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Stochastic fatigue damage modeling under variable amplitude loading

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Abstract

A general methodology for stochastic fatigue life prediction under variable amplitude loading is proposed in this paper. The methodology combines a nonlinear fatigue damage accumulation rule and a stochastic S–N curve representation technique to achieve this objective. The nonlinear damage accumulation rule proposed in this paper improves the deficiencies inherent in the linear damage accumulation rule and still maintains its simplicity in the application. The covariance structure of the stochastic damage accumulation process under variable amplitude loading is included into the methodology by a new stochastic S–N curve approach. A wide range of fatigue data available in the literature is used to validate the proposed methodology, which covers different type of metallic materials under constant and variable amplitude loadings. The prediction results are also compared with existing fatigue models.

Keywords: Fatigue; Stochastic; Damage accumulation; Variable loading

1. Introduction

The fatigue process of mechanical components under service loading is stochastic in nature. Life prediction and reliability evaluation is still a challenging problem despite extensive progress made in the past decades. A comprehensive review of early developments can be found in [42].

Compared to fatigue under constant amplitude loading, the fatigue modeling under variable amplitude loading becomes more complex both from deterministic and probabilistic points of view. An accurate deterministic damage accumulation rule is required first, since the frequently used linear Palmgren-Miner's rule may not be sufficient to describe the physics [5]. Second, an appropriate uncertainty modeling technique is required to include the stochasticity in both material properties and external loadings, which should accurately represent the randomness of the input variables and their covariance structures. In addition to the above difficulties, such a model should also be computationally and experimentally inexpensive. The last characteristic is the main reason for the popularity of simpler models despite their inadequacies.

In this paper, a general methodology for stochastic fatigue life prediction under variable loadings is proposed, which combines a nonlinear fatigue damage accumulation rule and a stochastic S–N curve representation technique. A brief review of fatigue damage accumulation rule is given first and a simple nonlinear damage accumulation model is proposed. Next, the uncertainty modeling is discussed and a stochastic S–N curve approach using the Karhunen–Loeve expansion technique is proposed to represent the randomness observed in the experimental data. Then, a wide range of experimental data is explored to validate the proposed methodology. The probabilistic fatigue life predictions of the numerical model are compared with those from experimental data under variable loadings.

The proposed model offers several advantages over existing approaches. It uses a nonlinear fatigue damage accumulation rule, which improves the accuracy of the Miner's rule by considering the load dependence effect of the fatigue damage. Unlike most of the previous nonlinear fatigue damage accumulation models, the proposed model

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does not require cycle-by-cycle calculation and can directly use the cycle counting results from the loading history, which significantly reduces the calculation effort, especially for the reliability evaluation. A stochastic S–N curve approach can capture the covariance structure of the fatigue damage process under different stress levels (which is usually ignored by other models), and thus makes the reliability evaluation more accurate compared to the existing models.

2. Fatigue damage accumulation

2.1. Existing models

Fatigue damage increases with applied loading cycles in both constant loading and variable loading. However, the characteristics of damage accumulation under different loadings are different. For more than eighty years, researchers have tried to find the best rule to describe the fatigue damage accumulation behavior. A comprehensive review is not the objective of this paper and can be found in [5]. Only a few damage accumulation rules are briefly described below.

Among all the fatigue damage accumulation rules, the LDR (linear damage accumulation rule), also known as Palmgren–Miner's rule, is probably the most commonly used. Miner [25] expressed the fatigue damage accumulation under variable loadings as

$$D = \sum_{i=1}^{k} \frac{n_i}{N_i} \tag{1}$$

where D is the fatigue damage of the material, n_i is the number of applied loading cycles corresponding to the ith load level, N_i is the number of cycles to failure at the ith load level, from constant amplitude experiments. Eq. (1) implies that fatigue damage accumulates in a linear manner.

If LDR is used for fatigue life prediction, it is usually assumed that the material fails when the damage D reaches unity. However, it has been shown that LDR produces a large scatter in the fatigue life prediction of both metal and composites [32,17]. Also, LDR cannot explain the load level dependence of fatigue damage observed in the experiments [12]. Despite all those deficiencies, LDR is still frequently used due to its simplicity.

In order to improve the accuracy of LDR, nonlinear functions have been proposed to describe the damage accumulation. Marco and Starkey [24] expressed the damage accumulation function as

$$D = \sum_{i=1}^{k} \left(\frac{n_i}{N_i}\right)^{C_i} \tag{2}$$

where C_i is a material parameter related to ith loading level. A similar formula named damage curve approach has been proposed by Manson and Halford [23]. Eq. (2) can reflect the load-level dependence and load-sequence dependence effects of the fatigue damage accumulation. It is shown that the Miner's sum $\sum_{i=1}^{k} \frac{n_i}{N_i} > 1$ for low-high load sequences and $\sum_{i=1}^{k} \frac{n_i}{N_i} < 1$ for high-low sequence 5]. As pointed out by Van Paepegem and Degrieck [37], this conclusion cannot be applied to all materials in the existing experimental data base in the literature.

Due to the nonlinearity of Eq. (2), the fatigue damage under service loading needs to be computed in a cycle-by-cycle manner, which requires a large amount of computational effort. This disadvantage can be circumvented by approximating the nonlinear function by double linear functions [12]. In each stage, a linear damage accumulation rule is applied. For two-block loading, the double linear damage model is easy to implement. For the multi-block loading or spectrum loading, the determination of the parameters in the model becomes complicated [12,9].

Several more complex fatigue damage accumulation functions have been proposed for increased accuracy. Halford and Manson [11] proposed a double damage curve approach, which combines the accurate parts of both the double linear damage approach and the damage curve approach. A similar result was obtained by using the fatigue crack growth concept [38]. A more recent approach for fatigue damage accumulation is to use a nonlinear continuum damage mechanics model [5,4,30]. Despite the different proposed damage functions, the basic idea is to calculate the fatigue damage in an evolutionary manner using a scalar damage variable. The main differences lie in the number and characteristics of the parameters used in the model, in the requirements for additional experiments, and in their applicability [5].

From the brief discussion above, it is found that most of the nonlinear fatigue damage models improve the deficiencies within LDR by considering additional loading effects. However, they are usually computationally expensive compared to LDR especially when the applied loading is repeated block loading or spectrum loading, since most of them require cycle-by-cycle calculation. This disadvantage makes it difficult to perform simulation-based reliability evaluation. Furthermore, the parameters calibration using experimental results is hard to perform for some nonlinear fatigue damage models. The model parameter calibration of double linear damage rule is difficult if only multi-block loading test is available. Some models contain several parameters and need couples of different variable amplitude loading data to solve these parameters uniquely [9]. In the following section, we are trying to develop a nonlinear damage accumulation model, which improves the deficiencies within the linear damage accumulation rule but still maintains its computational simplicity. Since the major deficiency of LDR is that it is independent of applied load levels, this paper attempts to modify the LDR to make it load level dependent and yet preserve the linear summation form to make the calculation easier.

