The Trumpet: Demystified Using Mathematics

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Abstract

Presented, is a model of how a trumpet works, starting from the simplest tube and increasing the complexity until faced with a realistic model. My approach is to use differential equations and standard laws of physics to describe what happens from when the player starts until the sound is heard by the listener. I model how the sound is generated, sustained, and changed by the player and the instrument.

In exploring these principles, I uncover many fundamental truths about the instrument and how it is played. Using these truths, I am able to identify the issues surrounded by why so many players find it difficult to play in the upper registers. The model also explores some of the apparent inconsistencies in teaching styles and philosophies towards playing the instrument.

1. Background

There are many misconceptions about playing the trumpet and how sound is produced on the instrument. This can be seen from the various opinions and methods of professional trumpet players. This is particularly true about the upper register where many opinions are completely opposite of each other. Before I can address these issues, I must introduce some basics about sound and waves.

Sound is the longitudinal compression of air. A speaker compresses and
uncompresses the air next to it which in turn compresses the air next to that. The molecules of the air vibrate back and forth at an average speed. This small back and forth movement creates a variance in pressure, increasing and decreasing, that is measured in Hertz (number of oscillations back and forth per second).

Here is a quick description of wave properties along a short string. Using Newton's Second Law:

\[ F = m \frac{d^2y}{dt^2} \]  \hspace{1cm} (1.1)

Take the wave to be in the standard xy-plane and assume: \[ \tan \Theta = \frac{\partial y}{\partial x} = m \]

because: \[ \tan \Theta \approx \sin \Theta \] for small angles. Take the length of the string segment we are dealing with to be \( \Delta x \) and \( \mu \) to be the mass per unit length. This makes \( \mu \Delta x \) the total mass. Plug in the known values into eq. 1.1:

\[ T \Delta m = \mu \Delta x \frac{\partial^2 y}{\partial t^2} \] divide by \( \Delta x \) : \[ T \frac{(\Delta m)}{(\Delta x)} = \mu \frac{(\partial^2 y)}{(\partial t^2)} \] (1.2)

Through a limiting process, we can see what happens at a single point on the string.

\[ \lim_{\Delta x \to 0} \left( \frac{\Delta m}{\Delta x} \right) = \frac{\partial m}{\partial x} = \frac{\partial^2 y}{\partial x^2} \]

Finally, plug this into eq. 1.2 and divide by \( T \):

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \] where \( c = \sqrt{T/\mu} \)  \hspace{1cm} (1.3)
Here, \( T \Delta m \) is the net vertical force on the string (\( T \) being the tension), \( m \) is the slope for all equations except eq. 1.1 (where it represents mass), and \( c \) represents the speed of the wave motion on the string. A general solution to this second order differential equation is:

\[
y = f_1(ct-x) + f_2(ct+x) \quad \text{where the sign on} \ x \ \text{denotes the wave’s direction and} \ f \ \text{is an arbitrary function.}
\]

2. Introduction

The model will be built up from a simple cylinder, describing how sound is produced on it and then modifying this for end corrections (mouthpiece and bell). At each stage, the player and his effects will be considered. In the end, the two will be combined to give a complete picture of the instrument-player system.

3. An Open-Ended Tube

First, consider a single tube with both of its ends open. Everything about it is in equilibrium with the environment. Seal one end with a hand, then pull it away quickly. This creates a lower pressure at the opening than what is in the space surrounding it because of the quick displacement of air. Air will rush from the tube and the room to equalize this point. There will be a lower pressure just inside the tube because some of the air left the tube to fill the low pressure point outside the tube. So air from outside the tube now enters it to equalize this lower pressure. This pulse of low pressure travels up
to the other end of the tube where air rushes in from the room to equalize this low pressure point. Since momentum must be conserved in this process (and no external forces are acting on the air – eq. 1.1), the air from the outside must continue moving through the tube to the other end in the form of a high pressure pulse. This process dies because more and more air molecules at the ends of the tube are set into vibration (which is heard as sound) as some molecules escape before being reflected back. This reflection process is known as resonance.

The fact that the tube is open at both ends is a good indication that the pressure there will be close to the ambient room pressure. This point is called a pressure node. A pressure anti-node is a point where the pressure varies maximally. If a standing wave were to be created inside the tube, the longest possible wave would be twice that of the length of the tube ($L$). Any other wavelength, and the node will not line up with the two ends of the tube. This can be seen in the figure below.

The lowest harmonic of the tube is the one that has a single pressure anti-node in the middle and a pressure node at each end.

$$f_1 = \lambda_1 = 2L \quad \text{where} \quad \lambda \quad \text{is the wavelength.}$$

This is called the fundamental harmonic (or just the fundamental: $f_1$). Because there is a pressure node at each end, it can be seen, intuitively, that the harmonics of the tube are every integer value of the fundamental harmonic.
4. How does it all work?

