HOW QUICK IS A QUANTUM JUMP?

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Although the only time scale one ordinarily associates with a quantum transition is its lifetime, observations of “quantum jumps” in recent years show that the actual transition time is much shorter. I define a “jump time” as the time scale such that perturbations occurring at intervals of this duration affect the decay. In terms of the “Zeno time” (related to the second moment of the Hamiltonian) the jump time is $\tau_J \equiv \tau_2^2/\tau_L$. Corroboration is given. I also show that observing the “jumping” will not seriously affect the system lifetime, but will affect the linewidth. This is consistent with Bohr’s ideas on measurement as well as with a heuristic time-energy uncertainty principle.

1 Introduction

In the early history of quantum mechanics, a transition from one quantum state to another was not thought to be subject to analysis beyond the calculation of the transition amplitude. This (or its magnitude) was the important “measurable” quantity. Thus the decay of an atomic metastable state and the concurrent emission of a photon were considered to be a “quantum jump.” Although the transition was presumed to occur on a time scale more rapid than the lifetime of the state, the actual time for the jump seem to be an issue related to the “measurement process” itself, hence not subject study with the quantum formalism.

Similar issues arise for the question of “tunneling time.” On the average a nucleus of Radium takes 1600 years to decay, but the time for the alpha particle to tunnel through the barrier (in Gamow’s picture) is surely less. This conference is devoted to exploring the meaning of that shorter time and in an appendix to this article I summarize previous work of my own on this subject, Ref. 1.

Here though I would like to raise the more general question of assigning a time to all quantum transitions. We have progressed beyond those earliest days of quantum mechanics and with respect to the subject at hand the mark of progress is the ability to observe “quantum jumps.” In 1986, several groups,2,3,4 using single atoms, were able to bracket atomic transitions within brief intervals, while showing that the lifetime of the decay was substantially unchanged
by this observation. No longer could one attribute the transition to the black magic of quantum measurement theory: one was now looking at a dynamical process that should be subject to the usual microscopic rules of quantum mechanics—either that or the rules themselves should be questioned. My own view is that there is no special dynamics for “measurements.” Maintaining this view in the face of the problems that led the wise founders of quantum mechanics to adopt their position on measurement is not trivial. (See, e.g., 5, 6.) However, in this instance analysis of the experiments has shown that there is no need to step outside the framework of normal quantum evolution.

The experiments themselves did not place a stringent bound on the duration of the transition. My goal in this article is to make an estimate of this time scale. It will also turn out that the mere observation of the jump should have experimental consequences—a position that Bohr would have found comforting. We will see why the lifetime is (substantially) unchanged by the observation; nevertheless, we will also show that there is a line broadening. It turns out that the minimum time scale for an atomic transition is quite short, but that other experimentally accessible transitions may have scales that will allow observation.

To be more specific, let me recall some details for one version of the experiment. The atom has a short-lifetime state and a long-lifetime state. It is illuminated by a laser at the frequency of (the transition energy for) the short-lived excited state, so that it is constantly cycling between this state and the ground state. To go up, it absorbs a photon from the laser field. In going down it emits a photon isotropically. Some of these fluorescence photons are observed away from the laser direction and one can confirm the cycling behavior. At some point the atom can drop (or be pushed) into the long-lived state. The fluorescence previously observed will suddenly stop. When the atom decays from the long-lived state the fluorescence resumes. The on/off record of fluorescence marks the “jumping.” The cited experiments confirmed that the lifetime of the state (the periods of no fluorescence) averaged to the known lifetime of the long-lived state. The topic of present concern is the suddenness of the jump. On the experimental graphs the transition is sharp to the extent measurable by the apparatus. What we want to know here is just how sharp it really was—or maybe just how sharp it could be, since as we shall see that sharpness will depend on the experiment itself.

2 Minimal jump time

The definition of the time scale for a quantum jump is based on the idea that if I disturb a system at intervals $\Delta t$, but those disturbances do not affect
the decay of the system, then $\Delta t$ must be longer than it takes the system to accomplish its jump. On the other hand, if I do manage thereby to affect the decay, then I have reached the time scale of the decay. As the disturbance to be used in the definition I will take a projection onto the original state—a check on whether the system has or has not decayed. In this way we arrive at a formal context rather similar to that used in the so-called quantum Zeno effect ("QZE").

