The Loneliest Number

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We examined the “Friendly Integer” problem first stated in [3]. A pair \((m, n)\) is called a friendly pair if \(\sigma(n)/n = \sigma(m)/m\) where \(\sigma(n)\) is the sum of all the divisors of \(n\). In this case, it is also common to say that \(m\) is a friend of \(n\), or simply that \(m\) and \(n\) are friends. For convenience, we define the function \(\hat{\sigma}(n) = \sigma(n)/n\). This is often called the abundancy or the index of a number. Perfect numbers have abundancy 2, and thus are all friends. Numbers with abundancy less than 2 are often called deficient, while numbers whose abundancy are greater than 2 are called abundant [6]. The original problem was to show that the density of friendly integers in \(\mathbb{N}\) is unity, and the density of solitary numbers (numbers with no friends) is zero. Also note, that any variables will be positive integers unless stated otherwise. The two letters \(p\) and \(q\) will represent prime integers throughout the paper.

We used two main approaches: one was an analysis of the \(\hat{\sigma}(n)\) function, while the other used number theoretic arguments to find a representation for a friend of 10. In [3], it was shown that 10 is the smallest number where it is unknown whether there are any friends of it. We will assume a basic understanding of the function \(\sigma(n)\). This can be found in [2]. For more on number theoretic techniques, see [1].

We discovered many useful properties of \(\hat{\sigma}(n)\):

1. \(\hat{\sigma}(nm) = \hat{\sigma}(n)\hat{\sigma}(m)\) if \(gcd(m, n) = 1\)
2. \(\hat{\sigma}(n) > 1\) for \(n > 1\)
3. For prime \(p\), integers \(a > b\), \(\hat{\sigma}(p^a) > \hat{\sigma}(p^b)\)
4. For primes \(p < q\), \(\hat{\sigma}(p^a) > \hat{\sigma}(q^a)\)
Proof. 1. \( \sigma(n) \) is weakly multiplicative, therefore \( \sigma(nm)/nm = (\sigma(n)/n)(\sigma(m)/m) = \sigma(n)\sigma(m) \) when \( \gcd(m, n) = 1 \)

2. This follows directly from \( \sigma(n) > n \) for \( n > 1 \)

3. In [3], it was shown that \( \hat{\sigma}(p^{a+1}) > \hat{\sigma}(p^a) \), this is a natural generalization

4. Show that \( \hat{\sigma}(p^a) \) decreases for larger primes, suppose \( q > p \):

\[
\hat{\sigma}(p^a) - \hat{\sigma}(q^a) = \frac{1 + p + p^2 + \ldots + p^a}{p^a} - \frac{1 + q + q^2 + \ldots + q^a}{q^a}
\]

It is enough to show that

\[
q^a(1 + p + p^2 + \ldots + p^a) - p^a(1 + q + q^2 + \ldots + q^a) < 0
\]

Regroup the terms

\[
(q^a - p^a) + qp(q^{a-1} - p^{a-1}) + q^2p^2(q^{a-2} - p^{a-2}) + \ldots + q^ap^a(1 - 1) < 0
\]

Some of these properties can be generalized further, but these will be our building blocks to prove bigger results.

**Proposition 1.** \( \hat{\sigma}(n) < \hat{\sigma}(an) \) for \( a > 1 \)

Proof. In general, \( a \) can share prime factors with \( n \). Let \( a = lm \) where \( \gcd(a, n) = l, \gcd(m, n) = 1 \). We thus have \( \hat{\sigma}(an) = \hat{\sigma}(ln)\hat{\sigma}(m) \) by property 1. Property 2 gives

\[
\hat{\sigma}(an) = \hat{\sigma}(ln)\hat{\sigma}(m) > \hat{\sigma}(ln)
\]

Now property 3 gives us

\[
\hat{\sigma}(ln) > \hat{\sigma}(n)
\]

Thus

\[
\hat{\sigma}(an) > \hat{\sigma}(n)
\]

□
**Corollary 1.** When \( a = n^b \) we get a generalization of property 3: \( \hat{\sigma}(n^j) < \hat{\sigma}(n^k) \) for \( j < k \).

Note that property 4 is not true for general \( x > y \).

Using these tools, we analyzed whether 10 has a friend and what forms that friend can and cannot take. Notice that \( \hat{\sigma}(10) = 9/5 \). We can easily deduce that a friend of 10 must be of the form \( n = 5^a m \) otherwise \( \hat{\sigma}(n) \) could not reduce to the fraction 9/5.

**Lemma 1.** A friend of \( m \) cannot be a multiple of \( m \). That is \( \hat{\sigma}(m) \neq \hat{\sigma}(am) \) for \( a > 1 \).