2.2. Fatigue damage accumulation under stationary loading

Fatigue cyclic loading is stochastic in nature. In this section, we first discuss the stationary loading process. Under laboratory conditions, the stationary loading process is usually approximated by repeated multi-block loading. Within each block, the loading is not stationary. But this assumption of stationary process holds when the material experiences many blocks before it fails (i.e. high-cycle fatigue problem). Under the stationarity assumption, the distribution of applied loading cycles is adequate to describe the loading process.

To make the discussion easier for the fatigue damage accumulation under stationary loading, let us consider a fatigue problem under a repeated two-block loading first. N_{f} is the total number of cycles to failure. If the linear damage accumulation rule is used, we obtain:

$$\begin{cases} \frac{n_1}{N_1} + \frac{n_2}{N_2} = 1 \\ n_1 + n_2 = N_{\mathrm{f}} \end{cases} \tag{3}$$

Eq. (3) can be rewritten as

$$\begin{cases} \frac{n_1}{N_1} + A_1 \frac{N_{\mathrm{f}} - n_1}{N_1} = 1 \\ A_1 = \frac{N_1}{N_2} \end{cases} \tag{4}$$

From Eq. (4) we can express the cycle ratio $\frac{n_1}{N_1}$ as a function of cycle distribution $\omega_1 = \frac{n_1}{N_{\mathrm{f}}}$ as

$$\frac{n_1}{N_1} = \frac{1}{\frac{A_1}{\omega_1} + 1 - A_1} \tag{5}$$

If the fatigue S–N curve under constant amplitude loading (s) is expressed as

$$N = g(s) \tag{6}$$

then, A_1 in Eq. (5) is a material parameter depending on the stress levels and equals $\frac{g(s_1)}{g(s_2)}$. Similarly, the cycle ratio of the second stress level can be expressed as a function of the cycle distribution as

$$\frac{n_2}{N_2} = \frac{1}{\frac{A_2}{\omega_2} + 1 - A_2} \tag{7}$$

where A_2 is a material parameter depending on the stress levels and equals $\frac{g(s_2)}{g(s_1)}$.

Substituting Eqs. (5) and (7) into Eq. (4), we obtain

$$\frac{n_1}{N_1} + \frac{n_2}{N_2} = \sum_{i=1}^{2} \frac{1}{\frac{A_i}{\omega_i} + 1 - A_i} \tag{8}$$

The above derivation is under the assumption of a linear damage accumulation rule. For the materials that follow this rule, the right hand of Eq. (8) equals unity. For materials that do not follow the linear rule, A_i cannot be determined only by constant amplitude experiments. They depend on the material properties and loading conditions. This parameter can be calibrated using one additional fatigue experiment under variable loading. In the proposed model, we plot the cycle ratio and cycle distribution together for each stress level. Then we compute the coefficients A_i in Eq. (5) through least square regression. Based on the experimental data collected in this study, the following empirical function is used to calculate A_i:

$$A_i = \alpha(s_i/\bar{s})^{\beta} \tag{9}$$

where α and β are material parameters; s_i is the current stress level and $\bar{s}$ is the mean value of all the stress amplitudes in each block. Notice that the modified LDR (Eq. (8)) is stress dependent.

Eq. (8) is extended for repeated multi-block loading as

$$\sum_{i=1}^{k} \frac{n_i}{N_i} = \sum_{i=1}^{k} \frac{1}{\frac{A_i}{\omega_i} + 1 - A_i} \tag{10}$$

For continuous stationary spectrum loading, Eq. (10) is expressed as

$$\int \frac{n(s)}{N(s)} \mathrm{d}s = \int \frac{1}{\frac{A(s)}{f(s)} + 1 - A(s)} \mathrm{d}s \tag{11}$$

where the cycle distribution ω_i (probability description for block loading) becomes the probability density function $f(s)$ of the applied continuous random loading (see Fig. 1). Eqs. (10) and (11) constitute the proposed fatigue damage accumulation model under stationary loading. Compared with the linear damage rule, the proposed model includes the effect of the stress levels. The Miner's sum is not a constant but depends on the cycle distribution of the applied loadings.

When using Eq. (10) (or Eq. (11)) for fatigue life prediction, the right-hand side of Eq. (10) (or Eq. (11)) is first calculated. For repeated multi-block loading, the cycle distribution of the different stress levels at failure can be approximated using the cycle distribution value in a single block. For high-cycle fatigue, this is a reasonable approximation. Then the fatigue life prediction is performed in the same way as the classical procedure using the linear damage rule.

From Eq. (10) (or Eq. (11)), it is seen that the proposed model still maintains the simplicity of the linear damage rule. It can directly use the cycle counting results and does not require cycle-by-cycle calculation. The proposed model includes the load level effect and load contents effects, which improve the deficiencies within the LDR model. In the later part of this paper, it is shown the proposed model give a more accurate prediction with similar calculation effort compared to the LDR model.

2.3. Fatigue damage accumulation under non-stationary loading

Fatigue damage accumulation under non-stationary loading is complicated compared with that under stationary loading. The proposed model described in Section 2.2 in only applicable to stationary loading as it only considers the cycle distribution of the applied loading. For non-stationary applied loading, the cycle distribution is not

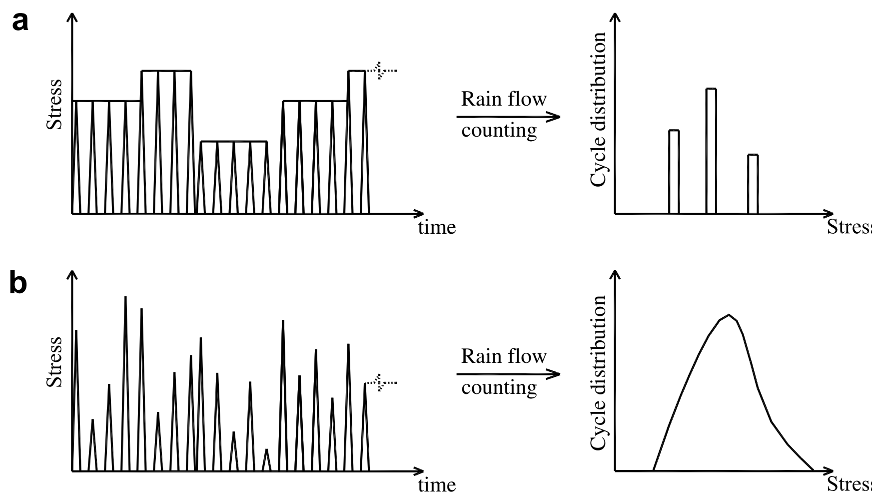

Fig. 1. Schematic illustration of cycle distribution using rain-flow counting method: (a) Block loading; (b) continuous random loading.