The trumpet player's lips act similar to a spring-mass system. Basically when there is a buildup of air pressure inside the mouth, the lips are forced open and the air pressure equalizes. Then there is a lower air pressure inside the mouth so the lips slam shut to allow the pressure to build back up (equalize). This happens very rapidly and creates a sound wave which is audible without the instrument at all. Of course this explanation is oversimplified, but suitable for the model.
“Buzzing” into the open ended tube that was previously described will change the model drastically. The tube becomes closed at the lip end and remains open at the other. On the end where the lips are, there is a huge change in pressure (creating a pressure anti-node) whereas the open end remains relatively constant (it remains a pressure node). The new fundamental is now $4L$. Now the only harmonics that can satisfy these conditions are odd integer multiples of the fundamental.

![Diagram showing the acoustic pressure and nodes in a tube.](Figure from www.phys.unsw.edu.au/music) [3]

Described below is a general spring-mass-damper system formula. The lips have a mass $m$ and dampening constant $R$ can be attributed to mouthpiece pressure (discussed next), while the spring constant $k$ is due to the tension in the lips. So, when there is no mouthpiece, $R=0$. Of course, these constants vary from moment to moment as the mouthpiece moves on the lips, pressure is added or reduced, and lip tension changes. The driving force function $f(t)$ is due to the pressure build up in the mouth and lungs.
(essentially this should be about the same value).

\[ m \frac{d^2x}{dt^2} + R \frac{dx}{dt} + kx = f(t) \quad (4.1) \]

This can be rewritten as: \[ m \frac{d^2x}{dt^2} = f(t) - R \frac{dx}{dt} - kx \]

Although the model is somewhat idealistic, it can describe many of the effects that players often have, discussed in the analysis. We can describe the pressure wave inside a cylindrical tube the same way that we can describe a mechanical wave along a string or rope because of the one-dimensional quality of both waves (there are waves in all directions, but the primary one is along the tube's primary axis). Assuming a pressure node at each end, just like a rope tied down on both ends, we can predict the motion of a standing wave. The two fixed points are \( x = 0 \) and \( x = L \), where \( L \) is the length of the tube/rope.

\[ y_n(x, t) = (A_n \sin \omega_n t + B_n \cos \omega_n t) \sin \omega_n \frac{x}{c} \]

where \( \omega \) is the circular frequency: \( \omega_n = \frac{(n \pi c)}{L} \) This can be converted into the form that was previously discussed in an intuitive sense where the frequency is: \( f_n = \frac{nc}{4L} \)

for our tube with one “tied down” end and one “loose end”, if compared with the string. In all these equations, \( c \) is the speed of wave propagation in air (see eq. 1.3). If we plug 1 in for \( n \), we get the value of the fundamental. Please note that there is a slight
discrepancy in the figures and my notation, as the figures were not drawn by me. The ν in the figure corresponds to my c, and the fundamental in the picture is \( f_0 \) while my fundamental is \( f_1 \).

This verifies mathematically our observation that every harmonic had to be an integer multiple of the fundamental. Also note that the acoustic length of the pipe is longer than the actual pipe, and this can be verified intuitively (it takes a short distance beyond the end of the tube before the pressure reflects back). This means the wave actually reflects a distance of approximately \(.61r\) beyond the opening, where \( r \) is the radius of the pipe. This yields the more precise equation: \[ f_n = \frac{nc}{4(L + .61r)} \]

5. Acoustic Impedance

Before moving on, we can describe the additional effects of our simple instrument. One primary detail to look at is the acoustic impedance.

\[ Z = \frac{P}{U} \] Where \( Z \) is the acoustic impedance and \( P \) is the pressure and \( U \) is the volume flow of air. This can be more accurately described in terms of the characteristic impedance of the pipe (sometimes referred to as wave impedance).

\[ Z_\theta = \frac{(\rho c_s)}{A} \] Where \( \rho \) is the mass density of air, \( c_s \) is the speed of sound, and \( A \) is the cross-sectional
area of the pipe.

\[ c_s = 331.3 + 0.6t \]

Where \( t \) is the temperature in degrees Celsius. So, the impedance value changes as the horn heats up and as the air inside the horn becomes saturated (increase in mass density of the air inside the pipe). A common misconception is that the pitch changes because the metal expands and contracts from temperature changes from playing. When, in reality, this effect is negligible compared to the effects of impedance.

6. The Bell and Mouthpiece

In a trumpet there is a flare and then a bell at the end of the pipe. The lower frequencies/longer waves are not able to follow along the bell as easily as a narrow tube, so they get reflected back earlier, making the tube effectively shorter. The shorter waves are better able to follow this curve and so are more easily radiated as sound and escape the instrument. The strongly radiated higher frequencies make the tube much louder because the pitches fall into the ear's more sensitive hearing range. As these waves get radiated more, there is less reflection and thus less resonance, so the standing wave becomes weaker.

