1 Universal Critical-Point Amplitude Relations

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1 Introduction

1.1 Opening remarks and outline

The phenomenological theory of scaling has been extremely useful in understanding critical phenomena in model systems and in real materials (e.g. Essam and Fisher, 1963; Domb and Hunter, 1965; Widom, 1965; Kadanoff, 1966; Patashinskii and Pokrovskii, 1966; Griffiths, 1967). A related concept, formulated as the hypothesis of universality, greatly reduces the variety of different types of critical behaviour by dividing all systems into a small number of equivalence classes (e.g. Fisher, 1966; Jasnow and Wortis, 1968; Watson, 1969; Griffiths, 1970; Betts et al., 1971; Kadanoff, 1971; Halperin and Hohenberg, 1967, 1969). The first characteristic of a universality class is that all the systems within it have the same critical exponents. In addition, the equation of state, the correlation functions and other quantities become identical near criticality, provided one matches the scales of the order parameter, the ordering field, the correlation length and the correlation time (e.g. Fisher and Burford, 1967; Ho and Lister, 1969; Vicentini-Missoni et al., 1969; Ferer et al., 1971a, b; Ferer and Wortis, 1972; Ritchie and Fisher, 1972; Levelt-Sengers, 1974; Tarko and Fisher, 1975; Aharony and Hohenberg, 1976).

A property of hyperscaling or hyperuniversality (two-scale-factor universality) applies to systems in the universality classes of fluctuation-dominated (i.e. non-mean-field) critical behaviour. These ideas were first developed phenomenologically and later confirmed by explicit renormalization group (RG) calculations (e.g. Widom, 1965; Kadanoff, 1966; Stauffer et al., 1972; Ferer et al., 1973; Aharony, 1974a; Fisher, 1974a, 1975a, b; Gerber, 1975; Hohenberg et al., 1976a). A scaling formulation for finite-size systems (Fisher, 1971; Fisher and Barber, 1972; reviews by Nightingale, 1982; Barber, 1983; Privman, 1990) has been extended to incorporate hyperscaling-type ideas relatively recently (Privman and Fisher, 1984; see also Brézin, 1982; Nightingale and Blöte, 1983; Binder et al., 1985). Another important new development has been the theory of conformal invariance and its applications to two-dimensional (2d) critical behaviour (see a review by Cardy, 1987).

The universality of scaling functions leads naturally to the consideration of universal critical amplitudes and amplitude combinations (Watson, 1969; Betts et al., 1971; Stauffer et al., 1972; Ahlers, 1973; Barmatz et al., 1975; Bauer and Brown, 1975; Aharony and Hohenberg, 1976; Hohenberg et al., 1976a, b; Privman and Fisher, 1984). The universality class to which a given experimental system belongs is thus not only characterized by its critical exponents but also by various critical-point amplitude combinations which are equally important. In particular, the variations in exponents between
different universality classes are often quite small, whereas amplitude ratios may vary by large amounts.

The aim of this review is to survey the current knowledge on universal relations among critical amplitudes. In the remaining sections of this introduction we consider a scaling formulation for bulk and finite-size systems, incorporating the universality of scaling functions in a particularly transparent form. No substantiation for the scaling ansatz will be given at this stage. Instead, we concentrate on its implications, specifically for the universality properties of critical amplitudes. A simple prescription for identifying universal amplitudes and amplitude combinations will be outlined, and several types of such universal quantities discussed.

Universal amplitude combinations emerge in all branches of phase transitions, critical phenomena and other related fields. It was therefore necessary for us to limit the scope of our review. The selection of topics has been determined in part by a desire to have a tutorial component, aimed at researchers who are less familiar with the nomenclature of universal amplitude ratios, etc. However, our goal is to review the topics selected in some detail, covering the most recent research results and surveying the current status of theory and experiment at the level of ongoing research. Thus, we hope that expert researchers will find this review a useful reference source. Obviously, the final selection of the topics covered has been influenced to some extent by authors' research interests.

In Section 2, we introduce the notation and definitions for bulk critical-point amplitudes and universal combinations. The scaling theory of critical-point behaviour, with emphasis on amplitudes, is discussed in Section 3, and its substantiation by RG methods is presented in Section 4. While we have attempted to keep the discussion as general as possible, these theoretical sections (2–4) are centred on statics and dynamics of the most studied $n$-vector models of ferromagnetic and liquid–gas critical point transitions. Section 5 briefly lists various methods of analytical and numerical estimation of bulk universal amplitude combinations, with comments on the relative advantages and limitations of different techniques.

A comprehensive summary of numerical results, for both statics and dynamics, for a class of selected models is given in Section 6. Discussion of experimental results is presented in Sections 7 and 8, for statics and dynamics, respectively.

Section 9 is devoted to polymer conformations: this topic is separated out mostly due to differences in notation and nomenclature. Finally, a rather detailed review of universal amplitudes and amplitude combinations in finite-size systems is given in Section 10.

As noted already, we had to limit the topical coverage of this review. Thus, certain subjects are not considered at all; these include, for example,
Kosterlitz–Thouless transitions and 2d melting, systems with anisotropic correlations ("directed models"), quantum critical phenomena at $T = 0$, etc. For some other topics we only list reference literature without actually reviewing the available results; this applies mostly to universal amplitude combinations involving surface properties (e.g. Section 10.3). Finally, for some theoretical developments, notably conformal invariance, we quote the appropriate results but do not review the underlying theory. (To some extent, these restrictions were influenced by the availability of comprehensive reviews in earlier Phase Transitions and Critical Phenomena volumes.)

1.2 Universality and scaling

The aim of this section is to present a scaling formulation which is particularly transparent and entails no excessive notation or conventions: the universality of amplitudes or amplitude ratios will emerge naturally in connection with the universality of scaling functions. Our purpose in this introductory section is to illustrate, not to substantiate, the scaling ideas. Thus, consider static critical behaviour of a finite-size system with periodic boundary conditions. We assume that the system is either finite in all dimensions, with characteristic size

$$L = V^{1/d},$$

(1.1)

where $V$ is the volume, or infinite in at most one dimension, with characteristic size in the other directions

$$L = A^{1/(d-1)},$$

(1.2)

where $A$ is the cross-sectional area. Such systems have no phase transitions as long as $L$ is finite.

We use here magnetic notation, with a prototype system having a ferromagnetic critical point, e.g. the Ising model. The formulation is, however, rather general. All quantities have been made dimensionless by suitable choices of scale factors. Thus, for the free-energy density, $f$, measured in units of $k_B T$, we expect

$$f(t, H; L) = f_s(t, H; L) + f_{ns}(t; L),$$

(1.3)

where

$$t = (T - T_c)/T_c$$

(1.4)

is the reduced temperature variable, $T_c$ refers to the $d$-dimensional infinite system, and $H$ is the ordering field. Here the "singular" (as $L \to \infty$) part, $f_s$, develops the thermodynamic singularities in the $L \to \infty$ limit, while $f_{ns}$ denotes
the non-singular "background", which can be chosen to have no field dependence.

