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PERFECT STATE TRANSFER ON SIGNED GRAPHS

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We study perfect state transfer of quantum walks on signed graphs. Our aim is to show that negative edges are useful for perfect state transfer. First, we show that the signed join of a negative 2-clique with any positive $(n, 3)$-regular graph has perfect state transfer even if the unsigned join does not. Curiously, the perfect state transfer time improves as n increases. Next, we prove that a signed complete graph has perfect state transfer if its positive subgraph is a regular graph with perfect state transfer and its negative subgraph is periodic. This shows that signing is useful for creating perfect state transfer since no complete graph (except for the 2-clique) has perfect state transfer. Also, we show that the double-cover of a signed graph has perfect state transfer if the positive subgraph has perfect state transfer and the negative subgraph is periodic. Here, signing is useful for constructing unsigned graphs with perfect state transfer. Finally, we study perfect state transfer on a family of signed graphs called the exterior powers which is derived from a many-fermion quantum walk on graphs.

Keywords: Quantum walk, perfect state transfer, signed graphs, exterior power
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1 Introduction

The study of quantum walks on finite graphs is important in quantum computing due to its promise as an algorithmic technique orthogonal to the Hidden Subgroup and Amplitude Amplification paradigms. Strong quantum algorithms based on quantum walks have been

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discovered in the ensuing years for diverse problems such as element distinctness, matrix product verification, triangle finding in graph, formula evaluation, and others. But recently, quantum walks have also proved crucial as a universal quantum computational model (see Childs [11]).

A property of quantum walks called perfect state transfer was originally studied by Bose [7] in the context of information transfer in quantum spin chains. Christandl *et al.* [13, 12] continued this investigation and proved strong results for various other graphs, notably the hypercubes. More recently, perfect state transfer was used by Underwood and Feder [23] in the simulation of universal quantum computation via quantum walks. This provides an alternative method to the graph scattering techniques used in [11].

The main problem in the study of perfect state transfer in quantum walks on graphs is to characterize graphs which exhibit this property. Much of the recent progress along these lines is described in Godsil [15]. Another related question is to ask for operations on graphs which create perfect state transfer. Examples of such operations include deleting edges [8], adding self-loops [9], and using arbitrary weights on edges [18, 14, 19, 20]. The work by Feder [14], which is relevant to our work here, is based on a many-particle quantum walk on graphs where the particles are bosons.

In this work, we study perfect state transfer of quantum walks on *signed* graphs. A signed graph is a graph whose edges are given ± 1 weights. The literature on signed graphs is vast (see Zaslavsky [25]). Our main goal is to understand the impact of negative edges on perfect state transfer on graphs. Our work is inspired by Pemberton-Ross and Kay [22] who provided an indication that negative unit weights are useful for perfect state transfer via dynamic couplings.[a] We show that negative unit edges are useful for creating perfect state transfer on certain classes of graphs even without dynamic couplings.

The first known family of signed graphs with perfect state transfer is arguably the weighted paths studied in [13, 12], albeit not explicitly stated as such (see also Lemma 2 in Kay [17]). Subsequently, a somewhat general scheme for crafting weights (possibly negative) to produce perfect state transfer graphs was proposed in [19]. Here, we study the effect of signed edges on graph products, on graph joins, on quotient graphs modulo equitable partitions, and on signed graphs with certain spanning-subgraph decomposition properties.

For graph joins, we show that the join between a negatively signed K_2 with any 3-regular unsigned graph has perfect state transfer; in contrast, the unsigned join lacks this perfect state transfer property. Curiously, the perfect state transfer time in the signed join decreases as the size of the 3-regular graph increases. Using the spanning-subgraph decomposition property of signed graphs, we show examples of signed complete graphs with perfect state transfer. This is in contrast to the known fact that unsigned complete graphs have no perfect state transfer (but are merely periodic). We also show the opposite effect by constructing *unsigned* graphs with perfect state transfer from double-coverings of signed graphs.

Finally, we consider an interesting graph operator called the exterior power. We observe that this operator creates, in a natural way, signed graphs from unsigned graphs. This operator was studied by Osborne [21] (and also by Audenaert *et al.* [2]) although not in the context of signed graphs. The exterior power of a graph is related to a many-particle quantum walk on graphs where the particle are fermions. More specifically, we show that the exterior

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[a] A similar technique was described earlier by Kay and Ericsson [18]; this was explored further in [20].

kth power of a graph G is the *quotient* graph corresponding to a k-fermion quantum walk on G. This complements the result of Bachman *et al.* [3] showing that Feder's weighted graphs (see [14]) are quotient graphs corresponding to a k-boson quantum walk on an underlying graph.

Our focus in this work is on the mathematical properties of perfect state transfer on signed graphs. We appeal to the standard reduction of state transfer of an arbitrary qubit state in a quantum network represented by a graph to a continuous-time quantum walk on the single excitation subspace of the underlying Hilbert space (see [7, 13, 12]). For relevant implementation issues related to signed graphs, we refer the interested reader to several systems suggested by Pemberton-Ross and Kay [22] (and the references therein).

2 Preliminaries

We describe some notation which will be used throughout the paper. For a logical statement S, we use $[\![S]\!]$ to mean 1 if S is true, and 0 otherwise. Given a positive integer n, the notation $[n]$ denotes the set $\{1, \ldots, n\}$. We use $A \uplus B$ to denote the disjoint union of sets A and B. The identity and all-one matrices are denoted I and J, respectively; the latter may not necessarily be square.

The graph $G = (V, E)$ we consider will be finite, undirected, and connected. The adjacency matrix $A(G)$ of G is defined as $A(G)_{u,v} = [\![(u, v) \in E]\!]$. A graph G is called k-regular if each vertex of G has exactly k adjacent neighbors. We say a graph G is (n, k)-regular if it has n vertices and is k-regular.

Let G and H be two given graphs. The *Cartesian product* $G \,\square\, H$ of graphs G and H has the vertex set $V(G) \times V(H)$ where the edges are defined as follows. The vertex (g_1, h_1) is adjacent to the vertex (g_2, h_2) if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$ or if $(g_1, g_2) \in E(G)$ and $h_1 = h_2$. The adjacency matrix is given by $A(G \,\square\, H) = A(G) \otimes I + I \otimes A(H)$. The *join* $G + H$ of graphs G and H is a graph whose complement is $\overline{G} \uplus \overline{H}$. The adjacency matrix of the join is given by

$$A(G + H) = \begin{bmatrix} A(G) & J \\ J & A(H) \end{bmatrix}. \tag{1}$$

The dimensions of the matrices I and J used above are implicit (but clear from context). Most notation we use above are adopted from [16].

Signed graphs A *signed* graph $\Sigma = (G, \sigma)$ is a pair consisting of a graph $G = (V, E)$ and a signing map $\sigma : E(G) \to \{-1, +1\}$ over the edges of G. For notational convenience, we may on occasion use G^σ in place of (G, σ).

We call G the underlying (unsigned) graph of Σ; we also use $|\Sigma|$ to denote this underlying graph. Two signed graphs Σ_1 and Σ_2 are called *switching equivalent*, denoted $\Sigma_1 \sim \Sigma_2$, if

$$A(\Sigma_1) = D^{-1} A(\Sigma_2) D, \tag{2}$$

for some diagonal matrix D with ± 1 entries. A signed graph Σ is *balanced* if $\Sigma \sim |\Sigma|$ and it is called *anti-balanced* if $\Sigma \sim -|\Sigma|$ where $-|\Sigma|$ refers to the all-negative signing of $|\Sigma|$.

