Mesh Equations

Introduction

The circuits in this problem set consist of resistors and independent sources. We will analyze these circuits by writing and solving a set of mesh equations. To write mesh equations, we

1. Express the element currents and voltages in terms of the mesh currents.

2. Apply KVL to the meshes of the circuit.

In particular, we will express resistor voltages in terms of the mesh currents. Consider the blue resistor in each of these two situations.

In (a) the blue resistor is located on the inside of the circuit. Consequently, it is in two meshes, mesh 1 and mesh 2 with corresponding mesh currents $i_1$ and $i_2$, in this case. Apply KCL in (a) to get

$$i_1 = i_R + i_2 \Rightarrow i_R = i_1 - i_2$$

Next, apply Ohm’s law to get

$$v_R = R i_R = R (i_1 - i_2)$$

In (b) the blue resistor is located on the outside of the circuit. Consequently, it is in only one mesh, mesh 1 with corresponding mesh current $i_1$, in this case. Apply KCL in (b) to get

$$i_R = i_1$$

Next, apply Ohm’s law to get

$$v_R = R i_R = R i_1$$
To summarize:

1. Consider a resistor having resistance $R$ and located on the inside of the circuit, in both mesh 1 and mesh 2. Label the resistor current $i_R$ and voltage $v_R$ to adhere to the passive convention. Suppose that the direction of mesh current $i_1$ is the same as the direction of $i_R$ but that the direction of mesh current $i_2$ is opposite the direction to $i_R$. The resistor voltage is given by $v_R = R(\pm i_1 - i_2)$. ("+" corresponds to “same” and “−” corresponds to “opposite”.)

2. Consider a resistor having resistance $R$ and located on the outside of the circuit, in mesh 1. Label the resistor current $i_R$ and voltage $v_R$ to adhere to the passive convention. When the direction of mesh current $i_1$ is the same as the direction of $i_R$, the resistor voltage is given by $v_R = Ri_1$. When the direction of mesh current $i_1$ is the opposite to the direction of $i_R$, the resistor voltage is given by $v_R = -Ri_1$.

We will also express the currents of current sources currents in terms of mesh currents. Consider the blue current source in each of these two situations.

In (a) the current source is located on the inside of the circuit. Consequently, it is in two meshes, mesh 1 and mesh 2 with corresponding mesh currents $i_1$ and $i_2$, in this case. Apply KCL in (a) to get

$$i_1 = i_s + i_2 \quad \Rightarrow \quad i_s = i_1 - i_2$$

There is no easy way to express the current source voltage in terms of the mesh currents.

In (b) the current source is located on the outside of the circuit. Consequently, it is in only one mesh, mesh 1 with corresponding mesh current $i_1$, in this case. Apply KCL to get

$$i_s = i_1$$
Again, there is no easy way to express the current source voltage in terms of the mesh currents.

To summarize:

1. Consider a current source having current \( I \) and located on the inside of the circuit, in both mesh 1 and mesh 2. Suppose that the direction of mesh current \( i_i \) is the same as the direction of \( I \) but that the direction of mesh current \( i_2 \) is opposite the direction of \( I \). The two mesh currents are related by the equation \( I = +i_i - i_2 \). (+ corresponds to “same” and – corresponds to “opposite”).

2. Consider a current source having current \( I \) and located on the outside of the circuit, in mesh 1. When the direction of mesh current \( i_i \) is the same as the direction of \( I \), the mesh current is given by \( i_i = I \). On the other hand, when the direction of mesh current \( i_i \) is opposite as the direction of \( I \), the mesh current is given by \( i_i = -I \).

3. There is no easy way to express the current source voltage in terms of the mesh currents.

**Worked Examples**

**Example:**

![Circuit Diagram]

Analyze this circuit by writing and solving a set of mesh equations.