We consider first systems with critical points having no logarithmic bulk \((L = \infty)\) singularities, i.e. with a non-integer specific heat exponent \(\alpha\), and below their upper critical dimension, which is, for example, \(d = 4\) for the ferromagnetic Ising spin model, etc. (see Fisher, 1974a). Then the singular part of the free energy can be described by the universal scaling form (Privman and Fisher, 1984)

\[
f_s(t, H; L) = L^{-d} Y(K_t t L^{1/\nu}, K_h HL^{\Delta/\nu}) + \ldots,
\]

which also entails hyperscaling (see below). Corrections to scaling in (1.5) are universal provided one allows for the system-dependent factors \(K_t > 0\) and \(K_h > 0\). For the correlation length \(\xi(t, H; L)\), we have a similar ansatz (Privman and Fisher, 1984),

\[
\xi(t, H; L) = L X(K_t t L^{1/\nu}, K_h HL^{\Delta/\nu}) + \ldots,
\]

with the appropriate universal scaling function \(X(x, y)\).

The "metric factors" \(K_t\) and \(K_h\) contain all the non-universal system-dependent aspects of the critical behaviour. The scaling functions \(X\) and \(Y\) are the same for all systems in a given universality class. Note that the finite-size scaling functions do depend on the boundary conditions and the sample shape. When the temperature and the field are at their critical-point values, \(t = 0, H = 0\), we identify the universal amplitudes \(X(0, 0)\) and \(Y(0, 0)\) from the relations

\[
\xi(0, 0; L) \approx X(0, 0)L,
\]
\[
f_s(0, 0; L) \approx Y(0, 0)L^{-d}.
\]

With the temperature not exactly at its critical value, i.e. \(t \neq 0\), consider the bulk limit of large \(L\). Thus, for \(L \gg |K_t t|^{-\nu}\), it is generally expected that the size dependence in, for example, (1.5), must disappear: we assume

\[
Y(x, y) \approx x^{d \nu} Q_+ (y x^{-\Delta}) \quad \text{for } x \to +\infty,
\]
\[
Y(x, y) \approx (-x)^{d \nu} Q_- [y (-x)^{-\Delta}] \quad \text{for } x \to -\infty,
\]

where the functions \(Q_+\) and \(Q_-\) are universal. This yields

\[
f_s(t, H; \infty) \approx |K_t t|^{2-\Delta} Q_\pm [(K_h K_t^{-\Delta}) H |t|^{-\Delta}],
\]

and the hyperscaling relation \(2 - \alpha = d \nu\). Note that we follow the conventional notation with \(\pm\) corresponding to \(t \geq 0\). A similar argument can be advanced for the correlation length, yielding

\[
\xi(t, H; \infty) \approx |K_t t|^{-\nu} S_\pm [(K_h K_t^{-\Delta}) H |t|^{-\Delta}],
\]
Scaling forms where $|H|$ enters nonlinearly, i.e. the "$H$-scaled" representation equivalent to (1.11)-(1.12), can also be derived. We thus recover the familiar bulk scaling laws with the correlation exponent $\nu$, the specific heat exponent $\lambda_1$, and the gap exponent $\gamma$.

\[ \Delta \equiv \beta \delta \equiv \beta + \gamma. \]  

(1.13)

The scaling forms (1.5), (1.6) and (1.11), (1.12) are convenient for the discussion of universal amplitude ratios, as will be illustrated in the next section. For example, one can see immediately that the critical-point combination

\[ \lim_{t \to 0^+} \left[ f_s(t, 0; \infty) \xi^d(t, 0; \infty) \right] = Q_\nu(0) S_\nu^d(0), \]  

(1.14)

is universal.

The reader must be cautioned, however, that the scaling ansätze (1.5)-(1.6), (1.11)-(1.12) implicitly contain many of the results and phenomenological assumptions of the scaling theory, and they obscure the importance of substantiation and verification by numerical and experimental tests. They include, for example, the equality of critical exponents for $t > 0$ and $t < 0$, and they assume the same metric factors for $f_s$ and $\xi$, etc. Furthermore, the notions of scaling, and hyperscaling, are not clearly separated. The following sections of our review address these issues. Note also that for the sake of simplicity we left out of consideration here several technical points, including the definitions of the singular versus the non-singular parts of the free energy (1.3), and of the correlation lengths for finite systems as well as consideration of different system shapes and non-periodic boundary conditions, etc.

As already mentioned, the scaling formulation presented in this section incorporates hyperscaling. However, the additional principle of conformal invariance (reviewed by Cardy, 1987) may impose further restrictions on the scaling functions. We do not discuss the conformal invariance formalism in this review, but only quote some results. Conformal invariance is most useful in two space dimensions and allows calculations of critical exponent values and certain universal amplitudes for many isotropic 2d models (see Cardy (1987) for details). Specifically, for the amplitudes introduced in this section (see (1.7)-(1.8), (1.14)) we have the following results: for the correlation length defined in the periodic strip geometry $L \times \infty$, by the exponential decay of the spin–spin correlation function, we have (Cardy, 1984a,b)

\[ X(0, 0) = (\pi\eta)^{-1} \quad (d = 2), \]  

(1.15)

where $\eta$ is the standard bulk critical correlation function exponent. Similar results can be derived for certain other definitions of $\xi$ and boundary
conditions, see section 10. For the amplitude in (1.8), again in the periodic-strip geometry, we have (Affleck, 1986; Blöte et al., 1986)

\[ Y(0, 0) = -\pi c/6 \quad (d = 2), \tag{1.16} \]

where \( c \) is the conformal anomaly number (Cardy, 1987), which is a universal quantity characterizing the 2d universality classes and usually sufficient for the determination of the critical exponents (Belavin et al., 1984; Friedan et al., 1984). Finally, if the correlation length is defined by the second moment of the energy–energy correlation function, which in 2d can be done unambiguously provided \( \alpha > 0 \), we have the following remarkable recent result by Cardy (1988a), for (1.14) and its \( t < 0 \) counterpart:

\[ Q_\pm(0)S_\pm^d(0) = -(c/12\pi)(2 - \alpha)(1 - \alpha)^{-1} \quad (d = 2, \alpha > 0). \tag{1.17} \]

Note that similar-looking amplitudes defined for \( t \to 0^+ \) and \( t \to 0^- \) are not necessarily equal on scaling grounds alone.

