1))$. The argument for non-external shattering is similar. □

3 Our Results

An easy extra fact is that the diameter of the complement of a graph with testing dimension k must also be 2. So these graphs form a special subset of the well-studied class of graphs with diameter 2.

Lemma 5 *(Deviation bounds) Let $G = (V, E)$ be a graph with tdim$(G) = k \geq 2$. Then* **(i)** *tdim$(G \setminus \{x\}) \geq k - 1$, for every $x \in V$.* **(ii)** *tdim$(G \setminus \overline{N}(x)) \geq k - 1$.* **(iii)** *tdim$(\overline{G}) = k + \{0, \pm 1\}$.*

Proof: **(i)** Take $S \subset V \setminus \{x\}$ of size $k - 1$ and an arbitrary $A \subset S$. Consider the vertex v that shatters $A \subset S \cup \{x\}$, where $x \notin A$, in G; note that $v \notin N(x)$. Thus v also shatters A in S (over $G \setminus \{x\}$). Thus S is shattered in $G \setminus \{x\}$. To prove **(ii)**, we can use the same proof idea as in **(i)**.

(iii) Let $S \subset V$ be an arbitrary subset of size $k - 1$ and let $A \subset S$. Take a vertex x not in S that yields the empty shattering for S in G, i.e., x is adjacent to none of the vertices in S. Consider the vertex v that shatters $(S \setminus A) \cup \{x\}$ in $S \cup \{x\}$ (over G). Note that v shatters A in S (over $\overline{G}$). So *tdim*$(\overline{G}) \geq k - 1$. Exchanging the roles of G and $\overline{G}$, the testing dimension could increase by at most one (or stays fixed). □

3.1 Improved bounds on $\delta(G)$ and $t(k)$

In this section we show a larger bound on the minimum degree $\delta(G)$ which in turn could be used to improve the lower bound on $t(k)$ in graphs G with *tdim*$(G) = k$.

Lemma 6 *Let $G = (V, E)$ be a graph with tdim$(G) = k \geq 3$. Then $\delta(G) \geq k 2^{k-2}$.*

Proof: Let $x \in V$ be arbitrary and let $S \subset V \setminus \{x\}$ be of size $k - 2$. Let $\hat{v}$ be a vertex in G that is adjacent to all of $S \cup \{x\}$. Fix a subset $T \subset S$. For each $y \in V \setminus S \cup \{x\}$, let $v_T(y)$ be the vertex that shatters $T \cup \{x\} \cup \{y\}$ in $S \cup \{x\} \cup \{y\}$. Note, if $D_T = \bigcup \{v_T(y) \mid y \notin S \cup \{x\}\}$, then $D_T \cup \{\hat{v}\}$ is a dominating set of G. Thus $|D_T| \geq k$. Also, note that $D_{T_1} \cap D_{T_2} = \emptyset$, if $T_1, T_2 \subset S$ and $T_1 \neq T_2$. Thus x is adjacent to at least $k 2^{k-2}$ vertices, for each subset T of S and each vertex in the dominating set D_T. □

Remark: The proof actually shows that $\delta(G) \geq (\gamma(G) - 1) 2^{k-2}$. We can improve $\kappa(G)$ by adapting the above proof.

Theorem 7 *For $k \geq 2$, $t(k) \geq 6 + (k-1)2^{k-1}$.*

Proof: Let $G = (V, E)$ be any graph with $tdim(G) = k$. First, we present a proof that gives the same asymptotic expression but does not give the sharpest bounds. Note that any vertex $x \in V$ is adjacent to at most $|V| - k2^{k-2}$ (since it has to be non-adjacent to at least $k2^{k-2}$ other vertices, by the same reasoning as in the proof of Lemma 6). Hence $k2^{k-2} \leq deg(x) \leq |V| - k2^{k-2}$, which implies $|V| \geq k2^{k-1}$.

The sharper proof begins by observing that $t(k) \leq t(k+1) - (k+1)2^{k-1}$, by the neighborhood deletion bound. Rearranging to $t(k+1) \geq t(k) + (k+1)2^{k-1}$, this yields $t(k) \geq t(2) + \sum_{j=3}^{k} j2^{j-2}$. Thus $t(k) \geq 6 + (k-1)2^{k-1}$. □

The next corollary shows that there is a linear gap left between the lower and upper bounds for $t(k)$. An obvious open question is if this gap could be narrowed even further.