sufficient to describe the loading process. Under laboratory conditions, step loading is also used for variable loading tests. The material is first pre-cycled under one or several stress levels. Then the material is cycled till failure at a certain stress level. This type of loading is non-stationary as the mean value and variance of the applied loading change corresponding to time. For this type of loading, load sequence effect of the fatigue damage accumulation is observed for some materials [5]. The high-low and the low-high loading sequences result in different Miner's sum. A comprehensive study of the general non-stationary loading needs further work. In the current study, several step loading experimental data are collected. The proposed model shown in Section 2.2 is modified to include the load sequence effect for step loadings.

The model coefficients A_i expressed in Eq. (9) are modified as

$$\log(A_i) = \alpha + \beta \log\left(\frac{s_i}{\bar{s}}\right) + \gamma \log\left(\frac{\bar{s}}{s_k}\right) \tag{12}$$

where γ is a material parameter to describe the load sequence effect of the material, and s_k is the stress amplitude at the last step. The third term in Eq. (12) represents the load sequence effect on the final fatigue damage of the material. When the material does not experience load sequence effect ($\gamma = 0$) or the applied loading is stationary $\left(\log\left(\frac{\bar{s}}{s_k}\right) = 0\right)$, Eq. (12) reduces to Eq. (9). The material parameters in Eq. (12) can be calibrated using the high-low and the low-high step loading experiments following the same procedure for repeated block loading.

Eqs. (10) and (12) are used together for fatigue life prediction under step loadings. For non-stationary loading, the cycle distribution at failure is not known before hand. Therefore a trial and error method can be used to find the solution of Eq. (10). The initial values for cycle distribution can be commutated using the LDR model. It is found that usually a few iterations are enough for convergence.

3. Uncertainty modeling

The large uncertainties within the fatigue problem come from several sources. Among them, external applied loading and material fatigue resistance are most important and thus are discussed in this section.

3.1. Uncertainty modeling of external loading

Two approaches are commonly used to describe the scatter in the random applied loadings. One is in the frequency domain and uses power spectral density methods. The other is in the time domain and uses cycle counting techniques. The major advantages of the frequency domain approach are that it is more efficient and can obtain an analytical solution under some assumptions of the applied loading process, such as Gaussian process, stationary and narrow banded. This of cause limits the applicability of the frequency domain approach to some real problems [14,36]. Also, most of the frequency domain approaches assume the linear fatigue damage accumulation rule [6,1,2], due to the loss loading sequence information during the computation of the power spectral density function from the loading history.

The time domain approach is used in this paper. Among many different cycle counting techniques, rain flow counting is predominantly used and is adopted in the proposed methodology. A detailed description of the rain flow counting method can be found in [34].

In the proposed fatigue damage accumulation model in the last section, the cycle distribution is required for fatigue life prediction. This information can be obtained by performing rain flow counting of the loading history. A schematic explanation is shown in Fig. 1 for two different loading history.

In the validation parts of this paper, only the zero-mean and multi-block (step) loading experimental data are collected. For these loadings, the rain flow counting method is not necessary since the peak and valley values are easy to obtain. However, the rain flow counting technique is still introduced here in order to make the methodology general for practical applications.

3.2. Uncertainty modeling of material properties

In classical fatigue life analysis, a fatigue damage accumulation rule together with the material properties under constant amplitude loading is used to predict the fatigue life under variable loadings. Many probabilistic frameworks and methods have been proposed to describe the statistics observed under constant amplitude fatigue tests and to evaluate the reliability under variable loadings. Depending on the method to handle the randomness in constant amplitude tests, the probabilistic methods can be grouped into two categories.

One type is to treat the fatigue lives at different stress levels as independent variables. The statistics of the random variables are described using a statistical distribution function, such as Weibull or lognormal. In this paper, we name this type of approach as statistical S–N curve approach. Liao et al. [21] used a model named dynamic interference statistical model to evaluate the reliability under spectrum loading, which assumes independent lognormally distributed life and linear fatigue damage accumulation. Kam et al. [15] compared several damage accumulation rules including the Miner's rule using both Lognormal and Weibull distribution of fatigue life for composites. Le and Peterson [20] also used lognormal distribution and the LDR for fatigue reliability analysis of engine blades. Shen et al. [31] used similar assumptions for fatigue life prediction under a narrow band Gaussian stochastic stress process. Kaminski [16] used a perturbation-based stochastic finite element method for fatigue analysis of composites, in which the LDR was also used. The restrictions of input random variables of fatigue life is looser since only the numerical characteristics of the random variables are required in the computational methodology, such as mean and variance.

The other widely used approach is to use a family of S–N curves corresponding to different survival probability of the material. This approach is also known as quantile or percentile S–N curve (referred as Q–S–N curve in this paper later on). Shimokawa and Tanaka [32] used the quantile S–N curve and the LDR to analyze the fatigue reliability under a two-step loading. Both lognormal and Weibull distribution assumption of fatigue life were explored. Kopnov [18,19] proposed a method named intrinsic fatigue curve (IFC), which is another format of the quantile S–N curve approach, combined with the LDR for fatigue analysis. The difference between quantile S–N curve and IFC is that Q–S–N uses a set of deterministic S–N curves and each represents a different survival probability level. IFC uses a single random function in which the realizations of the random function are the same as Q–S–N. A similar methodology for describing the scatter in the constant amplitude S–N curves was proposed by Pascual and Meeker [27] using a random fatigue-limit model. This random fatigue limit is explicitly included in the S–N model. Maximum likelihood methods are then used to estimate the parameters of the S–N equation as well as the parameters of the fatigue limit distribution. If the fatigue limit takes the quantile value from the distribution, the resulting S–N curve is the quantile S–N curve. Rowatt and Spanos [29] used Q–S–N curves and Markov chain models proposed by Bogdanoff and Kozin [3] for fatigue life prediction of composites. Ni and Zhang [26] used Q–S–N and LDR for fatigue reliability and compared their prediction results with the experimental results under two-step loading. Zheng and Wei [43] assumed that the constant fatigue life follows a lognormal distribution and used Q–S–N with LDR for probabilistic fatigue life prediction under repeated block loading. The model predictions were compared with experimental results for one type of steel.