Let the system be started in a state $\psi$ or $|\psi\rangle$ and let the full Hamiltonian be $H$. After a time $\Delta t$ the system evolves to the state $\exp(-iH\Delta t/\hbar)|\psi\rangle$. One then checks whether the system is still in the original state by taking an inner product with $\psi$. The probability that it is indeed still in $\psi$ is

$$p(\Delta t) = |\langle\psi|\exp(-iH\Delta t/\hbar)|\psi\rangle|^2$$

(1)

Expanding in $\Delta t$ this gives

$$p(\Delta t) = \left[1 - i\frac{\Delta t}{\hbar}\langle\psi|H|\psi\rangle - \frac{1}{2}\left(\frac{\Delta t}{\hbar}\right)^2 \langle\psi|H^2|\psi\rangle + O(\Delta t^3)\right]^2$$

$$= 1 - \left(\frac{\Delta t}{\tau_Z}\right)^2 \langle\psi|(H - E_\psi)^2|\psi\rangle + O(\Delta t^4)$$

(2)

where $E_\psi = \langle\psi|H|\psi\rangle$. We rewrite this as

$$p(\Delta t) = 1 - \left(\frac{\Delta t}{\tau_Z}\right)^2 + O(\Delta t^4)$$

(3)

with

$$\tau_Z^2 = \frac{\hbar^2}{\langle\psi|(H - E_\psi)^2|\psi\rangle}$$

(4)

We call $\tau_Z$ the “Zeno time,” notwithstanding our lack of full concurrence with the classical allusion (but recognizing the futility of any attempt to change it).

Suppose many such checks (of being or not being in $\psi$) are made during a time period $t$, such checks being carried out at intervals $\Delta t$. Then (dropping order $(\Delta t)^4$ terms) at time-$t$ the probability of still being in the state $\psi$ is

$$p_t(t) = "p_{\text{interrupted}}" = (p(\Delta t))^{t/\Delta t} = \left[1 - \left(\frac{\Delta t}{t}\right)^2 \frac{\Delta t}{\tau_Z^2}\right]^{t/\Delta t} \approx \exp\left(-t\Delta t/\tau_Z^2\right)$$

(5)

Following our proposed definition, we want to know whether or not this differs from the standard decay scenario. That scenario gives the following probability for still being in the original state

$$p_t(t) = "p_{\text{uninterrupted}}" = \exp\left(-t/\tau_L\right)$$

(6)
with $\tau_L$ the usual lifetime. When the “Golden rule” can be used this is given by

$$\frac{1}{\tau_L} = \frac{2\pi}{\hbar} \rho(E_\psi) |\langle f | H | i \rangle|^2$$

(7)

with $|i\rangle$ ($= |\psi\rangle$) and $|f\rangle$ initial and final states, respectively, and $\rho$ the density of states (at the energy of the initial state). Comparing Eq. (5) and Eq. (6), we see that the interrupted decay will be slower for $\Delta t < \tau_z^2/\tau_L$. (Reversing the inequality suggests faster decay, but in fact signifies breakdown in the expansion, Eq. (2). Our argument requires $\Delta t < \tau_z$.) We are thus led to define the “jump time” as the time for which the slowdown would begin to be significant, namely

$$\tau_j \equiv \tau_z^2/\tau_L$$

(8)

In words, $\tau_j$ is the time such that if one inspected a system’s integrity at intervals of this duration, the decay would be affected.

With respect to Eq. (8), there are several tasks to be undertaken. First we look for corroborative evidence, situations where time scales of this sort have been seen. Second, we will show that $\tau_j$ has a simple interpretation in terms of uncertainty principle-like ideas. Actually, the ideas are not so much the uncertainty principle as Fourier analysis. Third, we would like to evaluate $\tau_j$ for typical atomic transitions as well as for other transitions. And finally, we would like to examine the experimental consequences associated with observations of jump time. That is, suppose we would ascertain that a system jumped within some time interval—as was done in $^2, ^3, ^4$—would this change the transition in any way? We will see that the answer is, “yes.”

### 3 Corroboration

In $^10$, $\tau_z$ is calculated (from matrix elements of $H$, as in Eq. (4) above) for a particle tunneling through a smooth barrier, “smooth” meaning validity of the semiclassical approximation. The emphasis there was on the “Zeno time,” since other characteristic times, lifetime, tunneling time and vibrational times in the well, were better understood. It was found that to a good approximation, $\tau_z = \sqrt{\tau_T \tau_L}$, where $\tau_L$ has the meaning we have already assigned it, and $\tau_T$ is tunneling time (time spent in the barrier).$^{11}$ For the quantum transition associated with this barrier penetration process, it is natural to consider the tunneling time the time for making the jump, so the above formula for $\tau_z$, solved for $\tau_T$, is the same as Eq. (8). (Note, this is not circular, since $\tau_T$ is a separately calculated quantity.)