Corollary 8 *For $k \geq 2$, $6 + (k-1)2^{k-1} \leq t(k) \leq k^2 2^k(1 + o(1))$.*

3.2 Hyperuniversality

In this section we show a close connection between a stronger notion of universality and the notion of externally shattered graphs.

Definition 1 *A graph $G = (V, E)$ is* k-hyperuniversal *if for each graph H with at most k vertices, for each subset $A \subset V(H)$, where $|A| \leq k$, and for each $B \subseteq V(G)$ for which there is an isomorphism $g : H[A] \to G[B]$, there is an embedding $f : H \hookrightarrow G$ that completes g, i.e., $f|_A = g$.*

Next we describe the notion of external shatterings. Recall that this is shatterings when only *open* neighborhoods are allowed. So we say that a subset $S \subset V$ is externally shattered if every shattering of subsets of S is yielded by vertices from $V \setminus S$. The external testing dimension $xtdim(G)$ is the largest k such that all k-subsets of G are externally shattered. We state and prove a tight characterization of totally (externally) shattered graphs with hyperuniversal graphs in the next result.

Theorem 9 *Let $G = (V, E)$ be a graph and let $k \geq 1$. Then G is k-hyperuniversal if and only if $xtdim(G) = k - 1$.*

Proof: (Only if) Let G be k-hyperuniversal. Let $S \subset V(G)$ be a subset of size $k-1$ and let $A \subset S$ be arbitrary. Set $H = ([k], E_H)$ to be a graph on k vertices such that $H[\{1, \ldots, k-1\}]$ and $G[S]$ are isomorphic through the isomorphism g. Then let vertex k in H be connected only to the vertices in $g^{-1}(A)$. Since G is k-hyperuniversal, there is an embedding $f : H \hookrightarrow G$ that completes g. Thus $f(k)$ is a vertex in G that shatters A in S. Thus $xtdim(G) = k - 1$.

(If) By induction on $k \geq 1$. The base case is trivial. Assume that the claim is true for $\ell \leq k$ and we now establish the claim for $\ell = k+1$. Let G be a graph with $xtdim(G) = k$. To prove that G is $(k+1)$-hyperuniversal, let H be any graph on $k+1$ vertices, let $A \subset V(H)$ be any subset with $|A| \leq k+1$, let $B \subset V(G)$ be such that there is an isomorphism g whereby $H[A] \equiv G[B]$. If $|A| = k+1$, we are done; otherwise let $x \in V(H) \setminus A$. By inductive hypothesis, there is an embedding $f : H[V(H) \setminus \{x\}] \hookrightarrow G$ that completes g. Set $S = f(V(H) \setminus \{x\})$; note that $S \subset V(G)$ is of size k and hence is externally shattered in G. Let $A = f(N(x))$ be the image set of the neighbors of x in H. Since $xtdim(G) = k$, there is a vertex $y \in V(G)$ that externally shatters A in S. We can now complete the embedding f to $\hat{f} : H \hookrightarrow G$ by setting $\hat{f}(v) = f(v)$, if $v \neq x$, and $\hat{f}(x) = y$. Thus G is k-hyperuniversal. □

3.3 Graphs with small testing dimension

In this section we consider $t(k)$, for $k = 2, 3$. The exact value for $t(2)$ was given in [9]. The exact values for $t(3)$ and beyond are unknown; although one can given some lower and upper bounds.

Lemma 10 *$t(2) = 8$.*

Proof: (sketch) The upper bound is given by two 3-regular graphs on 8 vertices as shown in Figure 1. The lower bound uses the fact that a graph G with $tdim(G) = k$ is 2-connected and has

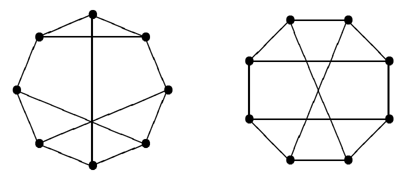

Figure 1: The (only) two minimal graphs with testing dimension 2.

$\delta(G) \geq 3$. Thus, by a result of Dirac, it must have a simple cycle of length at least $2\delta(G) \geq 6$. Then, one can show that graphs with 6 and 7 vertices that has a 6-cycle cannot shatter all 2-sets. □

To lower bound $t(3)$, we will use some counting arguments (especially on triples) on graphs with testing dimension 3. Our goal is to show that $t(3) > 17$. For upper bounds, it is known that the Paley graph P_{29} gives $t(3) \leq 29$.