**Solution:**

Label the label the mesh currents. Then, label the element currents in terms of the mesh currents:
Notice that the 2 A source on the outside of the circuit is in mesh 3 and that the currents 2 A and \( i_3 \) have the same direction. Consequently

\[
\begin{align*}
i_3 &= 2 \text{ A} \\
\end{align*}
\]

Apply KVL to mesh 1 to get

\[
25(i_1 - i_3) + 9(i_1 - i_2) + 8i_1 = 0
\]

In this equation \( 25(i_1 - i_3) \) is the voltage across the 25 \( \Omega \) resistor (+ on the left), \( 9(i_1 - i_2) \) is the voltage across the 9 \( \Omega \) resistor (+ on top) and \( 8i_1 \) is the voltage across the 8 \( \Omega \) resistor (+ on bottom). Substituting \( i_3 = 2 \) A and doing a little algebra gives

\[
42i_1 - 9i_2 = 50
\]

Next, apply KVL to mesh 2 to get

\[
14(i_2 - i_3) + 24 - 9(i_1 - i_2) = 0
\]

In this equation \( 14(i_2 - i_3) \) is the voltage across the 14 \( \Omega \) resistor (+ on the left), 24 is the voltage source voltage and \( 9(i_1 - i_2) \) is the voltage across the 9 \( \Omega \) resistor (+ on top). Substituting \( i_3 = 2 \) A and doing a little algebra gives

\[
-9i_1 + 23i_2 = -24 + 14(2) = 4
\]

The simultaneous equations can be written in matrix form.
\[ 42i_1 - 9i_2 = 50 \]
\[ -9i_1 + 23i_2 = 4 \]

\[ \Rightarrow \begin{bmatrix} 42 & -9 \\ -9 & 23 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 50 \\ 4 \end{bmatrix} \]

We can use MATLAB to solve the matrix equation:

```matlab
>> A = [ 42 -9
         -9 23 ];
>> b = [ 50
         4 ];
>> A\b
ans =
     1.3401
     0.6983
```

Then

\[ \begin{bmatrix} i_1 \\ i_2 \end{bmatrix} = \begin{bmatrix} 1.3401 \\ 0.6983 \end{bmatrix} \]

That is, the mesh currents are \( i_1 = 1.3401 \text{ A} \) and \( i_2 = 0.6983 \text{ A} \).
Example:

This circuit is represented by the simultaneous equations:

\[
\begin{align*}
    a_{11} i_1 + a_{12} i_2 &= 4 \\
    a_{21} i_1 + a_{22} i_2 &= 10
\end{align*}
\]

Determine the values of the coefficients \( a_{11} \), \( a_{12} \), \( a_{21} \) and \( a_{22} \).

Solution:

Label the mesh currents. Then, label the element currents in terms of the mesh currents:

Notice that the 0.4 A source on the inside of the circuit is in both mesh 1 and mesh 3. Mesh current \( i_1 \) is directed in the same way as current source current but the mesh current \( i_3 \) is directed opposite to the current source current. Consequently

\[ i_1 - i_3 = 0.4 \text{ A} \]
The current source is in both mesh 1 and mesh 3 so we apply KVL to the supermesh corresponding to the current source (i.e. the perimeter of meshes 1 and 3). The result is

\[ 10i_3 + 19(i_1 - i_2) + 10i_1 = 0 \]

In this equation \( 10i_3 \) is the voltage across the horizontal 10 \( \Omega \) resistor (+ on the left), \( 19(i_1 - i_2) \) is the voltage across the 19 \( \Omega \) resistor (+ on top) and \( 10i_1 \) is the voltage across the vertical 10 \( \Omega \) resistor (+ on bottom). Substituting \( i_3 = i_1 - 0.4 \) and doing a little algebra gives

\[ 39i_1 - 19i_2 = 4 \]

Next, apply KVL to mesh 2 to get

\[ 22i_2 - 10 - 19(i_1 - i_2) = 0 \]

In this equation \( 22i_2 \) is the voltage across the 22 \( \Omega \) resistor (+ on the left), \( 10 \) is the voltage source voltage and \( 19(i_1 - i_2) \) is the voltage across the 19 \( \Omega \) resistor (+ on top). Doing a little algebra gives

\[ -19i_1 + 41i_2 = 10 \]

To summarize, the circuit is represented by the simultaneous equations:

\[ 39i_1 - 19i_2 = 4 \]
\[ -19i_1 + 41i_2 = 10 \]

Comparing these equations to the given equations shows

\[ a_{11} = 39, \ a_{12} = -19, \ a_{21} = -19 \ \text{and} \ a_{22} = 41. \]