A second piece of evidence comes from the exploration of a specific decay model in which there was found to be quantum jumps.$^{12}$ In that study there
were certain particular initial states of a multilevel quantum system that were able to decay quite rapidly, in particular much more rapidly than random or non-special initial conditions. We refer the reader to that article for details and notation. For a non-special initial condition, in the units of that article, the Zeno time is roughly $\sqrt{N\pi}$, where $N$ is the dimension of the initial multilevel system. The lifetime is not fully well-defined in that model because the decay is not exponential. However, the survival probability goes from 1 to a small number in approximately $2\pi N$, so that it would be reasonable to call $\tau_L = \pi N$. In that article we were able to build states that dropped from being fully undecayed to being (essentially) fully decayed in $2\pi$ time units, confirming the validity of this scale.

4 An interpretation of $\tau_J$

Let us look at the matrix elements that appear in the definition of $\tau_J$. We can write

$$\langle \psi | (H - E) \psi \rangle = \int d\alpha \langle \psi | (H - E) \alpha \rangle \langle \alpha | (H - E) \psi \rangle = \|(H - E) \psi \rangle \|^2$$ \quad (9)

where $\{\alpha\}$ ranges over a complete set of states. We see that the essential difference between the lifetime, given through Eq. (7), and the Zeno time, is that the lifetime uses the on-shell matrix elements—those at the same energy as the initial state—while the Zeno time uses all accessible energies, so it is substantially off-shell. Note that the states $|\alpha\rangle$ could also be labeled $|E, k\rangle$, perhaps adding other quantum numbers as well. In this case we could write

$$\langle \psi | (H - E) \psi \rangle = \int dE \rho(E) \langle E | (H - E) | \psi \rangle \|^2$$ \quad (10)

where the density of states includes all available states at the (integration variable) energy $E$ and the matrix element is assumed independent of those other quantum numbers.

Combining Eqs. (10), (7) and (8), we obtain

$$\tau_J = \frac{1}{\int dE \frac{\rho(E) \langle E | (H - E) | \psi \rangle \|^2}{\rho(E) \langle f | H | \psi \rangle \|^2}}$$ \quad (11)

Now one could throw a “$-E$” into the Golden rule matrix element, for free. This is because of the orthogonality of the initial and final states. Thus the ratio

$$\frac{\rho(E) \langle E | (H - E) | \psi \rangle \|^2}{\rho(E) \langle f | H | \psi \rangle \|^2}$$ \quad (12)
takes the value 1 when $E$ passes through $E_\psi$ and presumably drops off to smaller values as one moves away from the on-shell values. In effect, this ratio measures the ability of the Hamiltonian $H$ to move the system away from its initial state. One thus has a band of accessible transition states, the accessibility measured by the above ratio.

This allows us to interpret Eq. (11). The time $\tau_J$ is an integral over energies (or frequencies, when one allows for the $1/\hbar$) of a function showing how the band of accessible states is modulated. It follows that $\tau_J$ is the inverse bandwidth for the transition.

From the standpoint of a time-energy uncertainty principle, or more precisely, Fourier analysis, this is a completely reasonable conclusion. You would like to generate a situation where the system’s transition is sudden. The extent to which you can succeed is governed by the frequencies you have available. Those frequencies, or their availability, is precisely what the definition of bandwidth is all about.

5 $\tau_J$ for atomic and other transitions

The Hamiltonian for the decay of an atomic level is

$$H = \frac{(p - eA/c)^2}{2m} + V + H_{\text{EM, free}}$$

Under reasonable circumstances, and without paying attention to polarization (since we are only after rough estimates), this can be written

$$H = E_g|g\rangle\langle g| + E_\psi|\psi\rangle\langle \psi| + \sum_k \omega_k a_k^\dagger a_k + \sum_k [\phi_k a_k^\dagger |g\rangle\langle \psi| + \text{adjoint}]$$

where $g$ stands for the ground state, only two atomic states are considered, $k$ runs over photon states, and $\phi_k$ is the matrix element for the transition.