From a statistical point of view, both statistical S–N curve and Q–S–N curve have an implicit assumption in representing the set of random variables. The statistical S–N curve approach assumes the covariance function of these variables is zero and the quantile S–N curve approach assumes the covariance function as unity. Either assumption can be barely achieved in the realistic condition. A more appropriate approach is to propose an S–N curve representation technique which can include the covariance structure of the constant amplitude fatigue lives. A schematic comparison of the various methods for representing the S–N curves is plotted in Fig. 2.

In this paper, the fatigue lives N under different constant amplitude tests are treated as random fields/processes with respect to different stress levels s. Stochastic expansion techniques are very successful in describing the variation in the corresponding random field/process. Several methods are available, such as spectral representation method [10,33], Karhunen-Loeve (KL) expansion method [22], polynomial chaos expansion [7,8] etc. In this study, the KL expansion technique is used and a new stochastic S–N curve method is proposed based on the KL expansion technique.

The fatigue lives under constant amplitude loading are assumed to follow the lognormal distribution for the sake of illustration. As mentioned earlier in this section, both lognormal and Weibull distributions are commonly used in the literature. The lognormal assumption makes the $\log(N(s))$ a Gaussian process with mean value process of $\log(\overline{N}(s))$ and standard deviation of $\sigma_{\log(N)}(s)$, where $\log(\overline{N}(s))$ is the mean S–N curve obtained by regression analysis. It needs to be pointed out that the Gaussian assumption is not a requirement in the proposed methodology. Non-Gaussian methods for random field representation are available and can be applied to the problem without difficulty. In the next section, it is shown that the

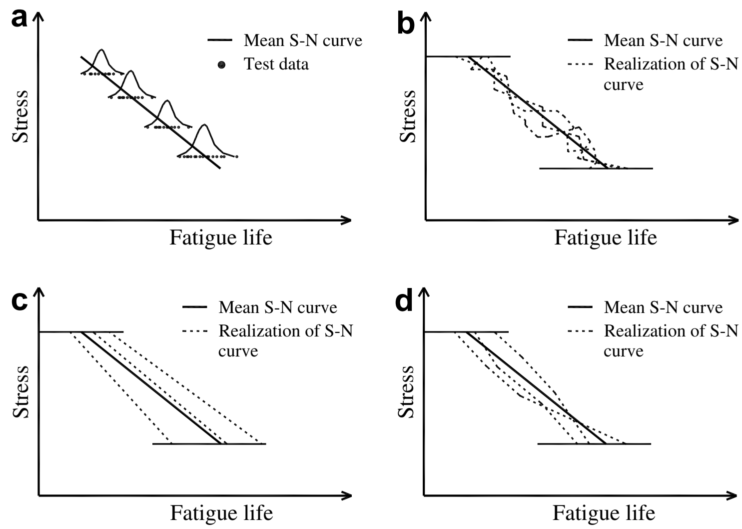

Fig. 2. Schematic comparisons of different approaches in representing the fatigue S–N curve: (a) S–N curve of experiments; (b) statistical S-N curve approach; (c) Q–S–N curve approach; (d) stochastic S–N curve approach (proposed).

lognormal distribution gives a satisfactory prediction for the materials used in this paper.

It has been shown that the variance is not a constant but a function of stress level s [27]. The $\sigma_{\log(N)}(s)$ represents the scatter in the data and can be obtained by classical statistical analysis. Based on the above assumption, the process

$$Z(s) = \frac{(\log N(s)) - \log(\overline{N}(s))}{\sigma_{\log(N)}(s)} \tag{13}$$

is a normal Gaussian process with zero mean and unit variance.

From a physical standpoint, the autocovariance function of the fatigue lives should decrease as the difference between stress levels increases. An exponential decay function is proposed for the covariance function $C(s_1, s_2)$ of $Z(s)$ as

$$C(s_1, s_2) = e^{-\mu|s_1-s_2|} \tag{14}$$

where μ is a measure of the correlation distance of $Z(s)$ and depends on the material. In classical S–N fatigue experiments, the specimen is tested until failure or runout at a specified stress level and cannot be tested at the other stress levels. Due to the non-repeatable nature of fatigue tests, the covariance function cannot be determined by constant amplitude fatigue experimental data alone. Since in the proposed methodology, the nonlinear fatigue damage accumulation model also needs one additional variable loading fatigue test to calibrate the model parameters α, β and/or γ. μ can be calibrated by the same variable loading fatigue test data as well.

In KL expansion, the random process/field $Z(s)$ can be expressed as a function of a set of standard random variables, or, in other words, expressed as a combination of several random functions. Generally, the expansion takes the form

$$Z(s) = \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)f_i(s) \tag{15}$$

where $\xi_i(\theta)$ is a set of independent random variables, satisfying

$$\begin{cases} E(\xi_i(\theta)) = 0 \\ E(\xi_i(\theta)\xi_j(\theta)) = \delta_{ij} \end{cases} \tag{16}$$

where E denotes the mathematical expectation operator, and δ_{ij} is the Kronecker-delta function.

In Eq. (15), $\sqrt{\lambda_i}$ and $f_i(x)$ are the ith eigenvalues and eigenfunctions of the covariance function $C(s_1, s_2)$, evaluated by solving the homogenous Fredholm integral equation analytically or numerically:

$$\int_D C(s_1, s_2)f_i(s_2) = \lambda_i f_i(s_1) \tag{17}$$

In practical calculation, only a truncated number of terms in Eq. (15) is required to achieve the satisfied accuracy. Under the standard Gaussian assumption, 10–20 terms are adequate to get very precise results. The detailed computational procedure for KL expansion can be found elsewhere [28,13].

From Eqs. (13) to (17), we obtain

$$\log(N(s)) = \sigma_{\log(N)}(s) \sum_{i=1}^{\infty} \sqrt{\lambda_i}\xi_i(\theta)f_i(s) + \log(\overline{N}(s)) \tag{18}$$

Substituting Eq. (18) into Eq. (10) (or Eq. (11)), we can solve for the fatigue life under variable amplitude loading. No analytical solution exists and Monte-Carlo simulation is used to find the probabilistic fatigue life distribution.

The proposed uncertainty modeling method in this section includes the covariance structure in the fatigue analysis. The importance of the covariance structure on the final reliability evaluation can be illustrated using the example problem below.