Lemma 11 *(Triple inequalities) Let $G = (V, E)$ be a graph with $tdim(G) = k \geq 2$, where $|V| = n$, with degree sequence $d_1 \geq \ldots \geq d_n$. Then*

$$\binom{n}{3} \leq \sum_{i=1}^{n} \binom{d_i + 1}{3} \leq n\binom{\Delta + 1}{3}, \quad \binom{n}{3} \leq \sum_{i=1}^{n} \binom{n - d_i - 1}{3} \leq n\binom{n - \delta - 1}{3}.$$

Proof: Each triple of vertices must be shattered and hence must belong to some neighborhood of a vertex. Likewise, each triple of vertices must be avoid by some vertex (and hence its neighborhood). □

3.4 Preservation properties of external shatterings

We observe some well-behaved properties of externally shattered graphs.

Proposition 12 *Let $G = (V, E)$ be a graph with $xtdim(G) = k$. Then* **(i)** *$xtdim(G[N(x)]) = k-1$, for each $x \in V$.* **(ii)** *$xtdim(G[V \setminus \overline{N}(x)]) = k - 1$, for each $x \in V$.* **(iii)** *$xtdim(\overline{G}) = k$.*

Proof: **(i)** Let $x \in V$ be arbitrary. Let $S \subset N(x)$ be a subset of size $k - 1$ and let $A \subset S$ be an arbitrary subset of S. Consider the vertex v that externally shatters $A \cup \{x\} \subset S$ in G. Note that $v \in N(x) \setminus S$. Thus $xtdim(G[N(x)]) = k - 1$.
(iii) Since all shatterings are external, they exist symmetrically in $\overline{G}$. □

The next observation generalizes the neighborhood projection bounds to an arbitrary collection.

Proposition 13 *Let $G = (V, E)$ be a graph with $xtdim(G) = k$. Let $v_1, \ldots, v_t \in V$ be any collection of vertices. Let $\tilde{N}(v_i)$ be either the neighborhood of v_i or the complement of the neighborhood of v_i. Then $xtdim(G[\cap_{i=1}^{t} \tilde{N}(v_i)]) = k - t$.*

3.5 Internal versus External Shatterings

In this section we study some properties that link the two types of shatterings.

Proposition 14 *Let $G = (V, E)$ be a graph with $xtdim(G) = k$. If $tdim(\overline{G}) = k + 1$ then $xtdim(G) = k$.*

Let $xt(k)$ be the minimum n such that there is a graph G on n vertices with $xtdim(G) = k$.

Conjecture 15 $\lim_{k\to\infty} xt(k)/t(k) = 1$.

3.6 Quasirandomness

Quasirandom graphs were studied by Chung, Graham, and Wilson [8]. They listed six equivalent properties that define this class of graphs. In this section, we make the following observation.

Proposition 16 *If there is a family of graphs $\mathcal{G} = \{G_k\}_k$, where $tdim(G_k) = k$ and $|V(G_k)| = O(k2^k)$ then $\mathcal{G}$ is quasirandom.*

Not all family of quasirandom graphs have non-trivial testing dimension. Note that if one proves that totally k-shattered graphs are *not* quasirandom then $t(k) = \omega(k2^k)$ (an indirect argument for a better lower bound).

4 Explicit constructions

The proof using probabilistic methods showed that graphs with arbitrary testing dimension k exists for graph sizes beyond $n > k^2 2^k$. If one asks for an explicit construction, it turns out that we can appeal to Paley graphs [12]. A Paley graph $P_p = (V, E)$ is defined on $V = [p]$, where p is a prime congruent to 1 modulo 4, and where $(x, y) \in E$ if and only if $x - y$ is a quadratic residue modulo p. The following proposition is apparently implicit in Bollobas and Thomason [6] (a basic exposition of this idea was given in Alon [1]) whose main concern was to construct explicit universal graphs; our main source of reference is Blass, Exoo, and Harary [5].

