New Modeling Techniques for Particle Flows Using The Proper Orthogonal Decomposition

B. J. O'Donnell* and B. T. Helenbrook
Department of Mechanical and Aeronautical Engineering
Clarkson University, Potsdam, NY 13699

Abstract
The Proper Orthogonal Decomposition (POD) is a reduced-order modeling technique that is used to compactly represent unsteady flows. In this paper, we use the POD to capture the parametric variation of a flow with Reynolds number. We study incompressible, axisymmetric, steady flow over spherical particles at various Reynolds numbers in order to give an alternative to correlation-based approaches for predicting the drag on a sphere. In previous applications of the POD, only the velocity field was studied; The pressure field was neglected. Since we are interested in drag, which is dependent on the pressure, we formulate the method to properly include the pressure field of an incompressible flow. The POD modes are then derived from numerical flow solutions obtained using an $hp$-finite element method. A reduced-order model is created by performing a streamwise-upwind-Petrov-Galerkin (SUPG) projection of the incompressible Navier-Stokes equations onto the space spanned by the POD modes. The importance of using this approach versus the more standard Galerkin projection is demonstrated. An efficient numerical implementation is also developed using a Taylor expansion of the SUPG projection of the Navier-Stokes equations. Finally, values of drag are computed from the reduced-order model. Drag can be calculated to within 1.0% of the direct numerical simulations using only a small number of modes while still retaining all of the essential physics around the particle.

*Corresponding Author
Introduction

The study of drops and particles is of importance to many natural and industrial processes [1], [2]. Such processes include raindrops and hailstones, atmospheric particle transport, particle transport by rivers and streams, sprays, and aerosols. To simulate these problems, often one uses an Eulerian grid to solve the governing equations for the fluid motion and a Lagrangian approach to track the particles. The coupling between the particle and the flow is predicted using a simplified model for the fluid drag on the particle. These simplified models have been developed through analytic, experimental, and numerical studies of individual drops and particles [3].

In Stokes flow, an analytic equation for predicting the particle evolution is available called the Basset-Boussinesq-Oseen (BBO) equation [4], [5]. The BBO equation includes many terms that govern the motion of the particle. These include: forces due to pressure gradients, an “apparent mass” term, a drag force, a lift force, a Basset time history term, and gravitational effects. Unfortunately, at higher Reynolds numbers, there is no analytic model for the terms in the BBO equation. There are empirical laws available for predicting the drag of a sphere [3], but it is not known how to model many of the additional effects accurately. Numerical studies have been performed [6], [7] at finite Reynolds numbers to find the forces related to the particle evolution equation. However, it is difficult to assemble this information into a reliable model for predicting particle evolution.

In this paper, we propose a new modeling technique that can accurately describe the fundamental physics of particle flows through the use of reduced-order modeling. One widely used technique is the Proper Orthogonal Decomposition (POD) [8]. The goal of this technique is to reduce the degrees of freedom needed to numerically solve a system of partial differential equations by introducing basis functions specifically adapted to the system. The POD is typically used to represent unsteady turbulent flow solutions using a compact basis. Our idea is to use similar techniques to develop an accurate sub-grid scale model of particle flows that can be used over a range of Reynolds numbers.

The POD has been used in a variety of applications to analyze experimental and numerical data in order to extract dominant features and trends. It is typically applied to unsteady data obtained in flow experiments or numerical simulations to generate a set of orthogonal basis functions that optimally represent the flow’s kinetic energy [8]. Typically the POD is used to study turbulent flows, but it can also be used to extract a basis for flows with varying parameters or geometry [9].

Using the POD, the governing equations can be projected onto a finite number of these basis functions to generate a reduced-order model of the flow. With this reduced-order model, a low-dimensional simulation can be performed that keeps the essential physics of the flow. Low-dimensional simulations have been performed for a variety of applications [10] that span from laminar to turbulent flows and compressible to incompressible flows. Aubry et al. [11] considered the motion of coherent structures in the wall region of a turbulent boundary layer. Rempe and Fasel [12] performed numerical studies to capture the evolution of coherent structures in a flat-plate boundary layer. The method has also been applied for simulating three-dimensional flow around a cylinder [13], [14] along with other complex geometries such as grooved channels [15]. The general aim of most of the studies was to either understand the essential physics of the flow or to develop reduced-order models for flow control.