Consider a two-block loading case under linear damage accumulation assumption. The covariance function for statistical S–N approach, Q–S–N and the proposed stochastic S–N approach can be expressed as

$$\begin{cases} C(s_1, s_2) = 0 & \text{statistical S–N curve} \\ C(s_1, s_2) = 1 & \text{Q–S–N} \\ C(s_1, s_2) = e^{-\mu|s_1 - s_2|} & \text{stochastic S–N curve} \end{cases} \quad (19)$$

For fatigue damage accumulation

$$D = \frac{n_1}{N_1} + \frac{n_2}{N_2} = D_1 + D_2 \quad (20)$$

the mean value of the fatigue damage is

$$E(D) = E(D_1) + E(D_2) \quad (21)$$

and the variance of the fatigue damage is

$$\text{Var}(D) = \text{Var}(D_1) + \text{Var}(D_2) + 2\rho\text{Var}(D_1)\text{Var}(D_2) \quad (22)$$

where ρ is the correlation coefficient of the random variables D_1 and D_2. It is seen that the different approaches have no effect on the mean value of the fatigue damage but have effect on the variance. Thus,

$$\text{Var(statistical S–N)} \leqslant \text{Var(stochastic S–N)} \leqslant \text{Var(Q–S–N)} \quad (23)$$

A schematic representation of the failure probability with respect to time from the three methods is shown in Fig. 3. The mean value of the fatigue life is 5 (log scale) and standard deviation is 0.1, 0.15 and 0.2 (log scale) for statistical S–N, stochastic S–N and Q–S–N, respectively. These amounts of difference in the standard deviation are observed for the collected materials in this study. Thus different approaches give different fatigue reliability estimates. The difference is especially significant around the tail region. For design and maintenance against fatigue, it is usually required that the mechanical component stay at a very low failure probability (i.e. less than 0.1%). At this stage, the difference among the three approaches is around 0.16 in log scale, which is about 45% difference in real life cycles. Zheng and Wei [43] used Q–S–N approach and observed that the standard deviation of the predicted fatigue life of 45 steel notched elements under variable amplitude loading is longer than that of test results. The authors stated that the reason behind it should be further investigated. Eqs. (19)–(23) give a possible explanation for this phenomenon, i.e., the effect of the correlation structure. Since the fully uncorrelated and fully-correlated cases can be barely found in reality, the standard deviation of the experimental results should lie between those predicted by the statistical S–N approach and the Q–S–N approach. In the next section, it is shown that this phenomenon is not only for 45 steel but also for other materials.

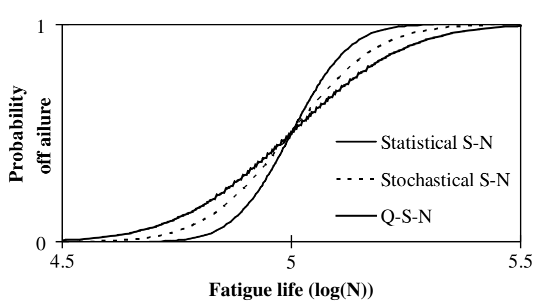

Fig. 3. Failure probability predictions by different approaches.

It is interesting to notice that the statistical S–N approach and Q–S–N are two special cases of the proposed method. If μ in Eq. (19) approaches positive infinity, the covariance function reduces to zero, giving the statistical S–N method. If μ in Eq. (19) approaches zero, the covariance function reduces to 1, giving the Q–S–N method.

4. Comparison with experimental data

In this section, the prediction results from the proposed method are compared with experimental data available in the literature. The objective is to show the applicability of the model to different materials and different loadings. The collected experimental data includes a wide range of metallic materials under step and multi-block loadings. Another guideline in collecting the data is that the experimental data should have enough data points both in the constant amplitude tests and variable amplitude tests, so that reliable statistical analyze and comparisons can be performed.

Table 1
Experimental description of collected materials

Material name	Reference	Types of variable loading[a]	Number of specimens[b]	
			Constant loading	Variable loading
Nickel-silver	Tanaka et al. [35]	TS	200	50
16Mn steel	Xie [40]	TS and MS	15	10
LY12CZ aluminum alloy	Wu [39]	MB	N/A	15–21
Carbon steel	Xie [40]	TS	15–18	13–15
45 steel-1	Zheng and Wei [43]	MB	10	9
45 steel-2	Yan et al. [41]	MB	10	6

[a] The abbreviation and schematic illustration of the type of the variable loading is shown in Fig. 4.
[b] The number of specimens indicates the number under the same stress level (constant loading) or the same type of variable loading.

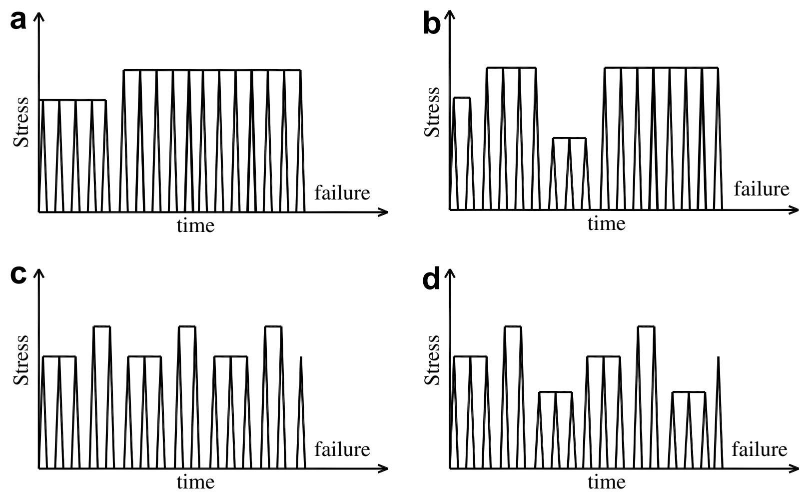

Fig. 4. Illustration of the type of variable loadings used in this study: (a) TS (two-step loading); (b) MS (multi-step loading); (c) TB (two-block loading); (d) MB (multi-step loading).