Another way to view a signed graph Σ is as two edge-disjoint spanning subgraphs of the underlying graph $|\Sigma|$; that is, $\Sigma = G^+ \cup G^-$, where the edges of G^+ are signed with

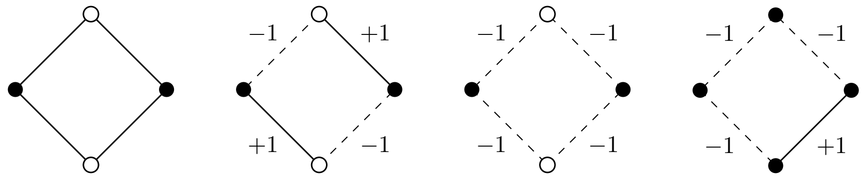

Fig. 1. Signed 4-cycles: (a) Unsigned; (b) Balanced; (c) Antibalanced; (d) Unbalanced. Dashed edges are negatively signed. Perfect state transfer occurs between vertices marked white within each case. Here, (b) and (c) belong to the same switching equivalence class.

+1 and the edges of G^- are signed with -1. The adjacency matrix of Σ is then given as $A(\Sigma) = A(G^+) - A(G^-)$.

A classic reference on signed graphs is the paper by Zaslavsky [24].

Quantum walks Given a graph G, a continuous-time quantum walk on G is described by the time-dependent unitary matrix

$$U(t) = \exp(-itA(G)). \tag{3}$$

We say a graph G has *perfect state transfer* from vertex a to b at time t if

$$|\langle b|U(t)|a\rangle| = 1. \tag{4}$$

On the other hand, the graph G is *periodic* at vertex a at time t if $|\langle a|U(t)|a\rangle| = 1$. Finally, G is called periodic if it is periodic at all of its vertices. For more background on state transfer on graphs, we refer the reader to Godsil [15].

3 Balanced Products

We state basic results for perfect state transfer on balanced and anti-balanced signed graphs.

Lemma 1 *If a graph G has perfect state transfer, then so does the signed graph $\Sigma = (G, \sigma)$ provided σ is a balanced or anti-balanced signing of G.*

Proof Suppose G has perfect state transfer from vertex a to b. If σ is a balanced or anti-balanced signing of G, then there is a diagonal ± 1 matrix D for which $A(G^\sigma) = \pm D^{-1}A(G)D$. Thus, we have

$$\langle b|e^{-itA(G^\sigma)}|a\rangle = \langle b|D^{-1}e^{\mp itA(G)}D|a\rangle = \pm\langle b|e^{\mp itA(G)}|a\rangle. \tag{5}$$

This shows that G^σ has perfect state transfer from a to b. $\square$

Remark: Since any signing of a path is balanced, the weighted paths described in [13, 12] retain their perfect state transfer properties under any sign switchings. This essentially provides the first example of signed graphs with perfect state transfer; see also Lemma 2 in Kay [17].

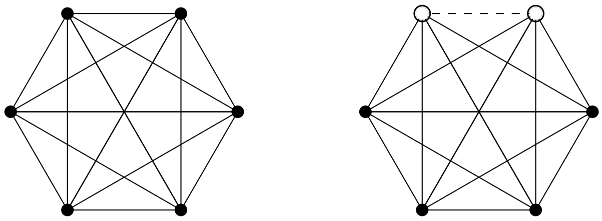

Fig. 2. Signed complete graphs. (a) K_6 has no perfect state transfer. (b) The signed join $K_2^- + K_4^+$ has perfect state transfer between vertices marked white. The dashed edge is negatively signed.

Corollary 1 *For a positive integer m, if for each $k \in [m]$, the graph G_k has perfect state transfer from vertex a_k to vertex b_k, then the signed graph $\square_{k=1}^m \Sigma_k$, where $\Sigma_k = (G_k, \sigma_k)$, has perfect state transfer from vertex $(a_1, \ldots, a_m)$ to vertex $(b_1, \ldots, b_m)$, provided each σ_k is a balanced or anti-balanced signing of G_k.*

Proof We note that $A(\square_{k=1}^m G_k)$ is a sum of k commuting terms:

$$A(\square_{k=1}^m G_k) = \sum_{k=1}^m (I \otimes \ldots \otimes I \otimes \overbrace{A(G_k)}^{k\text{th position}} \otimes I \otimes \ldots \otimes I). \tag{6}$$

Thus, we have

$$\bigotimes_{j=1}^m \langle b_j| \left[e^{-itA(\square_k G_k)} \right] \bigotimes_{\ell=1}^m |a_\ell\rangle = \prod_{k=1}^m \langle b_k| e^{-itA(G_k)} |a_k\rangle. \tag{7}$$

Now, we apply Lemma 1, using $A(G_k^{\sigma_k}) = \pm D_k^{-1} A(G_k) D_k$, to obtain:

$$\bigotimes_{j=1}^m \langle b_j| e^{-itA(\square_k \Sigma_k)} \bigotimes_{\ell=1}^m |a_\ell\rangle = \prod_{k=1}^m \langle b_k| D_k^{-1} e^{\mp itA(G_k)} D_k |a_k\rangle = \pm \prod_{k=1}^m \langle b_k| e^{\mp itA(G_k)} |a_k\rangle, \tag{8}$$

since $D_k|a\rangle = \pm|a\rangle$ for each vertex a. This proves the claim. □

4 Signed Joins

In this section, we consider perfect state transfer properties of binary graph joins where the two graphs are given opposite signs. So, we denote $G^+ \pm H^-$ to mean the join of graphs G and H where G is positively signed, H is negatively signed, and their connecting edges are all positively (or negatively) signed, respectively. We remark that $G^+ \pm H^-$ are switching equivalent and hence share perfect state transfer properties.

Theorem 1 *Suppose G_1 is a (n_1, k_1)-regular graph and G_2 is a (n_2, k_2)-regular graph. Let Σ be the signed join graph $G_1^- + G_2^+$ Then, for two vertices $a, b \in V(G_1)$, we have*

$$\langle b| e^{-itA(\Sigma)} |a\rangle = \langle b| e^{itA(G_1)} |a\rangle + \frac{e^{-it\tilde{\delta}_-}}{n_1} \left[\left(\cos(t\Delta) - i\frac{\tilde{\delta}_+}{\Delta} \sin(t\Delta) \right) - e^{-it\tilde{\delta}_+} \right] \tag{9}$$

where $\tilde{\delta}_\pm = -\frac{1}{2}(k_1 \pm k_2)$ and $\Delta = \sqrt{\tilde{\delta}_+^2 + n_1 n_2}$.

Proof For notational convenience, given $|u\rangle$ and $|v\rangle$ of dimensions n_1 and n_2, respectively, let $|u, v\rangle$ denote the (n_1+n_2)-dimensional "concatenated" column vector whose projection onto the first n_1 dimensions is $|u\rangle$ and whose projection onto the last n_2 dimensions is $|v\rangle$.

The adjacency matrix of Σ is

$$A(\Sigma) = \begin{bmatrix} -A(G_1) & J_{n_1,n_2} \\ J_{n_2,n_1} & A(G_2) \end{bmatrix} \tag{10}$$

If $|\alpha\rangle \neq |\mathbf{1}_{n_1}\rangle$ is an eigenvector of $A(G_1)$ with eigenvalue α, then $|\alpha, 0_{n_2}\rangle$ is an eigenvector of $A(\Sigma)$ with eigenvalue $-\alpha$. Similarly, if $|\beta\rangle \neq |\mathbf{1}_{n_2}\rangle$ is an eigenvector of $A(G_2)$ with eigenvalue β, then $|0_{n_1}, \beta\rangle$ is an eigenvector of $A(\Sigma)$ with eigenvalue β. The two remaining eigenvalues of $A(\Sigma)$ are the solutions of the quadratic equation

$$\lambda^2 + (k_1 - k_2)\lambda - (k_1 k_2 + n_1 n_2) = 0. \tag{11}$$

So, $\lambda_\pm = \tilde{\delta}_- \pm \Delta$ with the corresponding eigenvectors $|x_\pm \mathbf{1}_{n_1}, y_\pm \mathbf{1}_{n_2}\rangle$, where the two non-zero constants x and y are related through the equations

$$(\lambda + k_1)x = n_2 y, \quad (\lambda - k_2)y = n_1 x. \tag{12}$$

Letting $y = 1$, we get the normalized eigenvectors

$$|\lambda_\pm\rangle = \frac{1}{\sqrt{L_\pm}} |x_\pm \mathbf{1}_{n_1}, \mathbf{1}_{n_2}\rangle, \tag{13}$$

where $x_\pm = (\lambda_\pm - k_2)/n_1$ and $L_\pm = n_1 x_\pm^2 + n_2$.