Our goal is to predict how a particle evolves in a flow using reduced-order modeling. We will modify the POD technique to make it more suitable for particle modeling because a major goal of the modeling is to accurately predict the drag force, which is heavily dependent on the pressure. Typically the POD is applied to incompressible flows to derive a basis to represent the flow velocities optimally but the pressure is neglected. In the past, reduced-order models for the pressure have been found by applying the POD directly to the pressure field itself [16] or by solving a Poisson equation using the velocity eigenfunctions. This work provides a new simplified way to consistently incorporate the pressure in the POD formulation. This formulation is then used to develop an accurate model of flow over a sphere for a finite range of Reynolds numbers. This model can be used as a sub-grid scale model in a larger dilute particle-laden flow or spray simulation.

The paper begins with an overview of the POD formulation, and a new method is presented for incompressible flow that retains the velocity and the pressure. Next, a formulation of the governing equations for the particle flow problem is given and the details of the numerical method used to solve this problem are presented. The method of combining the POD and our numerical method is then given. Details on how we develop efficient low-dimensional models are then presented from our choice of discretization schemes. Using these techniques, a reduced-order model of particle flows is generated and results for spherical particles are then
presented to demonstrate the accuracy of the technique.

**POD Based Modeling**

The POD was introduced by Lumley [8] as a means to extract the large scale structure of turbulent flows. It uses data obtained in experiments or numerical simulations to generate an orthogonal set of spatial basis functions that optimally describe the flow’s energy. To find the basis functions or “POD modes,” we first need to introduce \( R(\vec{x}, \vec{x}') \), a time-averaged autocorrelation tensor

\[
R(\vec{x}, \vec{x}') = \langle \vec{u}(\vec{x}, t)\vec{u}^T(\vec{x}', t) \rangle,
\]

where the angled brackets indicate a time average and \( \vec{u}(\vec{x}, t) \) represents the flow velocity. The modes are then found by solving the Fredholm equation

\[
\int_{\mathcal{X}} R(\vec{x}, \vec{x}') \vec{\psi}(\vec{x}') d\vec{x}' = \lambda \vec{\psi}(\vec{x}),
\]

where the eigenvalues, \( \lambda \), represent the mean kinetic energy captured by each mode or eigenfunction, \( \vec{\psi} \). Once the modes are found using these equations, the flow field can be reconstructed using a linear combination of the modes.

When applied to data obtained from numerical simulations or experiments, the discrete form of Eq. 2 becomes an \( N \) dimensional eigenvalue problem where \( N \) is the number of data points. Often numerical and experimental data sets are highly resolved (\( N \gg 1 \)), which makes it difficult to solve this problem. A more efficient method proposed by Sirovich [17] allows us to reduce the size of the eigenvalue problem. The main idea of Sirovich’s method is to write the POD modes as a linear combination of “snapshots” of the instantaneous flow field

\[
\vec{\psi} = \sum_{k=1}^{M} \psi_k \vec{u}_k,
\]

where the sum is performed over the number of snapshots, \( M \), used in the discrete time average for Eq. 1. By substituting this equation into Eq. 2 we arrive at the following analogous eigenvalue problem

\[
C \vec{\psi} = \lambda \vec{\psi},
\]

whose discrete matrix entries are given as

\[
C_{kl} = \frac{1}{M} \int_{\mathcal{X}} u_i(\vec{x}, t_k)u_i(\vec{x}, t_l) d\vec{x},
\]

where the repeated \( i \) subscript implies summation over the velocity components. This method involves solving an eigenvalue problem that has dimension \( M \). The snapshot method is thus more efficient whenever the number of snapshots, \( M \), is smaller than the number of grid points, \( N \).