### 1.3 Multitude of universal amplitude combinations

In this section we consider several types of universal amplitude combinations, focusing for simplicity on the static bulk critical behaviour. The first class of amplitude ratios is conveniently associated with the scaling relations among critical exponents. For illustration, consider the \( H = 0 \) free energy,

\[ f_\pm(t, 0; \infty) \approx F_\pm|t|^{2-\alpha}, \tag{1.18} \]

where

\[ F_\pm \equiv K_\pm^2t^{-\alpha}Q_\pm(0) \tag{1.19} \]

by (1.11). Obviously, the ratio

\[ F_-/F_+ = Q_-(0)/Q_+(0) \tag{1.20} \]

is universal, and is naturally associated with the equality of the \( t > 0 \) and \( t < 0 \) free-energy critical exponents (denoted both by \( 2 - \alpha \) here). Consider another familiar scaling relation, \( \alpha + 2\beta + \gamma = 2 \), which can be rewritten in the form

\[ (2 - \alpha) - 2\beta + (-\gamma) = 0. \tag{1.21} \]

Restricting our consideration to \( t < 0 \) to have non-zero spontaneous magnetization in the limit \( H \to 0^+ \), relation (1.11) yields

\[ M(t, 0^+; \infty) = -\frac{\partial f}{\partial H} \approx -K_hK_\beta Q'_-(0^+)(-t)^\beta, \tag{1.22} \]
while for the initial susceptibility we get

$$\chi(t, 0; \infty) = \frac{\partial M}{\partial H} \approx -K_t^{\gamma}K_t^{\gamma}Q^\nu(0)(-t)^{-\gamma}.$$  \hspace{1cm} (1.23)

Thus, we can associate with (1.21) the universal amplitude combination

$$\lim_{t \to 0^-} \left[ f_s(t, 0; \infty) M^{-2}[t, 0^+; \infty] \chi(t, 0; \infty) \right] = -Q^\nu(0)Q'^\nu(0)/[Q'(0^+)]^2.$$  \hspace{1cm} (1.24)

By combining various derivatives of the free energy, both above and below $T_c$, one can construct an unlimited number of amplitude ratios of this type.

The second class of universal bulk amplitude combinations can be associated with the hyperscaling relations among critical exponents. Relation (1.14) provides an example of such a universal quantity associated with the hyperscaling relation $2 - \alpha = dv$. Both scaling and hyperscaling will be discussed further in Sections 3 and 4. At this point, we can classify all the exponent and universal amplitude relations containing the spatial dimensionality $d$ explicitly, as hyperscaling type. Typically, both thermodynamic and correlation amplitudes must be combined to construct hyperscaling-type universal amplitude ratios, customarily termed “hyperuniversal” or “two-scale-factor universal” in the literature.

To proceed, we turn again to the scaling ansatz (1.5) for finite-size systems with periodic boundary conditions. Its RG interpretation is particularly simple. Indeed, the exponents in (1.5),

$$\lambda_t = 1/v > 0 \quad \text{and} \quad \lambda_h = L/v > 0,$$  \hspace{1cm} (1.25)

are just the relevant RG eigenexponents corresponding to the scaling fields of the RG transformation (Wegner, 1972)

$$g_t = c_t t + o(t, H), \quad g_h = c_h H + o(t, H),$$  \hspace{1cm} (1.26)

(see Section 4 for details). Relation (1.5) can be rewritten in terms of these quantities as

$$f_s(t, H; L) \approx L^{-d} Y[g_t(L/l)^{\lambda_t}, g_h(L/l)^{\lambda_h}].$$  \hspace{1cm} (1.27)

Here $l$ is some fixed (system-independent) reference microscopic length. Thus, $c_t = K_t^{\lambda_t}, c_h = K_h^{\lambda_h}. This scaling form can be extended to include arguments for additional relevant scaling fields ($\lambda > 0$), in the cases of multicritical phenomena, or associated with surface couplings (e.g. Diehl, 1986). Obviously, this yields another source of universal amplitude ratios. Such ratios are also associated with correction-to-scaling terms resulting from the nonlinearity of scaling fields (e.g. the $o(t, h)$ corrections in (1.26)) as well as from inclusion of additional arguments in (1.27), accounting for the RG-irrelevant ($\lambda < 0$)
scaling fields (e.g. Aharony, 1976a; Wegner, 1976; Aharony and Ahlers, 1980; Aharony and Fisher, 1983). These topics will be discussed in detail in Section 4.

The last class of universal amplitude ratios that we consider in this introductory survey includes amplitudes of logarithmic singularities arising when $\alpha = 0$ (Widom, 1965; Wegner, 1972); for logarithmic singularities in the limit $\alpha \rightarrow -1, -2, -3, \ldots$ consult, for example, Chase and Kaufman (1986). Consider again, for simplicity, bulk zero-field critical behaviour (1.18) (for finite-size scaling in the $\alpha \rightarrow 0$ limit, see Privman and Rudnick (1986) and Privman (1990)). The non-singular part of the free energy can be expanded as

$$f_{\text{ns}}(t; \infty) = F_0 + F_1 t + F_2 t^2 + o(t^2).$$

(1.28)

The mechanism for the emergence of the logarithmic singularity as $\alpha \rightarrow 0$ is via development of poles in the amplitudes $F_\pm$ and $F_2$. Thus,

$$Q_\pm(0) = - \frac{\bar{Q}_0}{\alpha} + \bar{Q}_\pm + O(\alpha),$$

(1.29)

in (1.19), with the universal coefficients $\bar{Q}_0$ and $\bar{Q}_\pm$. Note that it is necessary to take the same $\bar{Q}_0 > 0$ for $t < 0$ and $t > 0$ to cancel the contribution from the “background” which must emerge in the form

$$F_2 = \frac{\bar{Q}_0}{\alpha} K_t^{-2-\alpha} + \bar{F}_2 + O(\alpha).$$

(1.30)

Collecting terms and assuming that $F_0$ and $F_1$ have finite limits as $\alpha \rightarrow 0$, we get

$$f(t, 0; \infty) \approx F_0 + F_1 t + [\bar{F}_2 + \bar{Q}_\pm K_t^2] t^2 - \bar{Q}_0 K_t^2 t^2 \ln(1/|t|).$$

(1.31)

The coefficient of the $t^2 \ln|t|$ term now involves the square of the non-universal metric factor $K_t$ and can therefore be used in constructing universal amplitude ratios. In particular, the critical specific heat amplitudes are the same for $t > 0$ and $t < 0$. One can also use, for example, the difference of the $t > 0$ and $t < 0$ amplitudes of the $t^2$ term in the free energy, involving the universal factor $\bar{Q}_+ - \bar{Q}_-$ (see the next section (relation (1.36)).

Another source of logarithmic corrections occurs at the upper critical dimensionality, $d_\succ$, above which mean-field theory applies (Larkin and Khmel’nit’zkii, 1969). An exact expansion of the RG recursion relations in $d = d_\succ - \varepsilon$ dimensions yields (Nelson and Rudnick, 1976) an expression of the form

$$f_s(t, 0; \infty) \approx - \frac{\mathcal{A}}{u} t^2 \left\{ 1 + \frac{\mathcal{B} u}{\varepsilon} \left( t^{-\varepsilon/2} - 1 \right) \right\}^x - 1$$

(1.32)
where $u > 0$ is a coupling constant in the model. For finite $\epsilon$, this behaves as $t^{2-\alpha}$, with $\alpha \simeq x\epsilon/2$. For $\epsilon \to 0$, however, one gets $t^2|\ln t|^\alpha$.