Suppose $A(G_1) = \sum_\alpha \alpha E_\alpha$ is the spectral decomposition of $A(G_1)$. The quantum walk on Σ from vertex a to b (within the G_1 subgraph of Σ) is given by

$$\langle b, 0| e^{-itA(\Sigma)} |a, 0\rangle = \sum_{\alpha \neq k_1} e^{it\alpha} \langle b|E_\alpha|a\rangle + \sum_\pm \frac{x_\pm^2}{L_\pm} e^{-it\lambda_\pm}. \tag{14}$$

We complete the first term using the fact that for the dominant eigenvalue $\alpha = k_1$ we have $E_\alpha = (1/n_1)|\mathbf{1}_{n_1}\rangle\langle \mathbf{1}_{n_1}|$, and simplify the second, using $\lambda_\pm = \tilde{\delta}_- \pm \Delta$. This yields

$$\langle b, 0| e^{-itA(\Sigma)} |a, 0\rangle = \langle b| e^{itA(G_1)} |a\rangle - \frac{e^{itk_1}}{n_1} + e^{-it\tilde{\delta}_-} \left[\frac{1}{L_+ L_-} \sum_\pm e^{\mp it\Delta} x_\pm^2 L_\mp \right] \tag{15}$$

The last term may be simplified further using the following identities (whose proofs may be found in the Appendix):

$$\sum_\pm x_\pm = \frac{2\tilde{\delta}_+}{n_1}, \tag{16}$$

$$\prod_\pm x_\pm = -\frac{n_2}{n_1}, \tag{17}$$

$$\prod_\pm L_\pm = 4\Delta^2 \, \frac{n_2}{n_1}, \tag{18}$$

$$x_\pm^2 L_\mp = L_\pm \, \frac{n_2}{n_1}. \tag{19}$$

Thus, we have

$$\langle b,0|e^{-itA(\Sigma)}|a,0\rangle = \langle b|e^{itA(G_1)}|a\rangle - \frac{e^{itk_1}}{n_1} + \frac{e^{-it\tilde{\delta}_-}}{n_1}\left(\cos(t\Delta) - i\frac{\tilde{\delta}_+}{\Delta}\sin(t\Delta)\right). \tag{20}$$

By combining the last two terms using $k_1 + \tilde{\delta}_- = -\tilde{\delta}_+$, we obtain the claim. □

In what follows, we state some immediate corollaries of Theorem 1.

Corollary 2 *Let G_1 be a (n_1, k_1)-regular graph with perfect state transfer from vertex a to vertex b at time $t = \pi/D$, for some positive integer D. Let G_2 be a (n_2, k_2)-regular graph. Then, the signed graph $\Sigma = G_1^- + G_2^+$ has perfect state transfer from vertex a to vertex b at time $t = \pi/D$ if one of the following conditions hold:*

- $\Delta \equiv 0 \pmod{2D}$ *and* $k_1 + k_2 \equiv 0 \pmod{4D}$*;*
- $\Delta \equiv D \pmod{2D}$ *and* $k_1 + k_2 \equiv 2D \pmod{4D}$*,*

where $\Delta = \frac{1}{2}\sqrt{(k_1+k_2)^2 + 4n_1n_2}$.

Proof Consider a subpart of the second summand in Equation (9):

$$\Lambda \;=\; \cos(t\Delta) + \left(\frac{k_1+k_2}{2\Delta}\right) i\sin(t\Delta) - \exp\left(it\frac{(k_1+k_2)}{2}\right)$$

First, suppose that $\Delta \equiv 0 \pmod{2D}$ and $k_1 + k_2 \equiv 0 \pmod{4D}$. If $t = \pi/D$, then $\Delta \equiv 0 \pmod{2D}$ implies $t\Delta \in 2\mathbb{Z}\pi$ and $k_1 + k_2 \equiv 0 \pmod{4D}$ implies $t(k_1 + k_2)/2 \in 2\mathbb{Z}\pi$. This shows that $\cos(t\Delta) = 1$, $\sin(t\Delta) = 0$, and $\exp(it(k_1 + k_2)/2) = 1$, which implies $\Lambda = 0$.

Next, suppose that $\Delta \equiv D \pmod{2D}$ and $k_1 + k_2 \equiv 2D \pmod{4D}$. If $t = \pi/D$, then $\Delta \equiv D \pmod{2D}$ implies $t\Delta \in (2\mathbb{Z}+1)\pi$ and $k_1+k_2 \equiv 2D \pmod{4D}$ implies $t(k_1+k_2)/2 \in (2\mathbb{Z}+1)\pi$. This shows that $\cos(t\Delta) = -1$, $\sin(t\Delta) = 0$, and $\exp(it(k_1 + k_2)/2) = -1$, which implies $\Lambda = 0$.

In both cases, since $\Lambda = 0$, Equation (9) in Theorem 1 becomes

$$\langle b|e^{-itA(\Sigma)}|a\rangle = \langle b|e^{itA(G_1)}|a\rangle + \frac{e^{-it\tilde{\delta}_-}}{n_1}\Lambda = \langle b|e^{itA(G_1)}|a\rangle. \tag{21}$$

Therefore, Σ has perfect state transfer between a and b whenever G_1 does. □

Another immediate corollary of Theorem 1 is the following curious result which shows that the perfect state transfer time within a 2-clique can be made arbitrarily small by joining it with a suitably large 3-regular graph; see Figure 2.

Corollary 3 *Let G be a 3-regular graph on n vertices. Then, the signed graph $K_2^- + G^+$ has perfect state transfer between the two vertices of K_2 at time π/Δ where $\Delta = \sqrt{4+2n}$.*

Proof We apply Theorem 1 for $G_1 = K_2$ and G_2 is a $(n, 3)$-regular graph G. In this case, $\Delta = \sqrt{(k_1+k_2)^2/4 + n_1n_2} = \sqrt{4+2n}$ since $k_1 = 1$, $k_2 = 3$, $n_1 = 2$ and $n_2 = n$. Let a

and b denote the two distinct vertices of K_2. Then, Equation (9) for a quantum walk on $\Sigma = K_2^- + G^-$ at time $t = \pi/\Delta$ is given by:

$$\begin{aligned} \langle b|e^{-itA(\Sigma)}|a\rangle &= \langle b|e^{itA(K_2)}|a\rangle + \frac{e^{-it}}{2}\left[\left(\cos(t\Delta) + \frac{2i}{\Delta}\sin(t\Delta)\right) - e^{2it}\right] && (22)\\ &= i\sin(t) + \frac{e^{-it}}{2}\left[\left(\cos(\pi) + \frac{2i}{\Delta}\sin(\pi)\right) - e^{2it}\right] && (23)\\ &= i\sin(t) - \frac{e^{-it}}{2}\left[1 + e^{2it}\right] = -e^{-it}. && (24)\end{aligned}$$

This shows that Σ has perfect state transfer between vertices a and b at time $t = \pi/\Delta$. □

Remark: The above result does not hold on the unsigned join $K_2 + G$ if G is a $(n, 3)$-regular graph. For a (n, k)-regular graph G, Angeles-Canul *et al.* [1] proved that $K_2 + G$ has perfect state transfer between the vertices of the 2-clique provided $\Delta = \sqrt{(k-1)^2 + 8n}$ is an integer and that both $k - 1$ and Δ are divisible by 8. The last condition is clearly impossible when $k = 3$.

5 Decomposition

In this section, we exploit the fact that the adjacency matrix of a signed graphs may be decomposed into positive and negative parts. This decomposition defines the positive and negative subgraphs of a signed graph. We describe some results on perfect state transfer on signed graphs under certain assumptions on these subgraphs.