In most applications of the POD only the velocity field is decomposed and the pressure is ignored. For our application, which will be discussed in the sections to follow, pressure plays an important role and must be incorporated into the formulation. One way to do this is to define a vector \( q_i \) as the dimensionless velocity and pressure \( (u(\vec{x}, t), v(\vec{x}, t), p(\vec{x}, t)) \) for \( i = 1, 2, 3 \) respectively. We scale the velocity field by the mean flow, \( \bar{v}_\infty \), and the pressure by a factor of \( \rho v_\infty^2 \), where \( \rho \) is the density. Eq. 5 then becomes

\[
C_{kl} = \frac{1}{M} \int_{\mathcal{X}} q_i(\vec{x}, t_k)q_i(\vec{x}, t_l) d\vec{x}.
\]

The POD then generates modes that optimally capture the quantity \( u^2 + v^2 + p^2 \) instead of the kinetic energy. This is obviously a quantity with no physical significance. Another option is to define a vector \( q_i \) as \( (u(\vec{x}, t), v(\vec{x}, t), \sqrt{p(\vec{x}, t)/\rho}) \) for \( i = 1, 2, 3 \) respectively. In this formulation the eigenvalues capture the total energy per unit mass, \( u^2 + v^2 + p/\rho \). However, because in incompressible flow we generally examine the gauge pressure, we can have negative values under the square root. This problem can be circumvented by noting that an arbitrary constant can be added to the pressure in incompressible flow, but then the modes are dependent on an arbitrary constant. In the section to follow we resolve this problem and provide a consistent way of including the pressure when using the POD.

**Incompressible POD Formulation**

In order to ensure that the eigenvalues in the POD formulation capture a meaningful quantity, we examine the effect of adding a constant to the pressure, such as the atmospheric pressure, \( p_0 \). To simplify the presentation, we assume \( \rho = 1 \) such that \( p \) is actually \( p/\rho \).

We define a vector \( q_i \) as

\[
q_1(\vec{x}, t) = u(\vec{x}, t) - \langle u(\vec{x}, t) \rangle

q_2(\vec{x}, t) = v(\vec{x}, t) - \langle v(\vec{x}, t) \rangle

q_3(\vec{x}, t) = \sqrt{p(\vec{x}, t)/\rho} - \sqrt{\langle p(\vec{x}, t)/\rho \rangle}.
\]

We will use the same methodology here.

Using the new set of flow variables given in Eq. 7, we can perform a Taylor expansion of the
pressure term due to the fact that the pressure variations are much smaller than atmospheric pressure

\[
\sqrt{p(x, t) + p_0} = \sqrt{p_0} + \frac{p(x, t)}{2\sqrt{p_0}} + \mathcal{O}\left(\left(\frac{p(x, t)}{p_0}\right)^2\right).
\]

(8)

If we substitute this expansion for the pressure into Eq. 7, we arrive at the following set of flow variables

\[
q_1(x, t) = u(x, t) - (u(x, t)) = \tilde{u}(x, t)
q_2(x, t) = v(x, t) - (v(x, t)) = \tilde{v}(x, t)
q_3(x, t) = \frac{\tilde{p}(x, t) - (p(x, t))}{2\sqrt{p_0}} = \frac{\tilde{p}(x, t)}{2\sqrt{p_0}},
\]

(9)

where the tilde indicates the perturbation to the mean.

If we substitute these variables into the time-averaged autocorrelation tensor, we arrive at the following expression

\[
R(\vec{x}, \vec{x}') = \begin{bmatrix}
\tilde{u}(x, t)\tilde{u}(x', t) & \tilde{u}(x, t)\tilde{v}(x', t) & \tilde{u}(x, t)\tilde{p}(x', t) \\
\tilde{v}(x, t)\tilde{u}(x', t) & \tilde{v}(x, t)\tilde{v}(x', t) & \tilde{v}(x, t)\tilde{p}(x', t) \\
\frac{\tilde{p}(x, t)}{2\sqrt{p_0}}\tilde{u}(x, t) & \frac{\tilde{p}(x, t)}{2\sqrt{p_0}}\tilde{v}(x, t) & \frac{\tilde{p}(x, t)}{2\sqrt{p_0}}\tilde{p}(x, t)
\end{bmatrix}
\]

(10)