### 1.4 Selection of metric factors

Consider the bulk zero-field static critical behaviour. Relation (1.18) involves three parameters. Two of these are universal, $\alpha$ and $F_-/F_+$. The third is the non-universal "strength" or "scale" of the free energy; via (1.19) it is proportional to $K_i^{2-\alpha}$. The metric factor $K_i$ also enters the zero-field finite-size scaling form (see (1.5)):

$$ f_s(t, 0; L) \approx L^{-d} Y(K_i t L^{1/v}, 0). \tag{1.33} $$

When the free-energy parameters $F_\pm$ and $\alpha$ are obtained by numerical or experimental measurement of, for instance, the specific heat critical behaviour, there arises a question of a "natural" definition of a non-universal free-energy scale, $F > 0$, linear in $F_+$ and $F_-$, such that $K_i = F^{1/(2-\alpha)}$. A particular choice of $F$ implies certain restrictions of the universal scaling function values, which usually can be imposed without violating any of the universality properties discussed in the preceding sections. For example, if we take $F = |F_+| + |F_-|$, then $|Q_+(0)| + |Q_-(0)| = 1$. Obviously, we could take $F = |F_+|$ or $F = |F_+ - F_-|$, etc. However, we would like to have a "natural" definition of this scale, symmetric for $t > 0$ and $t < 0$, and having no pathologies in various special limits. For example, when $\alpha \to 0$, the amplitudes $F_\pm$ diverge. In some models, $F_+ = 0$ while $F_- \neq 0$. In both cases, for example, the definition $F = |F_\pm|$ fails.

Before taking up the issue of the selection of $F$, let us briefly consider a related problem of measuring the asymmetry of the $t > 0$ and $t < 0$ critical behaviour as $\alpha \to 0$. Indeed, the leading contribution $\propto \alpha^{-1}$, in (1.29), is the same for $t \geq 0$, in the $\alpha \to 0$ limit. Up to terms linear in $\alpha$, we have

$$ F_-/F_+ = 1 + P(0)\alpha + O(\alpha^2), \tag{1.34} $$

where one can define the universal asymmetry parameter

$$ P(\alpha) \equiv \frac{F_- - F_+}{\alpha F_-} = \frac{Q_-(0) - Q_+(0)}{\alpha Q_-(0)}, \tag{1.35} $$

so that

$$ P(0) = \frac{\bar{Q}_+ - \bar{Q}_-}{\bar{Q}_0} \tag{1.36} $$

(see (1.29)). Since the numerical value of $\alpha$ is small for many 3d models, this quantity has been frequently used in theoretical and experimental studies of
the specific heat amplitudes (e.g. Ahlers and Kornblit, 1975; Barmatz et al., 1975; Hohenberg et al., 1976a; Singsaas and Ahlers, 1984; Belanger et al., 1985; Chase and Kaufman, 1986).

We now turn to the issue of defining the free-energy scale. The guidance for the "symmetric" definition of $F$ comes from a rather unexpected source: study of the complex temperature plane zeros of the partition function, and the connection between the location of these zeros and the finite-size scaling form (1.33). This recent theoretical development is rather complicated and will not be reviewed here; only some of the results will be used (further details can be found in Glasser et al. (1987)). First we note that the complex-$t$ zeros accumulate near $T_c$ along a complex conjugate pair of straight lines forming an angle $\phi$ with the negative real-$t$ axis, given by

$$
\tan[(2 - \alpha)\phi] = \frac{[\cos(\pi\alpha) - F_-/F_+]}{\sin(\pi\alpha)}
$$

(see Itzykson et al., 1983; Glasser et al., 1987). Secondly, the following "natural" combination of amplitudes emerges in these studies (Glasser et al., 1987):

$$
F = [F_+^2 + F_-^2 - 2F_+F_-\cos(\pi\alpha)]^{1/2}.
$$

The results (1.37) and (1.38) are rather interesting. Relation (1.37) should be compared with (1.35). The right-hand side of (1.37) provides an alternative amplitude–exponent combination involving $F_-/F_+$ and $\alpha$, having a finite limit as $\alpha \to 0$. In fact, we have the limiting relation

$$
\tan(2\phi) = -P(0)/\pi \quad (\alpha = 0).
$$

In relation (1.38), $F$ remains finite in the various pathological cases mentioned above. Here we consider in detail only the limit $\alpha \to 0$.

Note first that, by (1.19),

$$
F = K_i^{2-\alpha}[Q_+^2(0) + Q_-^2(0) - 2Q_+(0)Q_-(0)\cos(\pi\alpha)]^{1/2}.
$$

Thus, defining $K_i$ via $K_i^{2-\alpha} = F$ involves a symmetric constraint on the scaling function values to have the argument of the square root in (1.40) equal 1. In the limit $\alpha \to 0$, (1.40) reduces to

$$
F = K_i^2\tilde{Q}_0\sqrt{P^2(0) + \pi^2} \equiv K_i^2[Q_- - Q_+^2 + (\pi\tilde{Q}_0)^2]^{1/2}.
$$

This can vanish only if $\tilde{Q}_0 = 0$ and $\tilde{Q}_+ = \tilde{Q}_-$, i.e. there is actually no singularity in $O(t^2)$ (see (1.31)).

In summary, it is hoped that this introduction gave the reader a flavour of "what the universal amplitude ratios are all about" and a general theoretical background helpful in following the detailed exposition in the body of the review.
2 Definitions and notation for bulk critical amplitudes

2.1 Units

Several styles of notation for critical-point amplitudes and, more generally, for representing scaling relations exist in the literature. In this section we summarize our definitions and notational conventions. These conform, as far as possible, to those customarily used in theoretical and experimental studies of bulk \((L = \infty)\) critical amplitudes.

For magnetic systems, we measure the magnetization density \(M\), in units of

\[
M_N = N_A s g \mu_B / v_m,
\]

where \(N_A\) is Avogadro’s number, \(s\) is the spin, \(g\) is the \(g\)-factor, \(\mu_B\) is the Bohr magneton and \(v_m\) is the molar volume at criticality. (The subscript \(N\) in (2.1) and below stands for “normalization”.) The magnetic field \(H\) will correspondingly be measured in units of

\[
H_N = k_B T_c / s g \mu_B,
\]

where \(k_B\) is Boltzmann’s constant. The specific heat is measured in units of

\[
C_N^{\text{mag}} = H_N M_N / k_B T_c = N_A / v_m.
\]

Note that we include an extra factor \(k_B^{-1}\) in the definition of the specific heat so that its units are inverse volume. Throughout our work energies are measured in units of \(k_B T\).

For fluids, \(2M\) is replaced by \((\rho - \rho_c)\), where the density \(\rho\) is measured in units of its value at criticality,

\[
\rho_N = \rho_c,
\]

while \(2H\) is replaced by \((\mu - \mu_c)\), with the chemical potential \(\mu\) measured in units of

\[
\mu_N = p_c / \rho_c,
\]

where \(p_c\) is the critical pressure. The specific heat is measured in this case in units of

\[
C_N^{\text{fl}} = \frac{\rho_N \mu_N}{k_B T_c} = \frac{p_c v_m}{N_A k_B T_c v_m}.
\]

The temperature will always enter via the reduced variable \(t\) (see (1.4)). Lengths will be measured in units of a basic microscopic distance denoted \(a = C_N^{-1/d}\), see further below. When discussing time-dependent phenomena we often measure inverse times in units of a microscopic relaxation rate \(\omega_0\).