Theorem 2 *Let $G = (V, E)$ be a graph with perfect state transfer from vertex a to vertex b at time t. Suppose H is a spanning subgraph of $\overline{G}$ and that H is periodic at vertex a with time t. Then, the signed graph $\Sigma = G^+ \cup H^-$ has perfect state transfer from a to b at time t provided $A(G)$ and $A(H)$ commute.*

Proof Suppose G has perfect state transfer from vertex a to vertex b at time t and H is periodic at vertex a with time t, where $e^{-itA(H)}|a\rangle = e^{i\phi}|a\rangle$ for some real number ϕ. The adjacency matrix of $\Sigma = G^+ \cup H^-$ is given by $A(\Sigma) = A(G) - A(H)$. Then, the quantum walk on Σ is

$$\langle b|e^{-itA(\Sigma)}|a\rangle = \langle b|e^{-itA(G)}e^{itA(H)}|a\rangle = e^{i\phi}\langle b|e^{-itA(G)}|a\rangle. \tag{25}$$

This proves the claim. □

In the following result, we apply Theorem 2 to signed complete graphs.

Corollary 4 *Let G be a regular graph which has perfect state transfer from vertex a to vertex b at time t. Suppose that $\overline{G}$ is periodic at vertex a at time t. Then, the signed complete graph $\Sigma = G^+ \cup \overline{G}^-$ has perfect state transfer from vertex a to vertex b at time t.*

Proof Since G is regular, $A(G)$ commutes with J. The adjacency matrix of $\overline{G}$ is $A(\overline{G}) = J - I - A(G)$ which clearly commutes with $A(G)$. Thus, we may apply Theorem 2. □

Remark: Consider the n-partite graph $G = K_{2,2,\ldots,2}$ (also known as the cocktail party graph) which has antipodal perfect state transfer between vertices in the same partition at time $\pi/2$

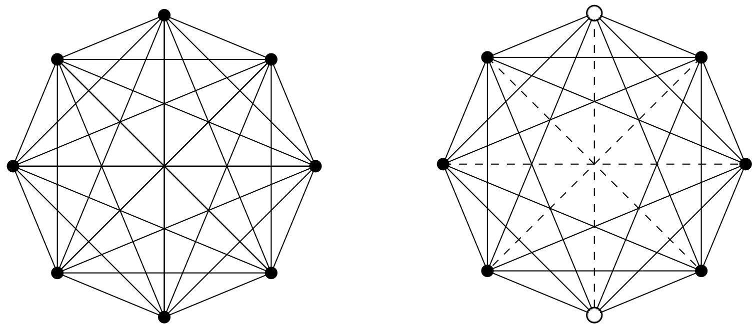

Fig. 3. More on signed complete graphs. (a) K_8 has no perfect state transfer. (b) The signed complete graph $K_8^{\pm}$ has perfect state transfer between vertices marked white. Dashed edges are negatively signed.

(see Bašić and Petkovic [4] and Angeles-Canul *et al.* [1]). If we view this graph as a circulant with $N = 2n$ vertices, then let P_σ denote the permutation $\sigma : x \mapsto x + N/2 \pmod N$. Note that σ is an automorphism of G. By Corollary 4, we have that the signed complete graph

$$K_N^{\pm} = G^+ \cup P_\sigma^- \tag{26}$$

has perfect state transfer between vertices x and $x+N/2$ at time $\pi/2$. Recall that no *unsigned* complete graph K_N has perfect state transfer for $N \geq 3$; see Figure 3.

Next, we use some useful constructions of perfect state transfer and periodic graphs from *cubelike* graphs which are Cayley graphs over the abelian group $\mathbb{Z}_2^d$.

Theorem 3 *(Cheung and Godsil [10])*
Let $C \subseteq \mathbb{Z}_2^d$ and let δ be the sum of the elements of C. If $\delta \neq 0$, then the Cayley graph $G = X(\mathbb{Z}_2^d, C)$ has perfect state transfer from u to $u + \delta$ at time $\pi/2$, for each $u \in \mathbb{Z}_2^d$. If $\delta = 0$, then G is periodic with period $\pi/2$.

Remark: Let $G = X(\mathbb{Z}_2^d, C)$ be a Cayley graph where δ, the sum of elements of C, is nonzero. Suppose P_δ is the $2^d \times 2^d$ permutation matrix representing the bijection $x \mapsto x + \delta$. Then, by Theorem 2, the signed cubelike graph

$$\Sigma = G^+ \cup P_\delta^- \tag{27}$$

has perfect state transfer from u to $u + \delta$ at time $\pi/2$, for each $u \in \mathbb{Z}_2^d$. But, the *unsigned* cubelike graph

$$\mathcal{G} = G^+ \cup P_\delta^+ \tag{28}$$

is merely periodic with period $\pi/2$; see Figure 4.

Double Covers Given a signed graph $\Sigma = (G, \sigma)$ where $G = (V, E)$, its *double cover* is an unsigned graph $\mathcal{G}$ with vertex set $V \times \{0, 1\}$ where for each $(u, v) \in E$ and $b \in \{0, 1\}$:

- (u, b) is adjacent to (v, b) if $\sigma(u, v) = +1$; and

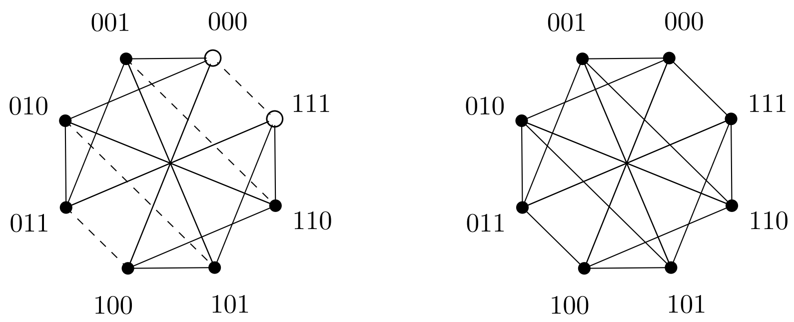

Fig. 4. (a) The cubelike graph $Q_3^+ \cup P_{111}^-$ has perfect state transfer between vertices marked white; dashed edges are negatively signed. (b) The (unsigned) cubelike graph $Q_3 \cup P_{111}$ is only periodic. Here, P_{111} represents the automorphism $x \mapsto x + 111$.

- (u, b) is adjacent to $(v, 1-b)$ if $\sigma(u, v) = -1$.

Alternatively, if the signed graph Σ has the decomposition $G^+ \cup G^-$, its double cover $\mathcal{G}$ is a graph whose adjacency matrix is

$$A(\mathcal{G}) = A(G^+) \otimes I + A(G^-) \otimes X. \tag{29}$$

Theorem 4 *Let $\Sigma = G^+ \cup G^-$ be a signed graph where $A(G^+)$ and $A(G^-)$ commute. Suppose that G^+ has perfect state transfer from vertex a to vertex b at time t and that G^- satisfies*

$$\langle b| \cos(A(G^-)t)|b\rangle = \pm 1. \tag{30}$$

Then, the double cover of Σ has perfect state transfer from $(a, 1)$ to $(b, 1)$ at time t.