Due to the form of the pressure term, we re-scale the components of our eigenfunctions as

\[
\tilde{\varphi} = \begin{bmatrix}
\varphi_u(\vec{x}) \\
\varphi_v(\vec{x}) \\
\frac{\varphi_p(\vec{x})}{2\sqrt{p_0}}
\end{bmatrix},
\]

(11)

so that \(\varphi_p(\vec{x})\) will be an \(O(1)\) quantity. If we substitute the new form of our eigenfunctions along with the new form of our autocorrelation tensor into Eq. 2 and neglect terms inversely proportional to \(p_0\), we arrive at the following result for the pressure

\[
\int_{\mathbb{R}^3} \tilde{p}(\vec{x}, t) \left(\tilde{u}(\vec{x}', t)\varphi_u(\vec{x}') + \tilde{v}(\vec{x}', t)\varphi_v(\vec{x}')\right) d\vec{x}' = \lambda \varphi_p(\vec{x}).
\]

(12)

We note that there is no \(\varphi_p\) eigenfunction dependency in this equation. This suggests that we can solve the eigenvalue problem using only the velocity field while ignoring the pressure. The eigenfunction associated with the pressure can be calculated subsequently by inserting the \(u\) and \(v\) velocity eigenfunctions into the equation above. This is the proper way to include the pressure in a POD analysis of an incompressible fluid. Using this technique one can derive modes that optimally capture the total energy of the flow.

As previously mentioned, if \(N \gg 1\) it is difficult to solve the discrete form of the eigenvalue problem given above. It is more efficient to consider the method of snapshots. Here we will perform a similar analysis of the snapshot method. Using our new set of flow variables (Eq. 9), the eigenvalue matrix entries in Eq. 6 become

\[
C_{kl} = \frac{1}{M} \int_{\mathbb{R}^3} \tilde{u}(\vec{x}, t_k)\tilde{u}(\vec{x}, t_l) + \tilde{v}(\vec{x}, t_k)\tilde{v}(\vec{x}, t_l) + \ldots \frac{\tilde{p}(\vec{x}, t_k)}{4p_0} d\vec{x}.
\]

(13)

If we again neglect terms that are inversely proportional to \(p_0\), the pressure dependence is eliminated and we arrive at the original formulation given below

\[
C_{kl} = \frac{1}{M} \int_{\mathbb{R}^3} \tilde{u}(\vec{x}, t_k)\tilde{u}(\vec{x}, t_l) + \tilde{v}(\vec{x}, t_k)\tilde{v}(\vec{x}, t_l) d\vec{x}.
\]

(14)

To determine the pressure mode, we should then substitute the eigenvectors of this matrix into the snapshot formulation given by Eq. 3. To confirm that this is consistent with our previous analysis, we substitute the form for the POD modes given by Eq. 3 into Eq. 12. This gives the following discrete time formulation

\[
\frac{1}{M} \sum_{l=1}^{M} \tilde{p}(\vec{x}, t_l) \int_{\mathbb{R}^3} \left(\sum_{k=1}^{M} \psi_k(\tilde{u}(\vec{x}', t_l)\tilde{u}(\vec{x}, t_k) + \ldots \tilde{v}(\vec{x}', t_l)\tilde{v}(\vec{x}, t_k))\right) d\vec{x}' = \lambda \varphi_p(\vec{x}).
\]

(15)

This equation contains the snapshot eigenvalue matrix entries (Eq. 14). This allows us to simplify the above equation as follows

\[
\sum_{l=1}^{M} \tilde{p}(\vec{x}, t_l) \sum_{k=1}^{M} C_{lk}\psi_k = \lambda \varphi_p(\vec{x}),
\]

(16)

where according to Eq. 5 the \(C_{lk}\psi_k\) term is equal to \(\lambda \psi_l\). From this we obtain the following result for the POD mode associated with the pressure

\[
\varphi_p(\vec{x}) = \sum_{l=1}^{M} \psi_l\tilde{p}(\vec{x}, t_l),
\]

(17)

which confirms that the snapshot approach gives results consistent with Eq. 12.