For other systems, such as binary fluids, superfluid \(^4\)He, structural transitions, etc., the physical quantities playing the role of the order parameter
and ordering field will be different, and corresponding normalizations may be introduced. For universal amplitude combinations, units cancel out in most cases.

2.2 Statics

We will be concerned with the equation of state in the form

\[ H = H(t, M), \]

and with the order parameter susceptibility

\[ \chi = \left( \frac{\partial M}{\partial H} \right)_t, \]

as well as other thermodynamic quantities, e.g. the specific heat \( C \).

We also consider the order parameter–order parameter correlation function:

\[ G(r, t, M) = n^{-1} \sum_{\alpha=1}^{n} \left[ \langle \psi_\alpha(r) \psi_\alpha(0) \rangle - \langle \psi_\alpha(r) \rangle \langle \psi_\alpha(0) \rangle \right]. \]

where \( \psi_\alpha(r) \) is the \( \alpha \)th component of the fluctuating \( n \)-component order parameter at the site \( r \).

For exponentially decaying correlations, the correlation length \( \xi \) may be defined by, for example, the second moment of \( G \):

\[ \xi^2 = (2d)^{-1} \sum_{r} r^2 G(r, t, M) \left/ \sum_{r} G(r, t, M) \right. \]

Below \( T_c \), the correlations for \( n > 1 \) models (e.g. \( XY \) and Heisenberg systems) do not decay exponentially but, rather, according to a power law (Hohenberg and Martin, 1965; Bogolyubov, 1970; Brézin et al., 1973a,b; Patashinskii and Pokrovskii, 1973). Assuming that

\[ \langle \psi_\alpha \rangle = M \delta_{\alpha 1}, \]

the longitudinal and transverse correlations are different from each other:

\[ G_L(r, t, M) = [ \langle \psi_1(r) \psi_1(0) \rangle - M^2 ], \]

\[ G_T(r, t, M) = \langle \psi_\alpha(r) \psi_\alpha(0) \rangle, \quad \alpha \neq 1. \]

Clearly,

\[ nG(r, t, M) = G_L + (n-1)G_T. \]

For \( H = 0, t < 0 \), \( G \) is dominated by \( G_T \). The (zero-field) Fourier transform of \( G_T \) behaves, for small wavenumbers \( q \), as

\[ \hat{G}_T(q, t, M) \approx M^2/\rho_s q^2, \]
where \( \rho_s \) is a (temperature-dependent) stiffness constant (Hohenberg and Martin, 1965; Hohenberg et al., 1976a). It may also be represented in terms of the helicity modulus \( \gamma \) (Fisher et al., 1973), \( \rho_s = \gamma(T)/k_BT \) (where we restored full dimensions). The spatial dependence can be easily calculated:

\[
\frac{G_T(r, t, M)}{M^2} \approx \frac{\Gamma(d/2)\pi^{-d/2}}{2(d - 2)} \left( \frac{\xi_T}{r} \right)^{d-2},
\]  

(2.16)

where we have introduced a transverse correlation length

\[
\xi_T \equiv \rho_s^{-1/(d-2)}.
\]  

(2.17)

For the superfluid transition \( (d = 3) \) in \(^4\)He, we have \( \xi_T = m^2k_BT/\rho h^2 \), where \( \rho \) is the superfluid density. For the zero-field longitudinal correlation function \( G_L \), for \( t < 0 \), similar considerations apply, as can be seen from the following general large-\(|r|\) relation (Fisher and Privman, 1985):

\[
G_L(r, t, M) \sim t^{(n-1)}[G_T(r, t, M)/M]^2,
\]  

(2.18)

which is in fact also valid for small non-zero fields \( H \), near the \( t < 0 \) phase boundary. Note that for small \( H \neq 0 \), one can use (2.15) with

\[
q^2 \to q^2 + H(t, M)M/\gamma;
\]  

(2.19)

the correlations then decay exponentially.

At \( T_c \) and \( H = 0 \), the correlation function decays algebraically for general \( n \), where it has the form

\[
G(r, t = H = 0) \sim 1/|r|^{d-2+\eta}.
\]  

(2.20)

The asymptotic singular behaviour of various quantities near the critical point may now be written as follows.

**Critical isochore:** \( t > 0, H = 0, q = 0 \)

\[
C \approx (A/\alpha)t^{-\alpha} + C_B,
\]  

(2.21)

\[
\chi \approx \Gamma t^{-\gamma}
\]  

(2.22)

\[
\xi \approx \xi_0 t^{-\nu}.
\]  

(2.23)

**Phase boundary:** \( t < 0, H = 0^+, q = 0 \)

\[
C \approx (A'/\alpha')(-t)^{-\alpha'} + C_B,
\]  

(2.24)

\[
\chi \approx \Gamma'(-t)^{-\gamma} \quad (n = 1),
\]  

(2.25)

\[
M \approx B(-t)^\theta,
\]  

(2.26)

\[
\xi \approx \xi_0(-t)^{-\nu} \quad (n = 1),
\]  

(2.27)

\[
\xi_T \approx \xi_0^T(-t)^{-\nu_T} \quad (n > 1),
\]  

(2.28)

\[
\xi_L \approx \xi_0^L(-t)^{-\nu_L} \quad (n > 1).
\]  

(2.29)
Critical isotherm: \( t = 0, \ H \neq 0, \ q = 0 \)

\[
C \approx (A_c/\alpha_c)|H|^{-\alpha_c} + C_B, \tag{2.30}
\]

\[
\chi \approx \Gamma_c |H|^{-\gamma_c}, \tag{2.31}
\]

\[
H \approx D_c M |M|^\delta - 1, \tag{2.32}
\]

\[
\xi \approx \xi_c |H|^{-\nu_c}. \tag{2.33}
\]

Critical correlation function: \( t = 0, \ H = 0, \ q \neq 0 \)

\[
\hat{G}(q, t = M = 0) \approx D_\infty / q^{2-\eta}. \tag{2.34}
\]

Note that (2.34) follows by Fourier transform of (2.20).

As emphasized earlier, the critical exponents above and below \( T_c \) are usually equal. Some caution must be taken, however, since this property is a prediction of the scaling theories (thus our “primed” notation for exponents below \( T_c \)).

In fact, it may not apply in some cases. For example, the definition (2.17) for \( \xi_T \) at the \( t < 0 \) phase boundary yields \( \nu_T = \nu \) for \( d \leq 4 \), when hyperscaling applies. However, for \( d \geq 4 \) one has \( \nu_T = 1/(d - 2) \), while \( \nu = \frac{1}{2} \) (see Fisher et al. (1973) for further discussion).