Proof Let $\mathcal{G}$ be the double cover of Σ. Assume that G^+ has perfect state transfer from vertex a to vertex b at time t. More specifically, suppose that for a real number ϕ, we have

$$e^{-itA(G^+)}|a\rangle = e^{i\phi}|b\rangle \tag{31}$$

Therefore, the quantum walk on $\mathcal{G}$ from vertex a to vertex b is given by

$$\begin{aligned}
\langle b, 1|e^{-itA(\mathcal{G})}|a, 1\rangle &= \langle b, 1|e^{-itA(G^-)\otimes X}e^{-itA(G^+)\otimes I}|a, 1\rangle && (32)\\
&= e^{i\phi}\langle b, 1|e^{-itA(G^-)\otimes X}|b, 1\rangle && (33)\\
&= e^{i\phi}\langle b, 1|\left[\cos(tA(G^-)) \otimes I - i\sin(tA(G^-)) \otimes X\right]|b, 1\rangle && (34)\\
&= e^{i\phi}\langle b| \cos(tA(G^-))|b\rangle. && (35)
\end{aligned}$$

This yields the claim. □

Remark: Bernasconi *et al.* [5] showed that the cubelike graph $G_1 = X(\mathbb{Z}_2^d, C_1)$ has perfect state transfer from u to $u+\delta_1$ at time $\pi/2$, where δ_1 is the sum of all elements of C_1, provided $\delta_1 \neq 0$. When the sum of the generating elements is zero, Cheung and Godsil [10] proved that there are cubelike graphs $G_2 = X(\mathbb{Z}_2^d, C_2)$ that have perfect state transfer at time $\pi/4$. They showed that the latter cubelike graphs correspond to self-orthogonal projective binary codes that are even but not doubly even. Moreover, these cubelike graphs G_2 are periodic at time $\pi/2$ and satisfy

$$\langle u|\exp(-itA(G_2))|u\rangle = e^{-i\pi|C_2|/2}. \tag{36}$$

See Lemma 3.1 in [10] and the comments which followed it. So, G_2 is periodic with period ± 1 provided $|C_2|$ is even. Recall that the adjacency matrices of any two cubelike graphs commute since they share the same set of eigenvectors (namely, the columns of the Hadamard matrices). By Theorem 4, the *double cover* of the signed (multi)graph

$$\Sigma = G_1^+ \cup G_2^- \tag{37}$$

has perfect state transfer at time $\pi/2$ from u to $u + \delta_1$ for each $u \in \mathbb{Z}_2^d$.

6 Signed Quotients

Given a graph $G = (V, E)$, a vertex partition π given by $V = \biguplus_{k=1}^{m} V_k$ is called an *equitable partition* of G if for each j, k there are constants $d_{j,k}$ so that the number of neighbors in V_k of each vertex in V_j is $d_{j,k}$; this is independent of the choice of the vertex of V_j. That is, for each $x \in V_j$ we have

$$d_{j,k} = |N(x) \cap V_k|, \tag{38}$$

where $N(x) = \{y : (x, y) \in E\}$ is the set of neighbors of x. When the context is clear, we use π_j in place of V_j and $\pi(u)$ to denote the partition which contains vertex u. The size of the equitable partition is denoted $|\pi| = m$. Let P_π be the partition matrix of π defined as $\langle x|P_\pi|\pi_k\rangle = [\![x \in \pi_k]\!]$. It is more useful to work with the normalized partition matrix Q_π defined as:

$$Q_\pi = \sum_{k=1}^{m} \frac{1}{\sqrt{|\pi_k|}} P_\pi |\pi_k\rangle\langle\pi_k|. \tag{39}$$

For a signed graph $\Sigma = G^+ \cup G^-$, we say a vertex partition $\pi = \biguplus_{k=1}^{m} \pi_k$ is *equitable*[b] for Σ if π is an equitable partition for both G^+ and G^-. We use $d_{j,k}^+$ and $d_{j,k}^-$ to denote $d_{j,k}$ restricted to G^+ and G^-, respectively. Also, we let $d_{j,k}^\pm = d_{j,k}^+ - d_{j,k}^-$. Let $A(\Sigma/\pi)$ be a symmetric $m \times m$ matrix with rows and columns indexed by the partitions of π and whose entries are defined by

$$\langle\pi_j|A(\Sigma/\pi)|\pi_k\rangle = \frac{d_{j,k}^\pm}{|d_{j,k}^\pm|}\sqrt{|d_{j,k}^\pm d_{k,j}^\pm|}. \tag{40}$$

Here, $A(\Sigma/\pi)$ is the adjacency matrix of a weighted signed graph Σ/π (which we call the quotient of Σ modulo π).

The next lemma generalizes a result for unsigned graphs (see Godsil [15]).

Lemma 2 *Let $\Sigma = G^+ \cup G^-$ be a signed graph and π be an equitable partition of Σ with a normalized partition matrix Q_π. Then:*

1. $Q_\pi^T Q_\pi = I_{|\pi|}$.

[b] This might not be the only way to define equitable partitions for signed graphs.

2. $Q_\pi Q_\pi^T = \mathrm{diag}(|\pi_k|^{-1} J_{|\pi_k|})$.

3. $Q_\pi Q_\pi^T$ *commutes with* $A(\Sigma)$.

4. $A(\Sigma/\pi) = Q_\pi^T A(\Sigma) Q_\pi$.

Proof The first two properties hold since the columns of Q_π are the normalized characteristic vectors of the partition matrix P_π. The third property holds since for any a and b we have

$$\langle b|A(\Sigma)Q_\pi Q_\pi^T|a\rangle = \frac{1}{|\pi(a)|}\sum_{u\in\pi(a)}\langle b|A(\Sigma)|u\rangle = \frac{1}{|\pi(b)|}\sum_{v\in\pi(b)}\langle v|A(\Sigma)|a\rangle = \langle b|Q_\pi Q_\pi^T A(\Sigma)|a\rangle. \tag{41}$$

Given that π is equitable for both G^+ and G^-, we have

$$d_{j,k}^{\pm}|\pi_j| = d_{k,j}^{\pm}|\pi_k|. \tag{42}$$

Since $A(\Sigma) = A(G^+) - A(G^-)$, we have

$$\langle \pi_j|Q_\pi^T A(\Sigma)Q_\pi|\pi_k\rangle = \frac{\langle \pi_j|P_\pi^T A(\Sigma)P_\pi|\pi_k\rangle}{\sqrt{|\pi_j||\pi_k|}} = \frac{d_{j,k}^{\pm}}{\sqrt{|\pi_j||\pi_k|}} = \pm\sqrt{|d_{j,k}^{\pm}d_{k,j}^{\pm}|}, \tag{43}$$

where the sign is $+1$ if $d_{j,k}^{\pm} > 0$, and -1 otherwise. This proves property the last property. □

The following theorem generalizes a result in Bachman *et al.* [3] on the equivalence of perfect state transfer on unsigned graphs and their quotients. For completeness, we provide the proof which follows from Lemma 2.

Theorem 5 *Let* Σ *be a signed graph with an equitable partition* π *where vertices* a *and* b *belong to singleton cells. Then, for any time* t

$$\langle b|\exp(-itA(\Sigma))|a\rangle = \langle \pi(b)|\exp(-itA(\Sigma/\pi))|\pi(a)\rangle. \tag{44}$$

Proof Since $A(\Sigma)$ commutes with QQ^T, we have $(QQ^TA(\Sigma))^k = A(\Sigma)^k QQ^T$ for $k \geq 1$. Given that a and b are in singleton cells, $|\pi(a)\rangle = Q^T|a\rangle$ and $|\pi(b)\rangle = Q^T|b\rangle$. Thus, we have

$$\begin{aligned}
\langle \pi(b)|e^{-itA(\Sigma/\pi)}|\pi(a)\rangle &= \langle \pi(b)|e^{-itQ^TA(\Sigma)Q}|\pi(a)\rangle && (45)\\
&= \langle b|Q\left[\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}(Q^TA(\Sigma)Q)^k\right]Q^T|a\rangle && (46)\\
&= \langle b|\left[\sum_{k=0}^{\infty}\frac{(-it)^k}{k!}(QQ^TA(\Sigma))^k\right]QQ^T|a\rangle && (47)\\
&= \langle b|e^{-itA(\Sigma)}QQ^T|a\rangle, && (48)
\end{aligned}$$

which proves the claim since $QQ^T|a\rangle = |a\rangle$ because a belongs to a singleton cell. □

7 Many-Particle Quantum Walks

In this section, we describe graph operators which arise naturally from k-particle quantum walk on an underlying graph G. These operators include symmetric powers, Cartesian quotients, and *signed* exterior powers.

- The weighted *Cartesian quotients* were described by Feder [14] as a generalization of graphs studied by Christandl *et al.* [13, 12]. The construction in [14] is explicitly based on many-boson quantum walks on graphs. Subsequently, Bachman *et al.* [3] showed that these graphs $G^{\odot k}$ described by Feder are quotients of k-fold Cartesian product modulo a natural equitable partition.