In this notation, $1/\tau_L$ is essentially $\rho(E_\psi)|\phi(k_0)|^2$ and $\tau_x$ is most easily evaluated as $||(H - E_\psi)|\psi||^2$, which becomes $\sum_k |\phi_k|^2$. In the continuum limit this is $\int dE \rho(E)||E|H|\psi||^2$, and the final states are restricted to those orthogonal to the initial state. Before writing this explicitly we make some further changes. First the matrix elements: all that appears is the ratio, so factors like $e$ and $c$, and even more annoyingly, volume, if one is paying attention to the continuum limit, conveniently drop out for the calculation of $\tau_J$. The same holds for $\rho$. The integration over $E$ is an integral over possible final photon energies or wave numbers, $k$. So $dE$ becomes $ckd$. The ratio of density of states factors is simply $k^2$. By introducing a distance scale, “a”
the expression for $\tau_j$ can be written as the product of a dimensional quantity and a dimensionless integral. Thus

$$\tau_j = \frac{2\pi a}{c} \frac{1}{I}, \quad \text{with} \quad I = a \int dk \left( \frac{k}{k_0} \right)^2 \frac{|\phi_k|^2}{|\phi_{k_0}|^2}$$

In the above $k_0$ is the wave number corresponding to the $\psi \rightarrow g$ transition. The important matrix element is therefore of the form

$$\phi_k \sim \langle g|\vec{p}\cdot\vec{A}|\psi \rangle \sim \frac{1}{\sqrt{k}} \int d^3p \psi_g^*(p)pe^{-ik(\vec{a}/\partial\vec{p})}\psi(p)$$

where we have gone to momentum space for the atomic-electron coordinate. The $1/\sqrt{k}$ that appears in Eq. (16) arises from the $1/\sqrt{\omega_k}$ in the expression for the vector potential, $A$, in terms of photon creation and annihilation operators, $a_k^\dagger$ and $a_k$. It is further convenient to change the momentum variable using the distance, “$a$,” so as to get dimensionless integration as well. Moreover, taking the distance scale for this to be the Bohr radius simplifies the electron states.

With this choice the dimensional quantity for $\tau_j$ can be evaluated. Referring to Eq. (15), we see that the dimensional factor is simply the time for light to go round a circle whose radius is the Bohr radius. This time is about $10^{-18}$ s.

Using the dimensionless integration variable, $ak$, in $I$, we separate a dimensionless quantity, $k_0a$. This is a rather small number. Typically the wavelength for these transitions is about 5000 Å, while $a$ is about 0.5 Å. Thus $k_0a$ is about $2\pi/10^4$.

For the remaining integration, we note first that $e^{-ik(\vec{a}/\partial\vec{p})}\psi(p) = \psi(p+k)$. This allows us to write $I$ as

$$I = k \frac{1}{k_0} \left| \int d^3p \psi_g^*(p)\psi(p+k) \right|^2$$

For wave functions, bearing in mind our modest goal of getting orders of magnitude only, we use for $g$ the ground state of the hydrogen atom, $\psi_g \sim 1/(p^2+1)^2$. For the excited state we use a p-state, whose $p$-dependence is roughly $p_z/(p^2+1/4)^3$ (see, e.g.,14). This leads to integrals that can be performed numerically. Combining all relevant terms it turns out that $I \approx 10^2$ so that for $\tau_j$ we get the remarkably small value, $10^{-20}$ s.

With all the discussion about superluminal transmission at the present conference, it is amusing to note that although we cannot pinpoint any distance the electron has covered in this very short time, it is still a lot faster—by
a factor of 100—than one would have thought reasonable for this atomic transition.

For the purpose of attaching physical significance to the calculation we have just done, it would be worthwhile to find a transition for which the jump time is not so very brief. Guided by the heuristic discussion at the end of the last section, one should look at bandwidths. For many transitions it is possible to get much narrower bandwidths than that we have just considered. Consider what is probably a reasonable bandwidth for phonon mediated transitions, 100 meV. For this

$$\tau_J \sim \frac{\hbar}{100 \text{ meV}} \approx 10^{-14} \text{ s} \tag{18}$$

which is far more accessible than the time calculated for atomic transitions and may well be measurable with present day equipment.

There are also transitions where collective, condensed matter effects narrow the bands; these should present yet richer possibilities for the realization of effects arising from the finiteness of the jump time. In Sec. 7 we will discuss systems that seem to be at the opposite extreme, molecular isomers. Apparently, chiral molecules are stabilized because $\tau_J$ is so long that environmental interruptions prevent the transition to a symmetric ground state or to the opposite chirality from ever happening.