In summary, one can find a POD mode associated with the pressure by solving the snapshot eigenvalue problem using only the velocity field. The
eigenvectors obtained can then be used to find the
POD modes for both the velocity and the pressure
\[ \mathbf{\tilde{\phi}} = \sum_{k=1}^{M} \psi_k \begin{bmatrix} \tilde{u}_k \\ \tilde{v}_k \\ \tilde{p}_k \end{bmatrix}. \] (18)

This approach optimally captures the total energy of the flow, which will be shown in the results.

The analysis presented here may also prove useful for low Mach number compressible flow problems as well. Current research efforts [18] use ad-hoc approaches to avoid taking the square root of the enthalpy. The form of the equations is as follows

\[ 0 = \sum_{e=1}^{n_e} \int_{\Omega} \left\{ -\phi^T \theta - \frac{\partial \phi}{\partial \xi} \tilde{\varepsilon} - \frac{\partial \phi}{\partial \eta} \tilde{\gamma} \right\} d\Omega + \int_{\Gamma} \phi^T (\tilde{\varepsilon}, \tilde{\gamma}) \cdot \tilde{u}_f d\Gamma \]

\[ + \int_{\Omega} \left[ \frac{\partial \phi}{\partial \xi} \tilde{A}_\xi + \frac{\partial \phi}{\partial \eta} \tilde{A}_\eta \right] \]

\[ T \left[ -\tilde{\theta} + \frac{\partial}{\partial \xi} \tilde{\varepsilon} + \frac{\partial}{\partial \eta} \tilde{\gamma} \right] d\Omega \right\} \forall \phi \] (21)

where the sum is performed over all of the elements in the mesh, 1 to \( n_e \). The solution for \((u, v, p)\) is sought in the space \([\Phi]^3\). Each integral in the sum is performed over either the standard triangle area \( \Omega \) or the standard triangle perimeter \( \Gamma \). The matrices \( A_\xi, A_\eta \) and \( T \) are associated with the SUPG stabilization. For more information on these matrices see [19]. If the term involving \( T \) is removed, we recover the standard Galerkin formulation.

**Problem Description**

For the calculations presented in this study we use a basis composed of quartic polynomials on each element \((P = 4)\). Quartic polynomials are used because they allow rapid convergence to the exact drag value with increasing mesh resolution (5th order spatial accuracy). An unstructured triangular mesh on a rectangular domain is used in all the calculations. Figure 1 shows the element mesh used to calculate the detailed solutions for the POD study of flow over a sphere. For the calculations, an inflow condition is enforced at the lower boundary of the mesh with \((u, v) = (0, V_\infty)\) such that the flow is from bottom to top, where \(V_\infty\) is varied depending upon the Reynolds number. At the downstream and right boundary, the total stress is set to zero. Along the centerline, a symmetry boundary condition is imposed, and on the sphere, which is centered at \((0,0)\), a no slip boundary condition is imposed. The downstream boundary is from \((0, 30)\) to \((20, 30)\), and the inflow boundary is from \((0, -20)\) to \((20, -20)\) where the units are sphere diameters. It is shown in [23]
that we are able to obtain values of drag with at least 1% accuracy using this configuration.

**Figure 1.** Typical element mesh and magnified region around the sphere

**Implementation of POD**

In this section we give an overview of how the POD modes are determined. Next, a description of how to obtain a low-dimensional model is given. We examine the quasi-steady response to the Reynolds number over a range from 0.1 to 200. The technique can easily be applied to unsteady flows, but for this first demonstration the Reynolds number is the only parameter that varies.

Because we have a highly resolved data set on the order of 5000 unknowns, it is more efficient to consider the method of snapshots [17]. Thirty-four snapshots are taken in the Reynolds number range described to resolve the drag. Using the snapshot method we only need to solve a 34x34 eigenvalue problem as opposed to a 5000x5000 one. The POD modes are computed from the numerical data by rewriting the formulation of the eigenvalue problem as opposed to a 5000x5000 one. The POD modes are determined. Next, a description of how to obtain a low-dimensional model is given.

We use this approach because it yields better results than the standard Galerkin projection. In certain cases, the standard Galerkin projection suffers from decoupling problems.