- The *symmetric powers* $G^{\{k\}}$ were studied by Audenaert *et al.* [2] in the context of graph isomorphism on strongly regular graphs. These graphs are equivalent to the k-tuple vertex graphs introduced by Zhu *et al.* [26] and were were later studied by Osborne [21] in connection with quantum spin networks. It can be shown that these graphs are based on many-particle quantum walks with hardcore bosons.

- The *exterior powers* $\bigwedge^k G$ and their perfect state transfer properties as *signed* graphs will be our main focus in this section. We formally define these graphs in the next section and connect them with many-fermion quantum walks.

A common trait shared by these families of graphs is that they are derived from a k-fold Cartesian product $G^{\square k}$ of an underlying graph G. The k-fold Cartesian product represents, in a natural way, a k-particle quantum walk on the graph G where the particles are distinguishable. Here, the edges of the Cartesian product represent scenarios where a single particle hops from its current vertex to a neighboring one. Since quantum particles are either fermions or bosons, this influences the resulting graph-theoretic construction in an interesting manner.

In the fermionic regime, no two particle may occupy the same vertex due to the Pauli exclusion principle. Mathematically, this requires the removal of a "diagonal" set $\mathcal{D}$ consisting of all k-tuples with a repeated vertex. On the other hand, in the bosonic regime, the particles are allowed to occupy the same vertex. But in the so-called *hardcore bosonic* model, we are also required to remove the diagonal since these bosons are not allowed to occupy the same vertex although they lack the antisymmetric exchange property present in fermions.

Since quantum particles are indistinguishable, the final step in these constructions "collapses" certain configurations of the particles in the k-fold Cartesian product. Mathematically, this simply involves a conjugation by either the symmetrizer (bosonic) or the anti-symmetrizer (fermionic); more specifically, this is a projection to either the symmetric or anti-symmetric subspace of the underlying k-fold tensor product space (see Bhatia [6]).

Therefore, the full construction of a graph $\mathcal{G}$ based on a k-particle quantum walk on G may be described via its adjacency matrix as:

$$A(\mathcal{G}) = \mathcal{Q}^{\dagger} A(G^{\square k} \setminus \mathcal{D})\mathcal{Q}. \tag{49}$$

In Equation (49), the matrix $\mathcal{Q}$ is the symmetrizer $P_{\vee}$ in the case of Cartesian quotients, the anti-symmetrizer $P_{\wedge}$ in the case of exterior powers, or a hybrid in the case of symmetric powers. Here, we borrow the notation $P_{\vee}$ and $P_{\wedge}$ from Bhatia [6] (see also Audenaert *et al.*

Graph	Notation	Particle	Citation
weighted Cartesian quotients	$G^{\odot k}$	bosonic	Feder [14], Bachman *et al.* [3]
symmetric powers k-tuple vertex graphs	$G^{\{k\}}$ $U_k(G)$	hardcore bosonic	Audenaert *et al.* [2] Zhu *et al.* [26], Osborne [21]
signed exterior powers	$\bigwedge^k G$	fermionic	this work

Fig. 5. Graph operators from many-particle quantum walks on graphs.

[2]). The diagonal set $\mathcal{D}$ is removed in all constructions except for Cartesian quotients; for exterior powers, this diagonal removal is implicitly done via the anti-symmetrizer $P_\wedge$. Overall, we will view $\mathcal{G}$ as a "quotient" of $G^{\Box k}$ modulo a suitable equitable partition.

Cartesian quotients For a graph G on n vertices and a positive integer k, the Cartesian quotient $G^{\odot k}$ is a graph whose vertex set is the set of n-tuple of non-negative integers whose sum is k, that is,

$$V(G^{\odot k}) = \left\{ a \in \mathbb{N}^{V(G)} : \sum_{v \in V(G)} a_v = k \right\}, \tag{50}$$

and whose edges are the pairs (a, b) which satisfy

$$(\exists (u,v) \in E(G))[b_u = a_u - 1 \ \wedge \ b_v = a_v + 1 \ \wedge \ (\forall w \notin \{u,v\})[a_w = b_w]] \tag{51}$$

with an edge weight of $\sqrt{(a_u - 1)(a_v + 1)}$. This construction corresponds to a k-**boson** quantum walk on G where the adjacency matrix of $G^{\odot k}$ is given by

$$A(G^{\odot k}) = P_\vee^\dagger A(G^{\Box k}) P_\vee, \tag{52}$$

and the resulting graph is weighted and unsigned. This class of graphs was studied by Feder [14] and then by Bachman *et al.* [3].

Symmetric powers For a graph G on n vertices and a positive integer k where $1 \leq k \leq n$, the symmetric kth power $G^{\{k\}}$ of G is a graph whose vertices are the k-subsets of $V(G)$ and whose edges consist of pairs of k-subsets (A, B) for which $A \triangle B \in E(G)$. This construction corresponds to a k-**hardcore boson** quantum walk on G where bosons are not allowed to occupy the same vertex but without the fermionic antisymmetric exchange rule. Here, we have

$$A(G^{\{k\}}) = (P^{(k)})^\dagger A(G^{\Box k} \setminus \mathcal{D}) P^{(k)}, \tag{53}$$

where $P^{(k)}$ is a "signless" variant of $P_\wedge$ (see Audenaert *et al.* [2]) and $\mathcal{D}$ consists of k-tuples from V^k which contain a repeated vertex. This class of unweighted and unsigned graphs was studied by Audenaert *et al.* [2] and also by Zhu *et al.* [26] and Osborne [21].

Exterior powers For a graph G on n vertices and a positive integer k where $1 \leq k \leq n$, the exterior kth power $\bigwedge^k G$ is a graph whose vertices are the *ordered* k-subsets of $V(G)$ and whose edges consist of pairs of ordered k-subsets (A, B) for which $A \triangle B \in E(G)$. Moreover, if $\pi \in S_k$ is the unique permutation which yields $A_{\pi(i)} = B_i$, for all i except for one j where $(A_{\pi(j)}, B_j) \in E(G)$, then the edge (A, B) is assigned the sign $\mathrm{sgn}(\pi)$. This construction is based on a k-**fermion** quantum walk on G. Here, we have

$$A(\textstyle\bigwedge^k G) = P_\wedge^\dagger A(G^{\Box k}) P_\wedge, \tag{54}$$

where $P_\wedge$ implicitly removes a diagonal set consisting of k-tuples from V^k with repeated vertices. The resulting graph is *signed*. We remark that the graphs $U_k(G)$ studied in [26, 21] are simply the exterior powers $\bigwedge^k G$ with the signs ignored.

We will define the exterior power $\bigwedge^k G$ more formally in the following section.

7.1 Exterior Powers

Given a graph $G = (V, E)$ and a positive integer k, where $1 \le k \le |V| - 1$, the *exterior kth power* of G, which is denoted $\bigwedge^k G$, is defined as follows. Suppose $V = \{v_1, \ldots, v_n\}$ is the vertex set of G. We fix some (arbitrary) total ordering $\prec$ on V, say, $v_1 \prec \ldots \prec v_n$. Let $\binom{V}{k}$ be the k-subsets of V where each subset is ordered according to $\prec$. That is, we have

$$\binom{V}{k} = \{(u_1, \ldots, u_k) \in V^k : u_1 \prec \ldots \prec u_k\}. \tag{55}$$

The elements of $\binom{V}{k}$ are often denoted as a "wedge" product $u_1 \wedge \ldots \wedge u_k$, again under the condition that $u_1 \prec \ldots \prec u_k$.

