AE 429 - Aircraft Performance and Flight Mechanics

Level Turn, Pull Up and Pull Down

Turning Performance

- What is a turn?
  - a turn is a change in flight path direction
  - turn rate is the time rate of change in heading

\[ \Psi = \lim_{\Delta t \to 0} \frac{\Delta \Psi}{\Delta t} \]
Turning Performance

- **More definitions**
  - Turn radius, $R$, is the distance between the flight path and the instantaneous center of curvature

- **Load factor and turn radius**
  - Load factor $n$ is defined as
    $$ n \equiv \frac{L}{W} $$
    - In a level, un-accelerated turn
    $$ W = L \cos \phi $$
    $$ n \equiv \frac{L}{W} = \frac{1}{\cos \phi} $$
    - $N$ is a function of $\phi$ (bank angle) only in a steady, level turn
    $$ \cos \phi = \frac{W}{L} = 1/(L/W) = 1/n $$
    $$ \phi = \arccos(1/n) $$

- **Performance parameters**
  - Turn radius $R$
  - Turn rate $\omega = d\psi / dt$
  - $\psi$ local angular velocity along the curved flight path
  - Larger the magnitude of $F_r$: tighter and faster will be the turn

Note: $L$ and $\phi$ are not independent in level turn
Turn Radius

\[ m \left( \frac{V_m}{R} \right)^2 = L \sin \phi \]

\[ R = m \frac{V_m^2}{L \sin \phi} = \frac{W}{g \sin \phi} = \frac{1}{n} \frac{V_m^2}{g \sin \phi} \]

\[ \cos \phi = 1/n \quad \cos^2 \phi + \sin^2 \phi = 1 \]

\[ 1/n^2 + \sin^2 \phi = 1 \quad \sin^2 \phi = 1 - 1/n^2 \]

\[ \Rightarrow \quad \sin \phi = \sqrt{1 - 1/n^2} \]

\[ \Rightarrow \quad R = \frac{1}{n} \frac{V_m^2}{g \sqrt{1 - 1/n^2}} = \frac{V_m^2}{g \sqrt{n^2 - 1}} \]

Small R \( \Rightarrow \) high n (large L/W) 
\( \Rightarrow \) low Velocity

Turn Rate

\[ \omega = \frac{d\psi}{dt} = \frac{V_m}{R} \]

\[ R = \frac{V_m^2}{g \sqrt{n^2 - 1}} \quad \Rightarrow \quad \omega = \frac{d\psi}{dt} = \frac{V_m}{R} = \frac{g \sqrt{n^2 - 1}}{V_m} \]

High \( \omega \) \( \Rightarrow \) high n (large L/W) 
\( \Rightarrow \) low Velocity

High Performance: smallest R and largest \( \omega \) for largest n; lowers speed V

What is the higher possible n?

R and \( \omega \) are function of n and V \( \Rightarrow \) Do not depend on W/S, T/W, k, C_d, \( \rho \)

L \( \uparrow \) \( \Rightarrow \) \( \phi \) \( \uparrow \) \( \Rightarrow \) D \( \uparrow \) \( \Rightarrow \) T_R \( \uparrow \) but T < T_{max} A implying that for T_{max} A \( \Rightarrow \) \( \phi_{T_{max}} A \)

\[ n = \frac{1}{\cos \phi} \quad \Rightarrow \quad n_{max} = \frac{1}{\cos \phi_{max}} = \frac{1}{\cos \phi_{T_{max}} A} \]

Level turn: \( \quad D = T; \quad L = n W = \frac{1}{2} \rho V^2 S C_L \)

\[ L/D = n W/T \]

\[ n_{max} = \frac{1}{2} \rho V^2 S \left[ \frac{T}{W} \right]_{max} \sqrt{\frac{1}{2} \rho V^2 \frac{C_{th}}{W/S} + \frac{K(W/S)}{W/S}} \]

\[ 1 \leq n \leq n_{max} \quad n_{max} = \frac{1}{2} \rho V^2 \frac{C_{L_{max}}}{W/S} \]
Minimum Turn Radius

- Minimum turn radius
  - Stall speed in straight and level flight \( (L = W) \) is
    \[
    V_s = \sqrt{\frac{2W}{\rho \omega SC_{L_{max}}}}
    \]
  - In a level turn, stall speed becomes \( (L=rW) \)
    \[
    V_{s_{turn}} = \sqrt{\frac{2nW}{\rho \omega SC_{L_{max}}}} \quad V_{s_{turn}} = V_s \sqrt{n}
    \]
  - Which suggests that
    - Replacing \( V_s \) with \( V_{s_{turn}} \) in the turn radius equation gives the aerodynamic limit on minimum turn radius
    \[
    R_{min} = \frac{V_{s_{turn}}}{g \sqrt{n^2 - 1}} = \frac{V_s^2 n}{g \sqrt{n^2 - 1}} = \frac{V_s^2}{g \sqrt{1 - \frac{1}{n^2}}}
    \]