The solution for \((u, v, p)\) is sought in the space spanned by a subset of our POD modes, \(\varphi\). By inserting the expansion given in Eq. 27 into the total flux vectors of the SUPG formulation of the Navier-Stokes equation (Eq. 21) and also using the POD modes as the test functions, we obtain \(S\) scalar equations for the expansion coefficients \(a_n\). If we define \(R(a_n)\) as the residual error of the Navier-Stokes equations evaluated using the current estimate of the solution, our system becomes

\[
R(a_n) = 0, \quad (28)
\]
which is solved using a Newton-Raphson iterative technique. After we find these coefficients, we are able to reconstruct the flow field (Eq. 27) and calculate the drag. We perform the same technique for a standard Galerkin projection by setting the stabilization term in the SUPG formulation equal to zero.

**Fast Numerical Implementation**

The low-dimensional model mentioned above involves integrating over all of the elements in the mesh in the SUPG formulation each time a simulation is performed. This is essentially the same amount of work as a full detailed simulation, therefore the computational savings is not substantial enough to be considered a reduced-order model. To further reduce the computational costs of the low-dimensional model, we write the residual error of the Navier-Stokes equations using a second order Taylor series approximation

\[
R_i \bigg|_k = R_i \bigg|_b + \frac{\partial R_i}{\partial a_j} \bigg|_k (a_j - b_j) + ... \\
\frac{1}{2} \frac{\partial^2 R_k}{\partial a_j \partial a_k} \bigg|_k (a_j - b_j)(a_k - b_k). \tag{29}
\]

To evaluate the partial derivatives in the expansion we use a finite difference approximation. These terms are pre-calculated such that they do not affect the implementation of the work. The idea is to pre-calculate the residual error of the Navier-Stokes equations about some base state, \( \vec{b} \), and perturb it depending on the Reynolds number. The goal is to obtain a model that rivals the computational efforts of a correlation based model, yet still retain all of the essential physics of the flow.

The formulation given here is a quadratic representation of the residual error. For the case of a Galerkin projection we expect it to fully reproduce the coefficients found from the system in Eq. 28 because it is quadratic in terms of the flow field. However, this is not the case for the SUPG projection. Equation 21 is cubic in terms of our flow variables. We analyze the ability of this quadratic representation to reproduce the detailed results in the section to follow.

**Results**

In this section, we analyze the ability of the low-dimensional model to predict the flow over a sphere. We begin by looking at how much energy is captured by the POD modes. Next, the shape of the modes are examined in relation to capturing the physics of the flow. Lastly, we compare drag results from the two low-dimensional models to the solution from the direct numerical simulation. Our main point of comparison is the prediction of the drag on a sphere as a function of the Reynolds number. In addition to providing a methodology for incorporating more complex physics into models, the POD approach also predicts the entire flow field around the particle. This may allow better coupling to the Eulerian flow solution than can be achieved with point-particle approaches.

The POD modes are computed from the numerical data using thirty-four solutions by solving the eigenvalue problem shown in Eq. 4. Figure 2 is a plot of the eigenvalues versus the mode number. From this figure we can see that the first mode captures most of the energy in the flow, and there is an exponential decay as we increase the mode number. This shows that the series can be truncated with a relatively small number of modes and yet still capture most of the energy of the flow.

![Figure 2. POD Mode Eigenvalues](image)

Figures 3 & 4 show the streamlines of the first two normalized modes. Both figures give a close-up view to better show the structures around the sphere. The 0th POD mode is the mean of the flow over the 34 snapshots. From the streamlines of the 1st POD mode, we can see that it captures Reynolds number variation of two main features of the solution. The first is the thickness of the boundary layer that occurs near the surface of the sphere. The second is the recirculation zone at the aft end of the sphere that is also captured in the 0th POD mode. If the coefficient of the 1st mode is positive, we expect a sharper boundary layer and a recirculation
zone at larger Reynolds numbers. Thus, not only are the modes useful for compactly representing the solution, they also give physical insight into the dependence of the flow on Reynolds number.

