Example:

Represent this circuit by a second order differential equation.

Solution:
Use KCL to write

$$i(t) = \frac{v(t)}{R_2} + C \frac{d}{dt} v(t)$$

where $C \frac{d}{dt} v(t)$ is the current directed downward in the capacitor and $\frac{v(t)}{R_2}$ is the current directed downward in $R_2$. Use KVL to write

$$v_s = R_1 i(t) + L \frac{d}{dt} i(t) + v(t)$$

where $L \frac{d}{dt} i(t)$ is the voltage across the inductor and $R_1 i(t)$ is the voltage across $R_1$.

Substitute to get

$$v_s = \frac{R_1}{R_2} v(t) + R_1 C \frac{d}{dt} v(t) + \frac{L}{R_2} \frac{d}{dt} v(t) + C L \frac{d^2}{dt^2} v(t) + v(t)$$

$$= CL \frac{d^2}{dt^2} v(t) + \left( R_1 C + \frac{1}{R_2 C} \right) \frac{d}{dt} v(t) + \frac{R_1 + R_2}{R_2 C L} v(t)$$

Finally,

$$\frac{v_s}{CL} = \frac{d^2}{dt^2} v(t) + \left( \frac{R_1}{L} + \frac{1}{R_2 C} \right) \frac{d}{dt} v(t) + \frac{R_1 + R_2}{R_2 C L} v(t)$$
Example:

Represent this circuit by a second order differential equation.

Solution:

Use KVL to get

$$R_2 i(t) + L \frac{d}{dt} i(t) = v(t)$$

where \( L \frac{d}{dt} i(t) \) is the voltage across the inductor and \( R_2 i(t) \) is the voltage across \( R_2 \). Use KCL and KVL to get

$$v_s = R_1 \left( i(t) + C \frac{d}{dt} v(t) \right) + v(t)$$

where \( C \frac{d}{dt} v(t) \) is the current directed downward in the capacitor and \( i(t) + C \frac{d}{dt} v(t) \) is the current directed to the left in \( R_1 \). Substitute to get

$$v_s = R_1 v(t) + R_1 CR_2 \frac{d}{dt} i(t) + R_1 C \frac{d^2}{dt^2} i(t) + R_2 i(t) + L \frac{d}{dt} i(t)$$

$$= R_1 CL \frac{d^2}{dt^2} i(t) + \left( R_1 R_2 + L \right) \frac{d}{dt} i(t) + \left( R_1 + R_2 \right) i(t)$$

Finally

$$\frac{v_s}{R_1 CL} = \frac{d^2}{dt^2} i(t) + \left( \frac{R_2}{L} + \frac{1}{R_1 C} \right) \frac{d}{dt} i(t) + \frac{R_1 + R_2}{R_1 CL} i(t)$$
Example:
The input to the circuit this is the voltage of the voltage source, $v_s(t)$. The output is the voltage $v_2(t)$.

Derive the second order differential equation that shows how the output of this circuit is related to the input.

Solution:
KCL gives

$$\frac{v_s(t) - v_1(t)}{R_1} = C_1 \frac{d}{dt} v_1(t) \quad \Rightarrow \quad v_s(t) = R_1 C_1 \frac{d}{dt} v_1(t) + v_1(t)$$

and

$$\frac{v_1(t) - v_2(t)}{R_2} = C_2 \frac{d}{dt} v_2(t) \quad \Rightarrow \quad v_1(t) = R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t)$$

Substituting gives

$$v_s(t) = R_1 C_1 \frac{d}{dt} \left[ R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t) \right] + R_2 C_2 \frac{d}{dt} v_2(t) + v_2(t)$$

so

$$\frac{1}{R_1 R_2 C_1 C_2} v_s(t) = \frac{d^2}{dt^2} v_2(t) + \left( \frac{1}{R_1 C_1} + \frac{1}{R_2 C_2} \right) v_2(t) + \frac{1}{R_1 R_2 C_1 C_2} v_2(t)$$