As will be discussed in the following sections, scaling theory predicts various scaling and hyperscaling relations among critical exponents, e.g.

\[
\alpha = \alpha', \tag{2.35}
\]

\[
\gamma = \gamma', \tag{2.36}
\]

\[
\gamma = \beta(\delta - 1), \tag{2.37}
\]

\[
\alpha = 2 - 2\beta - \gamma, \tag{2.38}
\]

\[
\nu = \nu' \quad \text{or} \quad \nu = \nu_T = \nu_L, \tag{2.39}
\]

\[
2 - \alpha = d\nu, \tag{2.40}
\]

\[
\alpha_c = \alpha/\beta\delta, \tag{2.41}
\]

\[
\gamma_c = 1 - 1/\delta, \tag{2.42}
\]

\[
\nu_c = \nu/\beta\delta, \tag{2.43}
\]

\[
\gamma = (2 - \eta)\nu. \tag{2.44}
\]

Only two exponents are needed in order to determine all the other exponents. Associated with the scaling relations (2.35)–(2.44), we now define universal
combinations of amplitudes:

\[
A/A', \\
\Gamma/\Gamma', \\
R_x = \Gamma D_c B^{\delta-1}, \\
R_c = A\Gamma/B^2,
\]

\[
\xi_0/\xi_0', \quad (n = 1) \quad \text{or} \quad \xi_0/\xi_0^T, \quad (n > 1),
\]

\[
R_\xi^+ = A^{1/4} \xi_0, \\
R_A = A_c D_c^{-1}(1+\alpha_c) B^{-2/\beta}, \\
\delta\Gamma_c D_c^{1/\beta} = 1, \\
Q_2 = (\Gamma/\Gamma_c)(\xi_c/\xi_0)^{2-\eta}, \\
Q_3 = D_\infty \xi_0^{2-\eta}/\Gamma.
\]

The relation (2.52) follows trivially, via (2.8). Only two amplitudes are necessary to determine all the other amplitudes (Stauffer et al., 1972). This property traces back to scaling relations with two metric factors, alluded to in the introduction. Note that in order to avoid notational complications for some relations involving correlation lengths, e.g. (2.50), (2.54), it is important to select the microscopic length scale \( a \) in such a way that \( a^{-d} \) corresponds to the specific heat density units as defined, e.g. in (2.3). Thus, the choice

\[
a = C_N^{-1/d}
\]

is appropriate.

Although we do not consider surface and interfacial phenomena in detail in this review, it is important to mention hyperuniversal amplitude combinations constructed with the amplitude of the surface tension, \( \Sigma \), defined at the phase boundary \( (H = 0, t < 0) \) by

\[
\Sigma(t) \approx \sigma_0(-t)^\mu,
\]

where \( \sigma_0 \) has units of \( a^{-d} \), and the critical exponent satisfies the hyperscaling relation (Widom, 1965)

\[
\mu = (d-1)v.
\]

The appropriate universal amplitude combinations are defined by

\[
R_{\xi_0} = \sigma_0 \xi_0^{d-1}, \\
R_{\sigma_0} = A^{(d-1)/d}/\sigma_0.
\]
2.3 Correction terms

In the early 1970s it was recognized that singularities near critical points cannot always be adequately described by pure power laws. Instead, confluent singularities have to be included in order to fit experimental data of high precision and to extract from such data the parameters of the leading singularity. Evidence for the importance of these confluent singularities came, quite independently, from high-temperature series expansions for the Ising model (Wortis, 1970), from experiments on superfluid helium (Greywall and Ahlers, 1972, 1973) and on liquid–gas critical points (Balzarini and Ohrn, 1972). As discussed in Section 4 below, confluent singularities were also shown to be a necessary consequence of the RG theory (Wegner, 1972). Near the critical point, along the path \( t > 0, \, H = q = 0 \) for instance, a general thermodynamic function will be written

\[
\Phi_i = A_i t^{-\theta_i}(1 + a_i t^\theta + e_i t + \ldots), \tag{2.60}
\]

where \( \theta > 0 \) is the leading correction exponent, and \( a_i, \, e_i \) are singular and regular correction amplitudes, respectively. The terms omitted in (2.60) are higher corrections, of order \( t^{2\theta}, \, t^2 \) and \( t^\theta \), with \( 0 < \theta < \theta_1 < \theta_2 < \ldots \). A similar expression holds for other thermodynamic paths, e.g. \( t < 0, \, H = q = 0 \), with different amplitudes \( A_i', \, a_i' \), but with the same exponents \( \theta_i \) and \( \theta \). We use the natural notation \( a_C, \, a_z, \, a_\xi, \, a_C', \, a_z', \, a_\xi', \, a_M, \, a_T, \, a_L, \, a_C, \, a_z, \, a_\xi, \, \tilde{a}_H, \, a_\xi, \) for the correction amplitudes in (2.21)–(2.33), etc.

In some situations one or more correction exponents \( \theta_1, \, \theta_2, \ldots \) may be anomalously small, and it is important to retain more terms in (2.60). This situation occurs when the system is near a crossover from one type of universal behaviour to another (e.g. Fisher, 1974a). The dimensionless correction amplitudes \( a_i \), which are non-universal and which determine the size of the “critical region”, will in general become large near a crossover.

We will show below that ratios of the type \( a_i/a_i', \, a_i/a_j, \) etc., are universal, and that there exist universal relations among the set of amplitudes \( e_i \) (e.g. Wegner, 1976; Aharony and Ahlers, 1980; Aharony and Fisher, 1983).

2.4 Dynamics

In the vicinity of the critical point, anomalies occur in a number of dynamical properties, such as transport coefficients, relaxation rates, and the response to time-dependent perturbations (see, for instance, Hohenberg and Halperin, 1977). These properties are all derivable from time-dependent correlation
functions, such as

\[
G(r, t, M; \tau) = n^{-1} \sum_{\alpha=1}^{n} \left[ \langle \psi_{\alpha}(r, \tau) \psi_{\alpha}(0, 0) \rangle - \langle \psi_{\alpha}(r, \tau) \rangle \langle \psi_{\alpha}(0, 0) \rangle \right],
\]

(2.61)
a formula which generalizes (2.9) to finite-time difference \( \tau \). (Since \( t \) denotes the reduced temperature, we use \( \tau \) for the dimensionless time here, \( \tau = \omega_0 \tau \), where \( \omega_0 \) is a characteristic frequency. Note that we will often suppress the variables \( t \) and \( M \) in writing the correlation functions.) In order to evaluate (2.61), it is necessary to specify the dynamics of the system, either via equations of motion (classical or quantum), or via phenomenological stochastic models with built-in dissipation.