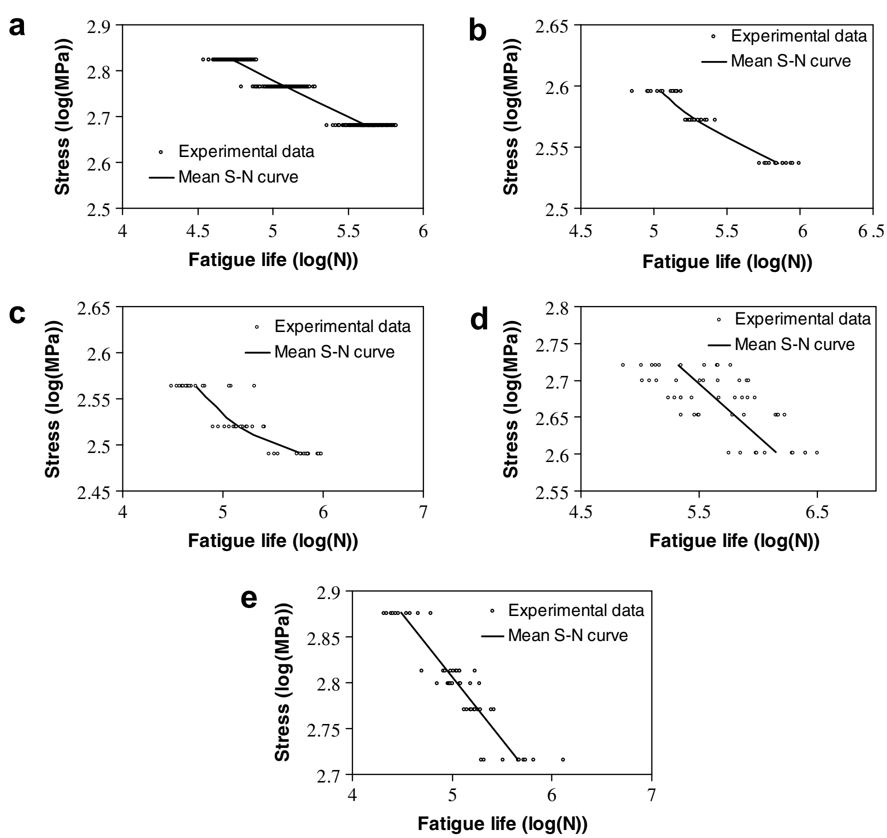

Fig. 5. Constant amplitude S–N curve data for different materials: (a) nickel-silver; (b) 16Mn steel; (c) Carbon steel; (d) 45 steel-1; (e) 45 steel-2.

4.1. Experiment description and material fatigue properties

A brief summary of the collected experimental data is shown in Table 1, which includes material name, reference, variable loading type, and specimen numbers at constant and variable loading tests. In Table 1, the abbreviation and schematic illustration of the applied variable loading is shown in Fig. 4.

The material fatigue properties (constant S-N experimental data) are plotted in Fig. 5. The statistics of the experimental data are shown in Table 2, which includes the mean value and standard deviation of the fatigue life

Table 2
Statistics of constant amplitude S–N curve data

Material	Stress amplitude (MPa)	Statistics of Fatigue life (log(*N*))		Material	Stress amplitude (MPa)	Statistics of fatigue life (log(*N*))	
		Mean	Standard deviation			Mean	Standard deviation
Nickel-silver	478	5.62	0.10	45 steel-1	525	5.33	0.32
	583	5.09	0.09		500	5.50	0.34
	666	4.73	0.07		475	5.59	0.29
16Mn steel	394	5.05	0.10		450	5.82	0.35
	373	5.29	0.06		400	6.15	0.26
	344	5.85	0.07	45 steel-2	750	4.49	0.15
LY12CZ	125.44	4.37	0.10		650	5.00	0.14
	101.92	4.76	0.04		630	5.04	0.12
	78.79	5.16	0.09		590	5.24	0.10
	49.98	5.65	0.15		520	5.65	0.24
	46.06	6.01	0.25	Carbon steel	366	4.73	0.19
	37.04	6.82	0.13		331	5.16	0.14
					309	5.79	0.16

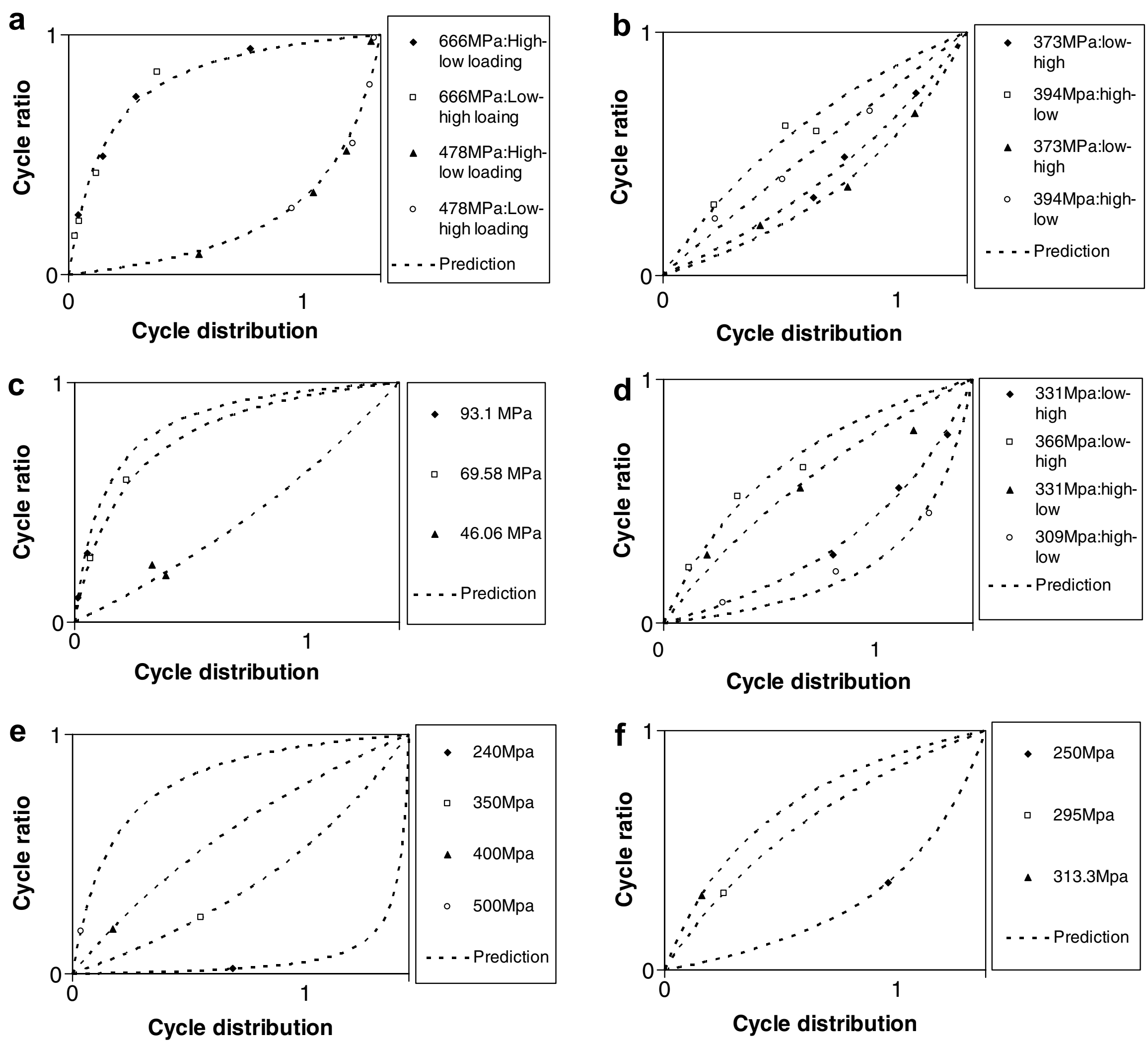

Fig. 6. Cycle ratio and cycle distribution relationship for different materials: (a) Nickel-silver; (b) 16Mn steel; (c) LY12CZ aluminum alloy; (d) carbon steel; (e) 45 steel-1; (f) 45 steel-2.

at different stress levels. The original data for LY12CZ aluminum alloy under constant amplitude tests are not available, thus only the statistics of the test data are listed in Table 2. The fatigue lives under constant amplitude tests are assumed to follow the lognormal distribution.