Figure 3. Streamlines of 0th Normalized POD Mode

Figure 4. Streamlines of 1st Normalized POD Mode

To obtain a reduced-order model of the flow we project the incompressible Navier-Stokes equations onto the space spanned by the POD modes. This enables us to find the expansion coefficients, \( a_n \), that describe the solution at any given Reynolds number. To quantitatively evaluate the POD technique, low-dimensional simulations are performed to determine how many POD modes are needed to accurately predict the drag. Solutions are obtained using \( S = 1, 2, 3, \ldots \) Figure 5 shows the percent error in drag from the detailed simulations versus the number of modes used for various Reynolds numbers. As we increase the number of modes, we see an exponential decrease in the error as shown by the semi-log scale. If we use all 34 modes and choose the Reynolds number to be one of the snapshot values, the simulation reproduces the full numerical solution to machine order accuracy, confirming the consistency of the formulation. An 11 mode simulation predicts the drag to within 1% over the entire range of Reynolds numbers. Thus, the POD approach can be used to generate compact and accurate models for particle flows.

Even though the above simulations produce accurate models compared to the full numerical solutions, it still involves an integration over all of the elements in the mesh each time an iteration is performed in the Newton-Raphson technique. Because our goal is to obtain a model that rivals the computational efforts of a correlation based approach, we examine the finite difference Taylor series model (Eq. 29) to see how well we can predict the drag.

Figure 5. POD Model - Percent Error vs. Number of Modes

Figure 6 shows the percent error in drag from the detailed simulations versus the Reynolds number. In this simulation we use all 34 modes to quantitatively evaluate the terms in the finite difference model while linearizing about a base state of \( Re = 0.1 \). Once these values are stored, we are able to perform a low-dimensional simulation at any
Reynolds number. We are able to predict the drag to within 1.0% over the entire range of Reynolds numbers and still retain all of the essential physics of the flow. Because the SUPG formulation is cubic in terms of the flow variables we are not able to predict the drag to within machine order accuracy using all 34 modes. One way to rectify this would be to consider a third order finite difference approximation to the residual error. However, the amount of initial storage would significantly increase along with the complexity of the code. Another way is to split the domain into piecewise linear or quadratic regions if more accuracy is desired. One could linearize about a base state of $Re = 0.1$ for $0.1 < Re < 1.0$, a base state of $Re = 1.0$ for $1.0 < Re < 10.0$, etc...

![Figure 6. Finite Difference Model - Percent Error vs. Reynolds Number](image)

**Conclusions**

We have developed a new modeling technique for particle flows using the Proper Orthogonal Decomposition (POD) that captures many physical effects of the flow. In this paper, we have only studied quasi-steady problems, but with this technique one could capture all of the effects of the BBO equation (time history, “apparent” mass, drag, lift, etc.) at finite Reynolds numbers. In the past, this has only been done through direct numerical simulations, which are computationally expensive. Because the POD is a reduced-order modeling technique, we can significantly reduce the computing cost, yet still retain the essential physics.

Using the POD we are also able to predict the entire flow field around the particle. Effects such as the boundary layer thickness, recirculation zone length, and the far-field wake structure can be found. We are also able to obtain the drag, which allows us to predict how a particle will evolve in a flow. The detailed flow information available using the POD approach may allow more accurate coupling of the model with an Eulerian-Lagrangian particulate flow simulation than point-particle approaches.

In the process of developing our POD model we have also extended the POD technique. For most applications of the POD, the pressure field is ignored and only the velocity field is decomposed. Because we were concerned with drag, we have formulated a method to properly include the pressure field using the POD. In the past the pressure was calculated given the velocity field via a Poisson equation, but that requires the solution of an elliptic equation on the same mesh used to obtain the detailed solutions. If that was required for each calculation of the drag the method could no longer be considered a reduced-order model. Our new formulation optimally captures the total pressure when pressure variations are small relative to the atmospheric pressure (i.e. incompressible flow). We are also able to work directly with the pressure and not the square root of the pressure, which simplifies the formulation. This result may also prove useful for low Mach number compressible flows to allow modes to be found that optimally capture the total energy of the flow while still working directly with the primitive variables.

To validate the approach, we applied the new POD formulation to create a model for particle flows that captures the parametric variation of a flow with Reynolds number. We were able to predict the quasi-steady drag to within 1.0% of the values found in direct numerical simulations. These findings show the potential the POD has to efficiently model complex flows.

**References**