Level Turn Chart
**Pull-Up**

- Consider a turn in the vertical plane: wing-level Pull-Up (instantaneous turn)
  - Different from level turn (constant flight properties)
  - The radial forces are:
    
    at t = 0; \( \theta = 0 \)
    
    \[ F_r = L - W = W(n-1) \]
    
    \[ F_r = m \frac{V_w^2}{R} = \frac{W V_{\infty}^2}{g R} \]
    
    - Solving for R:
      
      \[ R = \frac{V_{\infty}^2}{g(n-1)} \]
    
    - And for turn rate:
      
      \[ \omega = \frac{V_{\infty}}{R} = \frac{g(n-1)}{V_{\infty}} \]

**Pull-Down**

- Now, look at another instantaneous turning maneuver in the vertical plane -- a "split s"
  - Using the same approach as for a Pull-Up

    at t = 0; \( \theta = 0 \)
    
    \[ m \frac{V_{\infty}^2}{R} = L + W \Rightarrow R = m \frac{V_{\infty}^2}{L+W} \]
    
    \[ R = \frac{V_{\infty}^2}{g(n+1)} \]
    
    \[ \omega = \frac{g(n+1)}{V_{\infty}} \]
    
    - The rate is improved and the radius is enlarged over pull-ups
Limiting cases: n large

- Effect of W/S (wing loading) and $C_{l_{\text{max}}}$
  - When $n$ is large, $n+1 = n-1 = n \Rightarrow R = \frac{V_{\infty}^2}{gn}, \quad \omega = \frac{gn}{V_{\infty}}$
  - Recalling that $V_{\infty}^2 = \frac{2L}{\rho_{\infty}SC_{L}}$
  - Substituting, we obtain radius and rate of turn
    
    $$R = \frac{2L}{\rho_{\infty}SC_{L}g(L/W)} \quad \Rightarrow \quad R = \frac{2L}{\rho_{\infty}C_{L}g} \frac{W}{S}$$
    $$\omega = \frac{gn}{\sqrt{\frac{2L}{\rho_{\infty}SC_{L}}}} \quad \Rightarrow \quad \omega = g\sqrt{\frac{\rho_{\infty}C_{L}n}{2[W/S]}}$$

- For minimum turn radius and maximum turn rate
  - Maximize both $C_{L}$ and load factor

  $$R_{\text{min}} = \frac{2}{\rho_{\infty}gC_{L_{\text{max}}}} \frac{W}{S} \quad \omega_{\text{max}} = g\sqrt{\frac{\rho_{\infty}C_{L_{\text{max}}}n_{\text{max}}}{2[W/S]}}$$

  - Practical constraints on load factor
    - $n_{\text{max}}$ is a function of $C_{L_{\text{max}}}$
      - at low speeds it will be limited by the aerodynamic lifting capability (stall) of the lifting surfaces
        $n_{\text{max}} = \frac{1}{2} \rho_{\infty}V_{\infty}^2 \frac{C_{L_{\text{max}}}}{W/S}$
      - at high speeds, structural loads on the airframe may also limit $n_{\text{max}}$
    - for many airplanes, the other force balance ($T = D$) governs the minimum turn radius and the maximum turn rate -- turn performance is limited by available thrust
• constraints on $V$
  - $V$ as small as possible for $R_{\text{min}}$ and $\omega_{\text{max}}$

$$V = \frac{2nW}{\rho SC_L} \Rightarrow C_L = C_{L\text{max}} \Rightarrow V_{\text{Stall}} = \frac{2nW}{\rho SC_{L\text{max}}}$$

- $R_{\text{min}}$ does not necessarily correspond to $n_{\text{max}}$

$$1 \leq n \leq n_{\text{max}}$$

$$R = \frac{V^2}{g \sqrt{n^2 - 1}} = \frac{2q_{\infty}}{g \rho \sqrt{n^2 - 1}} \Rightarrow \frac{\partial R}{\partial q_{\infty}} = 0$$

$$R_{\text{min}} = \frac{4k(W/S)}{g \rho_{\infty}(T/W) \sqrt{1 - 4kC_{D\text{L}}(T/W)^2}}$$

$$\omega_{\text{max}} = q_{\infty} \sqrt{\rho_{\infty} l(W/S)(T/W)(2k) - (C_{D\text{L}}/k)^{1/2}}$$

$$n_{R_{\text{min}}} = \sqrt{2 - 4kC_{D\text{L}}(T/W)^2}$$

$$V_{=R_{\text{min}}} = \sqrt{4k(W/S) / (\rho_{\infty}(T/W))}$$

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V-n diagram

- the V-n diagram illustrates 2 of these constraints

![V-n diagram](image)
● Aerodynamic and structural limits on turn performance

\[ n_{\text{max}} = \frac{1}{2} \rho \infty V^2 \frac{C_{L_{\text{max}}}}{W/S} \]

Aerodynamics
Wing Design

Thrust Available
Drag

Structural
(Materials/
Wing Size)