The Fourier transform of (2.61) is related to the dynamical response function \( \chi(q; \omega) \) for complex frequency \( \omega \) in the upper-half plane, given in dimensional units by

\[
\hat{G}(q; \omega) = (2k_B T/\omega) \text{Im} \chi(q; \omega),
\]

(2.62)
whereas the equal time correlation function (2.9) is proportional to the zero-frequency response

\[
\hat{G}(q) = \hat{G}(q; \tau = 0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \hat{G}(q; \omega) = k_B T \chi(q; \omega = 0) = k_B T \chi(q).
\]

(2.63)
(We only consider classical systems.) Given the frequency spectrum of \( \chi(q; \omega) \) we may define the kinetic coefficient for the order parameter, \( \Gamma(q) \), by

\[
\frac{1}{\Gamma(q)} = i \frac{\partial \chi^{-1}(q; \omega)}{\partial \omega} \bigg|_{\omega=0}.
\]

(2.64)
The (dissipative) characteristic frequency for the order parameter is

\[
\omega_\psi(q) = \Gamma(q)/\chi(q).
\]

(2.65)
An alternative definition of a characteristic frequency makes use of the sum rule in (2.63). We define the median frequency \( \bar{\omega}_\psi(q) \) by

\[
\int_{-\bar{\omega}_\psi(q)}^{+\bar{\omega}_\psi(q)} \frac{d\omega}{2\pi} \hat{G}(q; \omega) = \frac{1}{2} \hat{G}(q).
\]

(2.66)
If the correlation function \( \hat{G}(q, \omega) \) has a Lorentzian spectrum centred about \( \omega = 0 \), the two definitions (2.65) and (2.66) are identical. For a spectrum having a sharp peak at a finite frequency the definition (2.65) is of course unsuitable. If the order parameter \( \psi \) is conserved by the dynamics then the
kinetic coefficient $\Gamma(q)$ is proportional to $q^2$ at small $q$, for $T \neq T_c$, and we may define a transport coefficient $\lambda(t)$ by

$$\lambda(t) = \lim_{q \to 0} q^{-2} \Gamma(q).$$

(2.67)

Similar definitions of response and correlation functions, kinetic and transport coefficients and characteristic frequencies can be introduced for other operators besides the order parameter $\psi$.

At the critical point the susceptibility $\chi(q, t, M)$ diverges, so it is expected that the characteristic frequency (2.65) will also have an anomaly. For a conserved order parameter we write

$$\omega(q) = D(t)q^2 \quad (q \to 0),$$

(2.68)

and we might expect, for $M = 0$, $t > 0$ for instance,

$$D(t) = \lambda(t)/\chi(q = 0, t) = D_0 t^{x_D}.$$  

(2.69)

Alternatively, if we define

$$\lambda(t) = \lambda_0 t^{x_\lambda},$$

(2.70)

then, according to (2.65), (2.67)–(2.69) and (2.22), we have

$$x_D = x_\lambda + \gamma.$$  

(2.71)

For $t = M = 0, q \neq 0$, we define

$$\omega(q) = \Omega_\infty q^z,$$

(2.72)

where $z$ is the characteristic dynamical exponent for the order parameter. We postpone the discussion of universal amplitude combinations involving the dynamical amplitudes introduced here to Section 3.5.

### 3 Scaling theory

#### 3.1 Equation of state

The basic statement of scaling theory is that asymptotically close to the critical point the equation of state may be written in the scaling form (Griffiths, 1967)

$$H \approx M |M|^{d-1} h(x),$$

(3.1)

where

$$x = t |M|^{-1/\beta}. $$

(3.2)
Note that for critical points in fluids the symmetry $M \leftrightarrow -M$ is only satisfied asymptotically close to the critical point. In that case the variables playing the roles of $H$ and $M$ are combinations of $\mu - \mu_c$ and $\rho - \rho_c$ (see Ley-Koo and Green, 1977; Balfour et al., 1978; Ley-Koo and Sengers, 1982). As is well known, the scaling assumption (3.1) implies the relations (2.35)-(2.38) and (2.41)-(2.42) among the thermodynamic critical exponents defined previously.

If we set $t = 0$ in (3.1) we recover (2.32) with

$$D_c = h(0) \equiv h_0. \quad (3.3)$$

Letting $H \to 0$ at $t < 0$, we expect that $M \neq 0$, and therefore $h(x)$ must vanish. Defining $x_0$ through

$$h(-x_0) = 0, \quad (3.4)$$

we recover (2.26) with

$$B = x_0^{-\beta}. \quad (3.5)$$

Both $x_0$ and $h_0$ are non-universal.

Taking the derivative of $H$ with respect to $M$, we can now obtain relations (2.22) and (2.25). Similarly, integrating $H$ with respect to $M$ and taking two $t$-derivatives, we can recover relations (2.21) and (2.24). However, it proves convenient to first rescale $h(x)$ by $h_0$ and $x$ by $x_0$. Thus, we define

$$\tilde{h}(\tilde{x}) = \frac{h(x/x_0)}{h_0^{-1} h(x)}. \quad (3.6)$$

The thermodynamic universality hypothesis states that the function $\tilde{h}(\tilde{x})$ is the same for all systems within a given universality class. With the definition (3.6), we can now express the amplitudes of interest (Griffiths, 1967; Barmatz et al., 1975):

$$\Gamma = \lim_{x \to \infty} \left[ \frac{x^\beta}{h(x)} \right] = x_0^2 h_0^{-1} \tilde{\Gamma}, \quad (3.7)$$

$$\Gamma' = \beta \frac{x_0^\beta}{h(-x_0)} = x_0^\beta h_0^{-1} \tilde{\Gamma}', \quad (3.8)$$

$$A = 2\beta \int_0^\infty h''(y) y^{\alpha - 1} \, dy = h_0 x_0^{\alpha - 2} \tilde{A}, \quad (3.9)$$

$$A' = 2\beta \left[ \int_{-x_0}^0 h''(y) |y|^{\alpha - 1} \, dy + x_0^{\alpha - 1} h'(-x_0) \right] = h_0 x_0^{\alpha - 2} \tilde{A}'. \quad (3.10)$$
Here we isolated the universal factors:

\[ \tilde{\Gamma} = \lim_{\tilde{x} \to \infty} \left[ \tilde{x}^7 / \tilde{h}(\tilde{x}) \right], \]
\[ (3.11) \]

\[ \tilde{\Gamma}' = \beta / \tilde{h}'(-1), \]
\[ (3.12) \]

\[ \tilde{\Lambda} = \alpha \beta \int_{0}^{\infty} \tilde{h}''(y) y^{z-1} \, dy, \]
\[ (3.13) \]

\[ \tilde{\Lambda}' = \alpha \beta \left[ \int_{-1}^{0} \tilde{h}''(y) y^{z-1} \, dy + \tilde{h}'(-1) \right]. \]
\[ (3.14) \]

Note that relations (3.9), (3.10) were obtained by integration by parts of \( \int_{M_0}^{M} H \, dM \). The particular expressions given here apply when \( \alpha > 0 \). For the appropriate \( \alpha < 0 \) relations for \( \tilde{\Lambda} \) and \( \tilde{\Lambda}' \), see Appendix C of Barmatz et al. (1975). If \( \alpha = 0 \), then we have (Griffiths, 1967)

\[ C \approx -A \ln t, \]
\[ (3.15) \]

with

\[ A = A' = \beta h''(0), \]
\[ (3.16) \]

\[ \tilde{\Lambda} = \tilde{\Lambda}' = \beta \tilde{h}''(0). \]
\[ (3.17) \]

Finally, integration of \( H \) at \( t = 0 \) similarly yields (2.30) with

\[ A_c = \delta^{-1} h_0^{1+z_0} x_0^{-2} \{ \tilde{h}''(0) + \alpha_c [\tilde{h}'(0)]^2 \} \]
\[ = h_0^{1+z_0} x_0^{-2} \tilde{A}_c. \]
\[ (3.18) \]

(See Barmatz et al. (1975), where a factor \( h_0^z \) is missing on the right-hand side of their equation (C.14).)