The vertex set of $\bigwedge^k G$ is $\binom{V}{k}$. The edge set of $\bigwedge^k G$ consists of pairs $\bigwedge_{j=1}^k u_j$ and $\bigwedge_{j=1}^k v_j$ for which there is a bijection π over $[n]$ so that $u_{\pi(j)} = v_j$ for all j except at one index i where $(u_{\pi(i)}, v_i) \in E(G)$; most importantly, the *sign* of this edge is defined to be $\mathrm{sgn}(\pi)$. Thus, $\bigwedge^k G$ is a *signed* graph on $\binom{n}{k}$ vertices. In summary, we have

$$V(\textstyle\bigwedge^k G) = \binom{V}{k} \tag{56}$$

$$E(\textstyle\bigwedge^k G) = \left\{ (\bigwedge_{j=1}^k u_j, \bigwedge_{j=1}^k v_j) : (\exists i)[(u_{\pi(i)}, v_i) \in E \ \wedge \ (\forall j \neq i)[u_{\pi(j)} = v_j]] \right\} \tag{57}$$

Example: Let $V = \{a, b, c, d\}$ be ordered as $a \prec b \prec c \prec d$. Then,

$$\binom{V}{2} = \{a \wedge b, a \wedge c, a \wedge d, b \wedge c, b \wedge d, c \wedge d\}. \tag{58}$$

Suppose G is a 4-cycle defined on V where a and d are not adjacent but are both adjacent to b and c; see Figure 6. Here, $\bigwedge^2 G$ is a signed $K_{2,4}$ with the bipartition given by $\{a \wedge d, b \wedge c\}$ and $\{a \wedge b, a \wedge c, b \wedge d, c \wedge d\}$. All edges are positive except for the edges between $a \wedge b$ and $b \wedge c$ (since this requires a transposition to "align" the common vertex b) and between $c \wedge d$ and $b \wedge c$ (by a similar reasoning).

The algebraic view of $\bigwedge^k G$ will be more convenient for our purposes. For a permutation $\pi \in S_k$ and a k-tuple $u = (u_1, \ldots, u_k)$, we denote $\pi(u)$ to mean the element $(u_{\pi(1)}, \ldots, u_{\pi(k)})$. We consider the *anti-symmetrizer* operator $\mathsf{Alt}_{n,k}$ defined as

$$\mathsf{Alt}_{n,k} = \sum_{v \in \binom{V}{k}, \pi \in S_k} \frac{\mathrm{sgn}(\pi)}{\sqrt{k!}} |\pi(v)\rangle\langle v|. \tag{59}$$

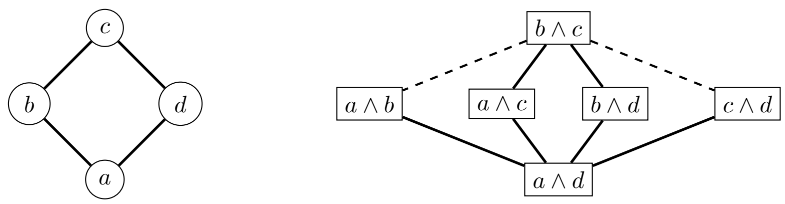

Fig. 6. (a) The four-cycle C_4; (b) Its exterior square $\bigwedge^2 C_4$ under the alphabetic vertex ordering $a \prec b \prec c \prec d$. Dashed edges are negatively signed.

So, $\mathsf{Alt}_{n,k}$ is an injective map from the exterior vector space $\mathbb{C}^{\binom{V}{k}}$ to the tensor product space $\mathbb{C}^{V^{\otimes k}}$. For each $v \in \binom{V}{k}$, we have

$$\mathsf{Alt}_{n,k}|v\rangle = \sum_{\pi \in S_k} \frac{\mathrm{sgn}(\pi)}{\sqrt{k!}}|\pi(v)\rangle. \tag{60}$$

Moreover, $\mathsf{Alt}_{n,k}^T$ is a surjective map from $\mathbb{C}^{V^{\otimes k}}$ to $\mathbb{C}^{\binom{V}{k}}$. For each $u \in V^{\otimes k}$,

$$\mathsf{Alt}_{n,k}^T|u\rangle = \frac{\mathrm{sgn}(\pi)}{\sqrt{k!}}|v\rangle, \tag{61}$$

where $v \in \binom{V}{k}$ satisfies $\pi(u) = v$; here, π is the permutation that orders u according to the total order $\prec$. Next, we show that $\mathsf{Alt}_{n,k}$ defines a "signed" equitable partition on $G^{\Box k}$. This shows that $\bigwedge^k G$ is the quotient graph of $G^{\Box k}$ induced by $\mathsf{Alt}_{n,k}$.

Lemma 3 *Let G be a graph on n vertices. For each positive integer k with $1 \leq k \leq n-1$:*

1. $\mathsf{Alt}_{n,k}^T \mathsf{Alt}_{n,k} = I_{\binom{n}{k}}$.
2. $\mathsf{Alt}_{n,k} \mathsf{Alt}_{n,k}^T$ *commutes with* $A(G^{\Box k})$.
3. $A(\bigwedge^k G) = \mathsf{Alt}_{n,k}^T A(G^{\Box k}) \mathsf{Alt}_{n,k}$.

Proof The first property holds since the columns of $\mathsf{Alt}_{n,k}$ are normalized and form a vertex partition. For the second property, we note that for each $a \in V^{\otimes k}$:

$$\mathsf{Alt}_{n,k}\mathsf{Alt}_{n,k}^T|a\rangle = \frac{1}{k!}\sum_{\pi \in S_k} \mathrm{sgn}(\pi) P_\pi |a\rangle, \tag{62}$$

where P_π denotes the permutation matrix which encodes the action of π on $V^{\otimes k}$. Moreover, $P_\pi A(G^{\Box k}) = A(G^{\Box k}) P_\pi$ for each permutation $\pi \in S_k$. Therefore, for each $a, b \in V^{\otimes k}$ we have

$$\begin{aligned} \langle a|\mathsf{Alt}_{n,k}\mathsf{Alt}_{n,k}^T A(G^{\Box k})|b\rangle &= \frac{1}{k!}\sum_{\pi \in S_k} \mathrm{sgn}(\pi)\langle a|P_\pi A(G^{\Box k})|b\rangle & (63)\\ &= \frac{1}{k!}\sum_{\pi \in S_k} \mathrm{sgn}(\pi)\langle a|A(G^{\Box k})P_\pi|b\rangle & (64)\\ &= \langle a|A(G^{\Box k})\mathsf{Alt}_{n,k}\mathsf{Alt}_{n,k}^T|b\rangle. & (65) \end{aligned}$$

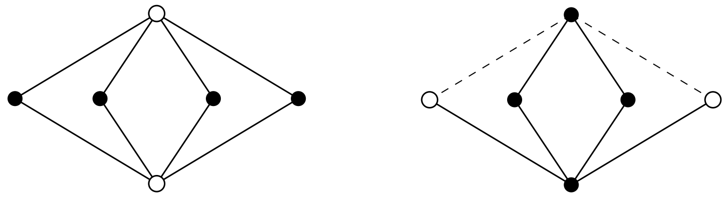

Fig. 7. Complete bipartite graphs. (a) The unsigned $K_{2,4}$ has perfect state transfer (since its quotient is P_3). (b) The exterior square $\bigwedge^2 C_4$ has perfect state transfer (by Theorem 6) but on a different pair of vertices; dashed edges are negatively signed. Perfect state transfer occurs between vertices marked white within each case.

The third property holds since for each $a, b \in \binom{V}{k}$ we have

$$\begin{aligned}
\langle b|\mathsf{Alt}_{n,k}^T A(G^{\square k})\mathsf{Alt}_{n,k}|a\rangle &= \frac{1}{k!}\sum_{\pi_1,\pi_2} \mathrm{sgn}(\pi_1 \circ \pi_2)\langle b|P_{\pi_1}^T A(G^{\square k})P_{\pi_2}|a\rangle && (66)\\
&= \frac{1}{k!}\sum_{\pi_1,\pi_2} \mathrm{sgn}(\pi_1 \circ \pi_2)\langle b|A(G^{\square k})P_{\pi_1\circ\pi_2}|a\rangle && (67)\\
&= \sum_{\pi} \mathrm{sgn}(\pi)\langle b|A(G^{\square k})P_{\pi}|a\rangle. && (68)
\end{aligned}$$

The last expression is either 0 or ± 1 and agrees with the definition of adjacency on $\bigwedge^k G$. $\square$

Next, we show that if a graph G has k disjoint pairs of vertices with perfect state transfer, then so does the exterior kth power of G.