*Proof.* This follows directly from Proposition 1. Since \( am \) is a multiple of \( m \), \( \hat{\sigma}(am) > \hat{\sigma}(m) \), so \( \hat{\sigma}(am) \neq \hat{\sigma}(m) \). \( \square \)

**Corollary 2** (Even Corollary). A friend of 10 cannot be of the form \( n = 2^a 5^b m \). Thus a friend of 10 cannot be an even integer.

*Proof.* For \( a, b > 1 \), \( 2^a 5^b \) is a multiple of 10, and thus so is \( n = 2^a 5^b m \). Therefore \( n \) is not a friend of 10. A friend of 10 must be of the form \( 5^b m \), so a friend of 10 cannot be an even integer. \( \square \)

**Corollary 3.** A friend of 10 must be the square of some number: \( n = 5^{2b} m^2 \).

*Proof.* Suppose \( \hat{\sigma}(n) = 9/5 \) and \( n = 5^b m \), \( m = p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k} \) then

\[
\frac{\sigma(n)}{n} = \frac{9}{5}
\]

\[
5\sigma(5^b \sigma(p_1^{e_1}) \sigma(p_2^{e_2}) \ldots \sigma(p_k^{e_k})) = (9)5^b p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}
\]

\[
5(1 + 5 + \ldots + 5^b)(1 + p_1 + \ldots + p_1^{e_1}) \ldots (1 + p_k + \ldots + p_k^{e_k}) = (9)5^b p_1^{e_1} p_2^{e_2} \ldots p_k^{e_k}
\]

From the Even Corollary, we have \( p_i > 2 \) for any \( i \leq k \). So the right side must be odd. To obtain this, we must have that every sum on the left side is also odd. Since each \( p_i^{e_i} \) is odd, we must have an odd number of terms in the sum for the whole sum to be odd. To obtain this, each \( e_i \) and \( b \) must be even. Thus \( n \) must be the square of some number. \( \square \)

**Proposition 2.** If \( \hat{\sigma}(n) = 9/5 \), then \( 3 \nmid n \).
Proof. From the even corollary, we have that if \( \hat{\sigma}(n) = \frac{9}{5} \) and \( 3 \nmid n \), then \( n = 3^{2a}5^{2b}m^2, 2, 3, 5 \nmid m \). It is easy to verify that \( \hat{\sigma}(3^45^2) \) and \( \hat{\sigma}(3^25^4) > \frac{9}{5} \). Combining this with Lemma 1, we have that \( \hat{\sigma}(3^45^2m^2), \hat{\sigma}(3^25^4m^2) > \frac{9}{5} \). Thus, the problem reduces to a single case: \( \hat{\sigma}(3^25^2m^2) = \frac{9}{5} \).

Combining this with Lemma 1, we have that \( \hat{\sigma}(3^45^2m^2), \hat{\sigma}(3^25^4m^2) > \frac{9}{5} \).

Thus, the problem reduces to a single case: \( \hat{\sigma}(3^{a_3}5^{a_5}m^{a_m}) \).

We can see that \( \sigma(13^c5^{d}) = 3^4(5)(13^{c-1})(31^{d-1})k^2 \)

\[
\frac{\sigma(k^2)}{k^2} = \frac{405}{\sigma(13^c)\sigma(31^d)} < \frac{405}{448} < 1
\]

\[
\frac{13^{c-1}}{13^{c-1}} \frac{31^{d-1}}{31^{d-1}} \geq 14 \geq 32
\]

This is a contradiction with property 2 for \( \hat{\sigma}(k^2) \). Hence, \( 3 \nmid n \) whenever \( n \) is a friend of 10. \( \square \)

Proposition 3. \( \lim_{k \to \infty} \hat{\sigma}(p^k) = p/(p - 1) \)

Proof.

\[
\lim_{k \to \infty} \frac{\sigma(p^k)}{p^k} = \lim_{k \to \infty} \frac{p^{k+1} - 1}{p^k(p - 1)} = \lim_{k \to \infty} \frac{p - 1/p^k}{p - 1} = \frac{p}{p - 1}
\]

\( \square \)

Proposition 4. If \( \hat{\sigma}(n) = \frac{9}{5} \), then \( n = 5^{2a}p_1^{e_1} \ldots p_m^{e_m} \) where \( m \geq 4 \).

Proof. Using the previous proposition and properties 3 and 4, we will construct the largest value of \( \hat{\sigma}(n^k) \) with 4 distinct primes. Let \( n = (5)(7)(11)(13) \).

Here is a largest value with four distinct primes because from proposition 2, we have that \( p_i \geq 7 \). To maximize the value of \( \hat{\sigma}(n^k) \) we let \( k \to \infty \). Hence,

\[
\lim_{k \to \infty} \hat{\sigma}(n^k) = (5/4)(7/6)(11/10)(13/12) = \frac{1001}{576} < \frac{9}{5}
\]

So there must be at least 5 distinct primes in the factorization of \( n \). \( \square \)
The rest of the paper will be refinements in our representation of \( n \), where \( n \) is a friend of 10.