The mouthpiece has many opposite effects than that of the bell. It lowers the frequency of the highest resonances and provides a counterbalance for how the bell raises the frequencies.
7. The Shape

The shape of the trumpet is complicated and difficult to describe using simple geometry; describing this and the effects of every detail are beyond the scope of this paper. The shape is crucial to the instrument because it is why the trumpet has a complete harmonic series (like the open ended tube) except for the fundamental. The higher frequencies combine in a process called mode locking which helps the lips vibrate at the missing fundamental. One can imagine the various different impedance values throughout the tubing. In mode locking, the input impedance (from the air column) will peak periodically to help the driving mechanism (the lips) oscillate at a steady frequency. When a player talks about a horn's “response”, this is usually what they are talking about, whether they realize it or not. Each frequency from this process is raised so that the ratios between them are integer values of the fundamental (4\(L\) is the “missing fundamental” and 2\(L\) is considered the lowest note) where the actual fundamental is about .7 \(f_1 = \frac{\lambda_1}{4} \approx 5.7L\). This longer wavelength is why “pedal C” is always flat on a standard trumpet. The actual note being played is not the first integer multiple of the rest of the harmonic series. Still, for practice purposes, many players “bend” the pitch up to make it “in tune” using their lips.

When a valve on trumpet is depressed, the length of the trumpet is increased somewhat. This is because the holes in the valves line up with tubes that actually add
extra piping. The minute details of this process are beyond the scope of this paper, but every valve lowers the pitch; none raise it.

8. Combining the Results with the Player

The player’s lips are usually vibrating at a different frequency than that of the bore of the instrument. The resonance of the bore is stronger (for all practical purposes) than that of the lips and takes control of the pitch, so to speak. This is why a player can be “buzzing” at any pitch and when he puts the horn to his lips, the pitch will “slot” into place at the nearest natural resonance of the instrument. This is not to say that the lips have no control; because in the previous example, the pitch would probably be “out of tune” if the frequency of the lips are off of the nearest horn resonance by a lot. Also vibrato is accomplish in many instances from changing pitch through changing lip tension.

9. Analysis

This is by no means a complete model of how the trumpet is played or works. A few weaknesses of the model that can be addressed are: doppler effect, shape of the instrument, precise measurements of valves and how they effect pitch, properties of sound waves (superposition, undertones/overtones, amplitude/volume, etc), material properties of the instrument (even though the effect is far less than in wooden instruments), and a more accurate model of the lip movements.
The benefits of this model are that we can use it to address the fundamental questions posed in the beginning. Why do so many trumpet players find the upper register difficult? As the frequency of the wave increases, we mentioned before that the standing wave becomes weaker. There is less energy being conserved within the tubing because much of it is lost to sound energy and being radiated out of the horn. At this point, the player must induce a much more powerful forced resonance than with the lower pitches of the instrument.

There are three main ways of creating a more powerful driving force for the lips to oscillate faster (which can be seen in eq. 4.1 or eq. 4.2). The most effective way would be to add air pressure, as this can be increased without changing anything else. If one adds mouthpiece pressure, the pitch will increase, but at the expense that it could create an overdamped effect and cause the oscillations of the lips to stop. This is an effect that many trumpet players often complain about. If one increases only lip tension, the results are identical; but with more lip tension and a stronger driving force (air pressure), the forces can balance out and the sound can be sustained. Also note that less mass within the mouthpiece and more air pressure will have a similar effect, but probably would not be as effective as this is a harder thing to control.

10. Conclusion

The actual sound wave is a longitudinal vibration in the air that is caused by
escaping air molecules from equalizing a sinusoidal pressure wave. When playing, things such as the tongue level (which is so highly debated) don't even matter. The air pressure being supplied behind the lips is approximately equal everywhere inside the body. It builds up at the smallest opening, which is the lips (the only opening). One can see that increasing the air pressure in the lungs alone will increase the pressure buildup behind the lips, resulting in a faster oscillation. Raising the tongue in the mouth will not effect this air pressure at all, just as lowering it would not. This argument alone goes against many teacher's opinions. One reason why there is so much confusion is probably that either lip tension, mouthpiece placement, lip position, or air pressure change when the tongue is moved. This would contribute to the effects of changing the pitch, and these things could happen unconsciously without the player realizing it.

In general, the pitch of the instrument is highly dependent on eq. 2.1. This shows that the four main things that play the instrument are lip tension, mouthpiece pressure, amount of mass inside the mouthpiece (mouthpiece placement), and air pressure behind the lips. Having too much of any one of the aforementioned quantities can cause an overdamped effect and cut off the sound. Having too little will have the same result except through an underdamped effect.

As a final statement, I would like to note that the principles applied in this paper can be applied to any brass instrument. There are minor differences in the shape and type
of tubing (conical versus cylindrical), but the conclusions in the end are the same.

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References


http://www.phys.unsw.edu.au/%7Ejw/brassacoustics.html

http://ccrma.stanford.edu/CCRMA/Courses/150/index.html


[8] Nave, Carl. The Trumpet. http://hyperphysics.phy-astr.gsu.edu/hbase/music/trumpet.html#c3

http://www.glenbrook.k12.il.us/gbsci/phys/Class/sound/soundtoc.html