Theorem 6 *Let G be a graph with perfect state transfer at time t from vertex a_j to vertex b_j, for each j with $1 \le j \le k$. Suppose that the collection $\{(a_j, b_j) : 1 \le j \le k\}$ are pairwise disjoint. Then, $\bigwedge^k G$ has perfect state transfer from vertex $\bigwedge_{j=1}^k a_j$ to vertex $\bigwedge_{j=1}^k b_j$.*

Proof Let $\hat{a} = \bigwedge_{j=1}^k a_j$ and $\hat{b} = \bigwedge_{j=1}^k b_j$. The quantum walk on $\bigwedge^k G$ from $\hat{a}$ to $\hat{b}$ is given by

$$\begin{aligned}
&\langle \hat{b}| \exp\left[-itA(\textstyle\bigwedge^k G)\right]|\hat{a}\rangle && (69)\\
&= \langle \hat{b}|\mathsf{Alt}_{n,k}^T \exp\left(-itA(G^{\square k})\right)\mathsf{Alt}_{n,k}|\hat{a}\rangle && (70)\\
&= \frac{1}{k!}\left[\sum_{\pi_1}\mathrm{sgn}(\pi_1)\langle\pi_1(\hat{b})|\right]\exp\left(-itA(G^{\square k})\right)\left[\sum_{\pi_2}\mathrm{sgn}(\pi_2)|\pi_2(\hat{a})\rangle\right] && (71)\\
&= \frac{1}{k!}\sum_{\pi_1,\pi_2}\mathrm{sgn}(\pi_1\circ\pi_2)\langle\pi_1(\hat{b})|\exp\left(-itA(G^{\square k})\right)|\pi_2(\hat{a})\rangle && (72)\\
&= \frac{1}{k!}\sum_{\pi}\langle\pi(\hat{b})|\exp\left(-itA(G^{\square k})\right)|\pi(\hat{a})\rangle. && (73)
\end{aligned}$$

This proves the claim. $\square$

8 Conclusions

In this work, we studied quantum walks on signed graphs. Our goal was to understand the effect of negatively signed edges on perfect state transfer. We explore this question in the cases of Cartesian products and graph joins. For the latter, we show that the join of a negative 2-clique with any 3-regular graph has perfect state transfer (unlike the unsigned join); moreover, the perfect state transfer time improves as the size of the 3-regular graph increases. Using the natural spanning-subgraph decomposition of signed graphs, we consider signed graphs obtained from regular graphs and their complements. This yields signed complete graphs which have perfect state transfer; in contrast, unsigned complete graphs do not exhibit perfect state transfer but are merely periodic. Most of our results show that signed graphs from multiple switching-equivalent classes of the same underlying graph may simultaneously have perfect state transfer. Finally, we consider a graph operator called exterior powers which had been studied elsewhere (but not in the context of signed graphs). We prove conditions for which the exterior power of a graph has perfect state transfer. The exterior graph powers may be of independent interest especially in relation to many particle quantum walk on graphs.

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Appendix A

Here, we derive the identities used in the proof of Theorem 1.

Claim 1 $\sum_{\pm} x_{\pm} = 2\tilde{\delta}_{+}/n_1$.

Proof Consider following derivation:

$$\begin{aligned} x_+ + x_- &= \frac{1}{n_1}(\lambda_+ + \lambda_- - 2k_2), \quad \text{since } x_\pm = (\lambda_\pm - k_2)/n_1 && \text{(A.1)} \\ &= \frac{2}{n_1}(\tilde{\delta}_- - k_2), \quad \text{since } \lambda_\pm = \tilde{\delta}_- \pm \Delta && \text{(A.2)} \\ &= \frac{1}{n_1}(-\delta_- - 2k_2), \quad \text{since } \tilde{\delta}_\pm = -\delta_\pm/2 && \text{(A.3)} \\ &= \frac{1}{n_1}(-\delta_+) = 2\tilde{\delta}_+/n_1, \quad \text{since } \delta_\pm = k_1 \pm k_2 && \text{(A.4)} \end{aligned}$$

□

Claim 2 $\prod_{\pm} x_{\pm} = -n_2/n_1$.

Proof We have the following derivation:

$$
\begin{aligned}
x_+x_- &= \frac{1}{n_1^2}(\lambda_+\lambda_- - k_2(\lambda_+ + \lambda_-) + k_2^2), \quad \text{since } x_\pm = (\lambda_\pm - k_2)/n_1 && \text{(A.5)}\\
&= \frac{1}{n_1^2}(\tilde{\delta}_-^2 - \Delta^2 - 2k_2\tilde{\delta}_- + k_2^2), \quad \text{since } \lambda_\pm = \tilde{\delta}_- \pm \Delta && \text{(A.6)}\\
&= \frac{1}{n_1^2}(\tilde{\delta}_-^2 - \Delta^2 + k_2(k_2 + \delta_-)), \quad \text{since } \tilde{\delta}_\pm = -\tfrac{1}{2}\delta_\pm && \text{(A.7)}\\
&= \frac{1}{n_1^2}(\tilde{\delta}_-^2 - \Delta^2 + k_1k_2), \quad \text{since } \delta_\pm = k_1 \pm k_2 && \text{(A.8)}\\
&= -n_2/n_1, \quad \text{since } \Delta^2 = \tilde{\delta}_+^2 + n_1n_2 = \tilde{\delta}_-^2 + k_1k_2 + n_1n_2 && \text{(A.9)}
\end{aligned}
$$

□

Claim 3 $\prod_\pm L_\pm = 4\Delta^2(n_2/n_1)$.

Proof The derivation is as follows:

$$
\begin{aligned}
L_+L_- &= n_1^2(x_+x_-)^2 + n_1n_2(x_+^2 + x_-^2) + n_2^2, \quad \text{since } L_\pm = n_1x_\pm^2 + n_2 && \text{(A.10)}\\
&= n_1^2\left(-\frac{n_2}{n_1}\right)^2 + n_1n_2((x_+ + x_-)^2 - 2x_+x_-) + n_2^2, \quad \text{by Claim 1} && \text{(A.11)}\\
&= n_2^2 + n_1n_2\left(\frac{4\tilde{\delta}_+^2}{n_1^2} + 2\frac{n_2}{n_1}\right) + n_2^2, \quad \text{by Claims 1 and 2} && \text{(A.12)}\\
&= 2n_2^2 + \frac{n_2}{n_1}(4\tilde{\delta}_+^2 + 2n_1n_2) = 2n_2^2 + \frac{n_2}{n_1}(2(\tilde{\delta}_+^2 + n_1n_2) + 2\tilde{\delta}_+^2) && \text{(A.13)}\\
&= 2n_2^2 + \frac{n_2}{n_1}(2\Delta^2 + 2\tilde{\delta}_+^2), \quad \text{since } \Delta^2 = \tilde{\delta}_+^2 + n_1n_2 && \text{(A.14)}\\
&= \frac{n_2}{n_1}\ (2n_1n_2 + 2\Delta^2 + 2\tilde{\delta}_+^2) = 4\Delta^2(n_2/n_1). && \text{(A.15)}
\end{aligned}
$$

□

Claim 4 $x_\pm^2 L_\mp = (n_2/n_1)L_\pm$.

Proof We have the following derivation:

$$
\begin{aligned}
x_\pm^2 L_\mp &= x_\pm^2(n_1x_\mp^2 + n_2), \quad \text{since } L_\pm = n_1x_\pm^2 + n_2 && \text{(A.16)}\\
&= n_1(x_+x_-)^2 + n_2x_\pm^2 = n_1\left(-\frac{n_2}{n_1}\right)^2 + n_2x_\pm^2, \quad \text{by Claim 2} && \text{(A.17)}\\
&= \frac{n_2}{n_1}(n_2 + n_1x_\pm^2) = (n_2/n_1)L_\pm. && \text{(A.18)}
\end{aligned}
$$

□