Theorem 17 *For all $k > 0$, if $p > k^2 2^{2k-2}$ then $tdim(P_p) \geq k$.*

Proof: (adapted from [1, 5]) Let $P_p = (V, E)$ be a Paley graph, for some prime p congruent to 1 modulo 4 (this can be generalized to prime powers as well). Take $S \subset V$ to be a subset of k vertices and let $A \subset S$ be arbitrary. Denote the quadratic character over $\mathbb{Z}_p$ by $\chi : \mathbb{Z}_p^* \to \{-1, 1\}$. Recall that x is adjacent to y if and only if $\chi(x - y) = 1$; this yields an undirected graph since -1 is a quadratic residue modulo p. We need the following deep theorem of A. Weil (1948).

Claim 18 *(A. Weil) Let $\chi : \mathbb{F}_q \to \mathbb{C}^*$ be a nontrivial character of the finite field $\mathbb{F}_q$ on q elements and let f be a degree d polynomial with at most t zeros. Then $\sum_{x\in\mathbb{F}_q} |\chi(f(x))| \leq (d - 1)\sqrt{q}$.*

We can express the condition for (external) shattering of A in S via a polynomial g defined as follows.

$$g(S) = \sum_{x \notin S} \prod_{y \in A} (1 + \chi(x - y)) \prod_{y \in S \setminus A} (1 - \chi(x - y))$$

So A is shattered in S if and only if $g(S) > 0$. Next consider another polynomial h that is defined as follows.

$$h(S) = \sum_{x \in \mathbb{Z}_p} \prod_{y \in A} (1 + \chi(x-y)) \prod_{y \in S \setminus A} (1 - \chi(x-y))$$

Note that $h(S) - g(S) \leq k2^{k-1}$. Also note that

$$h(S) = \sum_{x \in \mathbb{Z}_p} 1 + \sum_{x \in \mathbb{Z}_p} \sum_{y \in S} (-1)^{[y \notin A]} \chi(x-y) + \ldots = p + 0 + \ldots ,$$

since $\sum_z \chi(z) = 0$. Thus

$$\begin{aligned} |h(S) - p| &\leq \sum_{x \in \mathbb{Z}_p} \sum_{t=2}^{k} \sum_{T \subset S : |T| = t} |\prod_{y \in T} \chi(x-y)| \\ &= \sum_{t=2}^{k} \sum_{T \subset S : |T| = t} \sum_{p \in \mathbb{Z}_p} |\chi(\prod_{y \in T} (x-y))| \\ &\leq \sum_{t=2}^{k} \binom{k}{t} (t-1)\sqrt{p}, \quad \text{by Weil's theorem} \\ &= \sqrt{p} \sum_{t=2}^{k} \binom{k}{t} (t-1) = \sqrt{p}[(k-2)2^{k-1} + 1]. \end{aligned}$$

Therefore, $g(S) \geq h(S) - k2^{k-1} \geq p + \sqrt{p}[(k-2)2^{k-1} + 1] - k2^{k-1}$. By choosing $p > k^2 2^{2k-2}$, we can ensure that $g(S) > 0$ for arbitrary S and $A \subset S$. □

We found that $xtdim(P_p) = 3$, whenever $p \geq 29$; in fact, 29 is the smallest size graph of testing dimension 3 (external or otherwise) that we are aware of. We also verified that $xtdim(P_p) = 4$, whenever $p \geq 89$. These bounds are smaller than what is predicted using Theorem 17. We are not aware of any explanation of why the Paley graphs behave monotonically with respect to the testing dimension. Also, is there an elementary proof for why large Paley graphs have high testing dimension (without using Weil's theorem)?

5 Conclusions

We conclude this note with the following open questions and conjectures. **(i)** What is the correct asymptotic expression for $t(k)$? **(ii)** Is the minimal totally shattered graph *regular* or *Hamiltonian* or *vertex-transitive*? **(iii)** Is there *recursive, explicit* non-Paley construction for (minimal) totally shattered graphs? **(iv)** What is the complexity of computing the testing dimension of graphs? Papadimitriou and Yannakakis [14] proved that the complexity of computing the VC dimension of a (hyper)graph is LOGNP-complete. **(v)** There is a result of Prömel and Rödl [13] showing that if a graph and its complement don't contain $\log n$-clique then it is $\log n$-universal; these graphs are called *non-Ramsey* graphs. Is there a connection between non-Ramsey graphs and totally shattered graphs?

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