6 Experimental consequences of seeing the “jump”

What does it mean to have a very short jump time? Does there occur somewhere in the system dynamics a characteristic energy associated with this time? For the atomic transition, the time was on the order of $10^{-20}$ s. Does this mean that there are MeV scale energies when an atom decays? How could that be?

Actually, things are the way Bohr said they should be. If you do not try to catch the atom in the act, if you do not try to check whether it jumped or whether it took its time about the decay, then the only important time scale is the lifetime, $\tau_L$. However, if you have been intrusive, if you do an experiment of the sort we quoted, then indeed you would introduce energies related to the temporal precision of your probe.

Presenting the detailed calculations behind the above assertion would lengthen this article considerably. Such calculations appear in\(^9\). To use the language of that article, suppose a laser beam at the energy of the fast transition creates an initial photon field which, in an occupation number representation, would have a high quantum number $N_0$. If the initial atomic state is $\psi$, then the initial state of the coupled systems is of the form $\psi \otimes |N_0\rangle$. Suppose further that the fast transition has matrix elements $\Phi$, corresponding to the “$\phi$” that appears in the Hamiltonian, Eq. (13). Let $B^2 \equiv N_0|\Phi|^2$, with dimensions of
energy (we now take $\hbar = 1$). Then the decay rate for the slow-level decay is no longer $2\pi \rho(E_\psi) |\phi(E_\psi)|^2$, but is $\pi \rho(E_\psi + B) |\phi(E_\psi + B)|^2 + \pi \rho(E_\psi - B) |\phi(E_\psi - B)|^2$. Now if these matrix elements do not have a strong variation, then the overall decay rate will be substantially unchanged. (In $^9$ we are interested in fields strong enough to stop the decay, but those used in the quoted experiments are not that strong.) However, the energies of the photons that emerge from this decay will not have the usual distribution. On the contrary, they will be spread by the amount $B$ because the system has made its transition using different modes of the electromagnetic field. That’s why $E \pm B$ appear as arguments in the lifetime formula.

It would be necessary to evaluate “$B$” for the quoted experiments to determine whether it is feasible to detect this observational broadening of the line. However, in doing this measurement one would not at all need to work with a single atom (which I expect would make the observation of the few decay photons impossible). Rather, the more the better—up to the point where collisional or other broadening would mask the sought-after effect.

7 Discussion and conclusions

As if there weren’t already enough debate and disagreement over the concept of tunneling time, we have introduced yet another time scale. Since one can define anything one wants, the test of our definition is not whether it is right or wrong, but whether it is interesting, whether any worthwhile physics can emerge from it.

The definition is based on the idea that the time scale for a process is a time interval such that perturbing a system on that time scale can lead to changes in its behavior. In our case the change is the onset of QZE freezing of an unstable system. The Büttiker-Landauer time for tunneling is based on a similar philosophy of interfering with a system.

Although the time scale we have found is a property of the transition, in some sense, unless you actually check whether the system has been so quick, it does not actually accomplish the transition in this very brief time. Presumably this would also be true for tunneling time. Now the assertion we have just made may sound like metaphysics. However, in Section 6 we gave precise meaning to it, and proposed an experiment to check that assertion. In particular, we claim that determining that a system “jumps,” rather than allowing it to decay without monitoring, will affect the line shape of the emitted photons (or phonons, etc.). Thus the natural (i.e., no jump-checking) linewidth of the system will be broadened by an energy scale inverse to the time-localization of an observer who is checking for jump behavior. In the spirit of Bohr, I call
this “observational broadening.”

Besides the theoretical corroborative evidence for these ideas that I provided above, I would like to close with what I believe is a physical corroboration. The opening sentence of \cite{15} reads, “One of the earliest problems of molecular quantum mechanics was the nature and stability of chiral molecules,” and reference is made to a 1927 article of F. Hund (Z. Phys. 43, 805). The quantum transition—or more often, absence of transition—that supplies this “problem” is the transition between left and right handed isomers.\cite{16} A feature needing explaining\cite{17} is the remarkable stability of chiral, i.e., asymmetric, molecules. The answer, in these and earlier\cite{18,19} references, is that the quantum states are stabilized by the environment, meaning that the pushing and shoving of nearby molecules prevents the transition to the other chirality or the relaxation to the pure parity eigenstate. In fact, \cite{18} explicitly relates this effect to QZE, under its alias, “A Watched Pot Never Boils.” Although this calculation does not explicitly derive what we have called the “Zeno time” (nor the “jump time”) the stability deduced there implicitly shows that the time scale for environmental interruptions of the pure quantum tunneling is short compared to $\tau_j$.