**Proposition 5.** If \( \hat{\sigma}(n) = 9/5 \), then \( n = 5^{2a}p_1^{e_1} \ldots p_k^{e_k} \) where \( k \geq 5 \).

**Proof.** We can use the same technique as in the last proposition to show that only 3 cases could work if \( n \) were represented as 5 distinct primes. These are 
\[ n = (5^611^e13^d17^f)^2, \]
but this does not work because the smallest it could be is: \( \hat{\sigma}(5^611^e13^d17^f)^2 \) > 9/5. Case 2 gives 
\[ n = (5^611^e13^d19^f)^2, \]
but this does not work because the smallest it could be is: \( \hat{\sigma}(5^611^e13^d19^f)^2 \) > 9/5.

The final case takes a little more work: 
\[ n = (5^611^e13^d23^f)^2. \]
We can see that if \( a > 1 \), then \( \hat{\sigma}(n) > 9/5 \), so \( a = 1 \). Let us examine when \( \hat{\sigma}(n) = 9/5 \), then
\[
5\sigma(5^2)\sigma(7^2)\sigma(11^2e)\sigma(13^2f)\sigma(23^2f) = 9(5^2)(7^2)11^e13^d23^f = l
\]
Clearly, 31 \( \nmid \) \( l \). Since the left hand side is some integer, this results in a contradiction. Hence a friend of 10 must be composed of at least 6 distinct primes. \( \square \)

**Proposition 6.** If \( \hat{\sigma}(n) = 9/5 \), then \( n = 5^{2a}p_1^{e_1} \ldots p_i^{6e_i+2} \ldots p_k^{e_k} \) where \( k \geq 5 \) and \( p_i \equiv 1 \mod 3 \) for some \( i, 1 \leq i \leq k \).

**Proof.** Suppose we had a friend \( \hat{\sigma}(n) = 9/5 \), then the resulting equation occurs for \( n = 5^{2a}m^2 \)
\[
5\sigma(5^2a)\sigma(m^2) = 9(5^{2a})m^2
\]
\[
2\sigma(5^{2a})\sigma(m^2) \equiv 0 \mod 3
\]
Let \( q_1 \equiv 1 \mod 3 \), then \( \sigma(q_1^{6x+2}) \equiv 0 \mod 3 \), \( \sigma(q_1^{6x+4}) \equiv 2 \mod 3 \), \( \sigma(q_1^{6x}) \equiv 1 \mod 3 \). Let \( q_2 \equiv 2 \mod 3 \), then \( \sigma(q_2^{2y}) \equiv 1 \mod 3 \). Clearly for the above stated equation to be true, we must have some \( p_i \equiv 1 \mod 3 \) in the factorization of \( m \). Moreover, it must be of the form \( p_i^{6x+2} \). \( \square \)

**Proposition 7.** If \( \hat{\sigma}(n) = 9/5 \), then \( n = 5^{2a}p_1^{e_1} \ldots p_i^{6e_i+2} \ldots p_j^{2(2e_j+1)} \ldots p_k^{e_k} \)
where \( k \geq 5 \) and \( p_i \equiv 1 \mod 3 \) for some \( i, 1 \leq i \leq k \) and either \( a = 2x \) or \( \exists p_j, 1 \leq j \leq k \) such that \( p_j \equiv 1 \mod 4 \).

**Proof.** The arguments are identical to those of the previous proposition, except mod 4. Notice that an either/or condition results instead of a single fact. \( \square \)
Just for fun, we introduce a new definition.

**Definition 1** (Theoretical Friend). A sequence $n_k$ is a **theoretical friend** of $m$ if: $\lim_{k \to \infty} \hat{\sigma}(n_k) = \hat{\sigma}(m)$.

**Proposition 8.** 10 has at least one theoretical friend, namely $n_k = 3^k5$.

**Proof.**

$$\lim_{k \to \infty} \hat{\sigma}(n) = \lim_{k \to \infty} \frac{\sigma(3^k) \sigma(5)}{3^k}$$

$$= \left( \frac{3}{2} \right) \left( \frac{6}{5} \right) = \frac{9}{5} = \hat{\sigma}(10)$$

For further reading on the topic of $\hat{\sigma}(n)$ and $\sigma(n)$, see [4] and [5]. See [5] for information concerning when $\sigma(n) = k$ has exactly $m$ solutions (Sierpiński Conjecture). See [4] for a more indepth study of $\hat{\sigma}(n)$ and on the distribution and density of numbers of this form.

**Literatur**


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