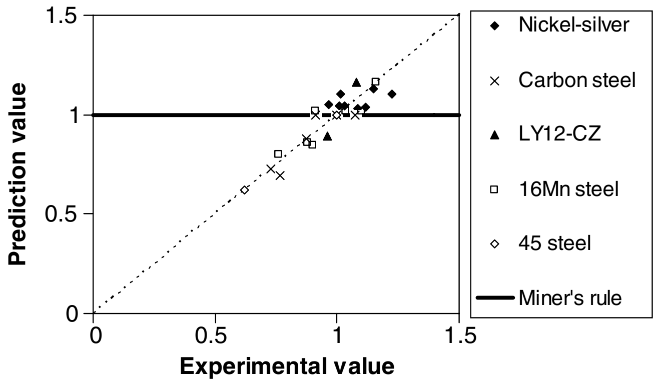

Fig. 7. Comparisons between predicted and experimental Miner's sum for different materials.

4.2. Validation of the nonlinear fatigue damage accumulation rule

The comparison of the prediction results using the proposed nonlinear fatigue damage accumulation rule and the experimental results is performed first. As shown in Section 3, the three uncertainty modeling techniques have no effect on the mean value of the fatigue damage. In order to minimize the randomness effect on the fatigue damage accumulation modeling, the mean values of the experimental data are used. The comparisons are plotted in Fig. 6. The x-axis is the cycle distribution and the y-axis is the cycle ratio as described in Section 2. The dashed curves are the prediction of the proposed method and the points are experimental results. From Fig. 6, it is seen that the proposed

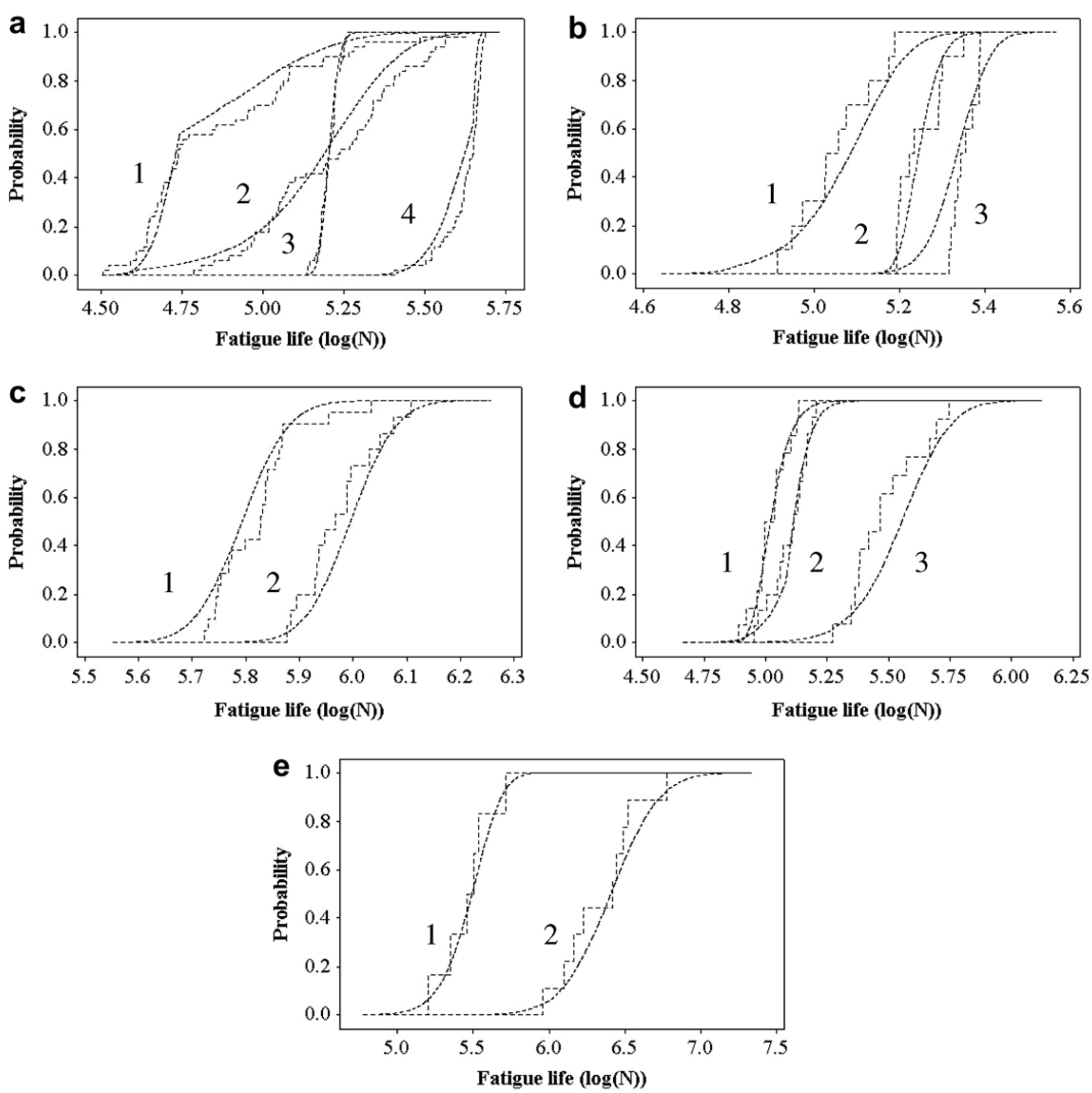

Fig. 8. Empirical cumulative distribution function comparisons between prediction and experimental results: (a) nickel-silver; (b) 16Mn steel; (c) LY12CZ aluminum alloy; (d) carbon steel; (e) 45 steel.