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Appendix A: Tunneling time using the path integral

This is a summary of the tunneling time related results of Ref.\cite{1}. The main goal of that article was to explore the asymptotic expansion of the path integral in the case when no classical path exists. However, an application of the methods developed for that purpose allowed a path integral definition of tunneling time to be made. Several other authors proposed similar definitions about the same time, and indeed some of that work is appearing in the current conference proceedings.

Consider the following Hamiltonian in one dimension

$$H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V, \quad V = V_0 \theta \left(\frac{a}{2} - |x|\right)$$  \hspace{1cm} \text{(19)}$$

This has a barrier of height $V_0$ between $-a/2$ and $a/2$. To discuss barrier penetration in the context of the path integral we consider the propagator between points that are separated by $vT$, at time $T$, and allow $T$ to be large.
This selects neighborhoods in path space of classical paths that have energy close to \( \frac{mv^2}{2} \).

We can write the propagator

\[
G \left( \frac{vT}{2}, T; -\frac{vT}{2} \right) = \sum_{s', s''} \sum_{x(s); s', s''} e^{iS[x(s)]/\hbar}
\]  

(20)

The notation means the following: the path, \( x(s) \), in the second sum does not enter the barrier region until the time \( s' \), and after the time \( s'' \) never again enters that region. The first sum then collects paths, \( x(s) \), for various \( s' \) and \( s'' \). In this way one can say that the most time the system spent “within” the barrier is \( s'' - s' \). (One could also sort the paths according to their last entrance into the barrier and their first departure. In the semiclassical approximation these should not differ by much.) We will interpret this as the time spent in the barrier, although one should recognize this as a loose interpretation. In particular, since we are only looking at amplitudes this separation ignores—improperly—the interference between different classes of paths. This point is made in 20.

That paths can be sorted as in Eq. (20) is a consequence of the “path decomposition formula”21. In this case the formula takes the form

\[
G(vT/2, T; -vT/2)
\]  

(21)

\[
= \int_0^T ds'' \int_0^{s''} ds' \left( \frac{\hbar}{2m} \right)^2 \frac{\partial}{\partial u} G' \left( \frac{vT}{2}, T - s''; u \right) \bigg|_{u=a/2} 
\]

\[
\times G \left( \frac{a}{2}, s'' - s'; -\frac{a}{2} \right) \frac{\partial}{\partial s'} G' \left( u, s'; -\frac{vT}{2} \right) \bigg|_{u=-a/2}
\]

The next steps in this problem make use of the explicit form of \( G \) for the barrier. Some of the estimates turn out to be rather delicate (considering how simple the basic problem is). The form of \( G \) that leads to the estimate of tunneling time is

\[
G \left( \frac{vT}{2}, T; -\frac{vT}{2} \right)
\]  

(22)

\[
\sim \int_0^T d\tau \int_0^\infty du \exp \left[ \frac{i\tau m}{2\hbar} (v^2 - u^2) - \frac{2}{\hbar} \sqrt{2m (V_0 - au^2)} \right]
\]

The variable “\( \tau \)” above is \( s'' - s' \), and so has the interpretation of time spent in the barrier. The variable “\( u \)” is a kind of velocity. To get the most important
contributions to the integral we do a steepest descent calculation, looking at \( \partial/\partial u \) and \( \partial/\partial \tau \). The principal contribution to the \( u \) integral is \( u = v \), while the stationary point in \( \tau \) occurs for

\[
\tau = \frac{-2ia}{\sqrt{(2V_0/m) - v^2}}
\]

(23)

Note that this is pure imaginary. What it means is that the most important contributions to \( G \) come from the range of \( \tau \) integration roughly around the absolute value of this expression. This expression is the semiclassical barrier traversal time. So in a way we do return to familiar ground. However, we emphasize that any attempt to interpret this in too literal a way is complicated first by the fact that the steepest descent point is \textit{not} on the real axis, and second by the reservations about interfering amplitudes mentioned earlier.

References


11. Actually $\tau_x$ is a combination of tunneling time and vibrational times in the well. For the potentials considered these times are roughly the same, certainly within orders of magnitude; in any case, our relation, $\tau_x = \sqrt{\tau_T \tau_L}$, is not expected to be better than that.


16. The word, “transition,” is being used loosely here; the true stationary states are presumably parity eigenstates.