Table 3
Experiments description shown in Fig. 8

Material	Symbol in Fig. 8	Variable loading[a]
Nickel-silver	1 – Fig. 8a	TS: 666 (5.54×10^4) → 478 (X)
	2 – Fig. 8a	TS: 666 (3.98×10^4) → 478 (X)
	3 – Fig. 8a	TS: 478 (1.15×10^5) → 666 (X)
	4 – Fig. 8a	TS: 478 (4.46×10^5) → 666 (X)
16Mn steel	1 – Fig. 8b	TS: 394 (7.5×10^4) → 373 (X)
	2 – Fig. 8b	TS: 373 (1.46×10^5) → 394 (X)
	3 – Fig. 8b	MS: 373 (10^5) → 394 (10^5) → 373 (10^5) → 344 (10^5) → 394 (10^5) → 344 (10^5) → 394 (X)
LY12CZ	1 – Fig. 8c	MB: 93.1 (2.64×10^3) → 69.58 (1.056×10^4) → 46.06 (1.848×10^4) → 23.52 (3.432×10^4)
	2 – Fig. 8c	MB: 93.1 (6.6×10^2) → 69.58 (3.3×10^3) → 55.86 (6.6×10^3) → 46.06 (1.584×10^4) → 37.24 (3.96×10^4)
Carbon steel	1 – Fig. 8d	TS: 331 (8.06×10^4) → 373 (X)
	2 – Fig. 8d	TS: 331 (1.21×10^5) → 373 (X)
	3 – Fig. 8d	TS: 331 (4.03×10^5) → 309 (X)
45 steel-1	2 – Fig. 8e	MB: 240 (10^5) → 350 (8×10^4) → 400 (2.5×10^4) → 500 (10^4) → 400 (2.5×10^4) → 350 (8×10^4) → 240 (10^5)
45 steel-2	1 – Fig. 8e	MB: 500 (1.5×10^4) → 590 (4×10^3) → 626.6 (5×10^3) → 590 (4×10^3) → 500 (1.5×10^4)

[a] The number before the bracket indicates the stress level and the number in the bracket is the applied cycle numbers. For the step loadings (TS and MS), the applied cycle number of the last stress level is not known as prior and thus an "X" is used.

function (Eq. (5)) give a satisfactory prediction and relate the cycle ratio and cycle distribution under different loading conditions.

The predicted Miner's sums under variable loadings are compared with experimental results for all the materials in Fig. 7. From Fig. 7, it is seen that the proposed method gives a better prediction compared to the LDR.

4.3. Validation of the uncertainty modeling

For the stochastic fatigue modeling methodology, the final objective is to predict the fatigue life distribution under different variable loadings. Thus it is easy to calculate the reliability of the mechanical components. In this section, the predicted fatigue life distributions are compared with the empirical fatigue life distribution of experimental data. Due to large number of experimental data collected in this study and the space limitations, we only show the comparisons under several loading conditions for each material. The comparisons are shown in Fig. 8 by plotting the predicted and experimental distribution together. The details of the plotted experimental distributions are listed in Table 3. All the prediction results are obtained using 10,000 Monte Carlo simulations. Since the proposed nonlinear damage accumulation model does not require cycle-by-cycle calculation, the computational time is very short and ranges from 5 to 20 s for 10,000 Monte Carlo simulations. In Fig. 8, the stepped lines are experimental results and the smooth lines are prediction results. It is observed that the prediction results agree with the experimental results very well for different variable loadings.

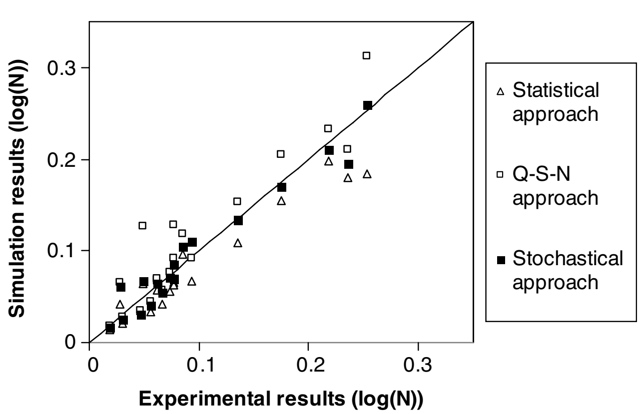

Fig. 9. Standard deviation comparison of different approaches.

The standard deviation of the fatigue life of experiments and predictions are plotted in Fig. 9, for all the experimental data collected in this study. The predictions using statistical S–N approach, Q–S–N approach and the proposed stochastic S–N approach are plotted together. The prediction results of all three approaches use 10,000 Monte Carlo simulations. Points lying close to the diagonal line indicate close agreement between the experimental results and the prediction results. From Fig. 9, it is seen that the statistical S–N approach tends give a smaller variance prediction, since most of the prediction results lie below the diagonal line. The Q–S–N tends to give a larger variance prediction, since most of them lie above the diagonal line. The proposed stochastic S–N approach is closer to the experimental results, since the variance prediction is between the statistical S–N approach and Q–S–N approach. This type of observation is consistent with Eq. (23) in Section 3. It can be concluded that the covariance structure in the S–N curve is important for probabilistic fatigue life prediction under variable loading and thus needs to be considered in analysis and design.

5. Conclusions

A general stochastic fatigue life prediction methodology under variable loading is proposed in this study. It combines a nonlinear fatigue damage accumulation rule and a stochastic S–N curve representation technique. The nonlinear fatigue damage accumulation rule improves the deficiency inherent in the Miner's rule but still maintain its simplicity in calculation. The verification with experiments shows that the proposed fatigue damage accumulation rule improves the mean value of the fatigue life prediction while only using the similar computational effort as that of the Miner' rule.

A new uncertainty modeling method for fatigue S–N curve representation is proposed. It uses the Karhunen–Loeve expansion technique to consider the covariance structures within different stress levels. It is shown that the available probabilistic fatigue life prediction methods are two special cases of the proposed method, which implicitly assume that the covariance function is either zero or unity. The verification with experiments shows that ignoring the covariance of the input variables results in different variance predictions of the fatigue life. The difference is especially significant at low failure probability stage, which is of more interest for practical design and maintenance decision with respect to fatigue.

Compared with traditional fatigue life prediction methods (i.e. using linear damage accumulation rule and ignoring the covariance of input random variables), the proposed methodology requires one additional set of experimental data under variable loading. The benefits are achieved both in the accuracy of the mean value and variance prediction of the fatigue life.

The proposed methodology is suitable and validated for stationary variable loading and certain types of non-stationary variable loading (step loadings). For general non-stationary variable loadings, further research work is required both for model development and experimental validation. The current validation is only for uniaxial loading. The proposed methodology needs to be extended to general multiaxial loading.

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