We have thus expressed the eight amplitudes \( A, A', \Gamma, \Gamma', B, D, A_c \) and, by (2.52), \( \Gamma_c \), in terms of the two non-universal coefficients \( h_0 \) and \( x_0 \), and in terms of various properties of the universal function \( \tilde{h}(\tilde{x}) \). The universality of the thermodynamic amplitude ratios (2.45)–(2.48) and (2.51) is now related straightforwardly (Aharony and Hohenberg, 1976) to the universality of \( \tilde{h}(\tilde{x}) \):

\[ A / A' = \tilde{\Lambda} / \tilde{\Lambda}', \]
\[ (3.19) \]

\[ \Gamma / \Gamma' = \tilde{\Gamma} / \tilde{\Gamma}', \]
\[ (3.20) \]

\[ R_x \equiv \Gamma D c B^{z-1} = \tilde{\Gamma}, \]
\[ (3.21) \]

\[ R_c \equiv A B^{-2} = \tilde{A} \tilde{\Gamma}, \]
\[ (3.22) \]

\[ R_A \equiv A_c D_c^{-1-z_0} B^{-2z_0} = \tilde{A}_c. \]
\[ (3.23) \]
One can also relate the non-universal constants $x_0$ and $h_0$ to the metric factors $K_t$ and $K_h$ used in Section 1. Up to universal proportionality constants, we have

$$x_0 \propto K_h^{-1/\beta} K_t^{-1}, \quad (3.24)$$

$$h_0 \propto K_h^{-1-\Delta/\beta}. \quad (3.25)$$

3.2 Correlation functions

The scaling assumption for the Fourier transform of the correlation function states that for $q, t, H \to 0$, one may write

$$\chi(q, t, M) \equiv \hat{G}(q, t, M)$$

$$\approx |t|^{-\gamma} Z(t|M|^{-1/\beta}, |t|^{-\gamma} q). \quad (3.26)$$

For simplicity, we will suppress in what follows the use of the absolute values for $t, M$, etc. In (3.26), we follow the unit conventions of Section 2, so that $\chi(q = 0)$ is the usual bulk susceptibility. (Note that we will also omit the vector notation for $q$.) As is well known, (3.26) implies exponent relations (2.43)–(2.44), as well as (2.39), provided certain precautions are taken on the low-temperature side for $n > 1, d > 4$, as mentioned in Section 2.

The relation $z = Z(x, y)$ generally suggests the use of three scales, namely those for $x, y$ and $z$. Given the form (3.26), it is natural to make a three-scale-factor universality hypothesis, stating that once the three scales are chosen in a specified way, then the properly rescaled scaling function is the same for all systems within a universality class. As discussed in the next section, Stauffer et al. (1972) proposed a stronger, hyperscaling-related hypothesis of two-scale-factor universality, which states that only two of the three scale constants are independent.

By considering (3.26) at $q = 0$, and using (3.1), one can identify

$$Z(x, 0) = [\delta x^{-\gamma} h(x) - x^{1-\gamma} h'(x)/\beta]^{-1}. \quad (3.27)$$

The function $Z(x, 0)$ is thus uniquely determined by $h(x)$, and scaling of $Z$ by $\Gamma$ and of $x$ by $x_0$ will define the universal function

$$\tilde{Z}(x/x_0, 0) = Z(x, 0)/\Gamma. \quad (3.28)$$

We need one more scaling factor, for the variable $y \equiv t^{-\gamma} q$. Following Fisher and Aharony (1973, 1974), we choose this scale by considering the small-$y$ behaviour of $Z(\infty, y)$. For a correlation function decaying exponentially in space, we have

$$Z(\infty, y) = \Gamma/[1 + (\xi_0 y)^2 + O[(\xi_0 y)^4]]. \quad (3.29)$$
As is well known, when (3.29) is used in (2.10), relation (2.23) is obtained. Having determined \( \xi_0 \) via the above convention, we now define
\[
\tilde{Z}(\tilde{x}, \tilde{y}) = \tilde{Z}(x/x_0, \xi_0 y) = Z(x, y)/\Gamma.
\] (3.30)

The function \( \tilde{Z}(\tilde{x}, \tilde{y}) \) is assumed to be universal.

For \( n = 1 \), the \( t < 0 \) equivalent of the relation (3.29) is the small-\( y \) expansion of \( Z(-x_0, y) \). In the scaled form, one can easily check that
\[
\tilde{Z}(-1, \tilde{y}) = \frac{\Gamma'}{\Gamma} \left[ 1 + \left( \frac{\xi_0}{\tilde{y}} \right)^2 \tilde{y}^2 + O(\tilde{y}^4) \right]^{-1},
\] (3.31)
which establishes the universality of the ratio \( \xi_0/\xi_0 \), see (2.49).

As long as \( G(r, t, M) \) decays exponentially, the behaviour of \( \hat{G}(q, t, M) \) for small \( \xi q \) may be written as
\[
\hat{G}(q, t, M) = \chi(t, M) \left\{ 1 + [\xi(t, M)q]^2 + O((\xi q)^4) \right\}^{-1}.
\] (3.32)

This relation can be formally solved for the correlation length \( \xi(t, M) \), and the following scaling relation established:
\[
\xi(t, M) \approx \xi_0 t^{-\nu} \tilde{X}(\tilde{x}).
\] (3.33)

The universal scaling function \( \tilde{X}(\tilde{x}) \) satisfies
\[
\tilde{X}(\infty) = 1 \quad \text{and} \quad \tilde{X}(-1) = \xi_0/\xi_0.
\] (3.34)

It is now straightforward to show that
\[
\xi_c = \xi_0 (h_0^{1/\nu}/x_0)^{\tilde{c}},
\] (3.35)
\[
\Gamma_c = \Gamma (h_0^{1/\nu}/x_0)^{\tilde{\Gamma}},
\] (3.36)
for the amplitudes in (2.31), (2.33). Here, the universal terms are given by
\[
\tilde{c} = \lim_{\tilde{x} \to 0} \tilde{X}(\tilde{x})/\tilde{x}^2,
\] (3.37)
\[
\tilde{\Gamma} = \lim_{\tilde{x} \to 0} \tilde{Z}(\tilde{x}, 0)/\tilde{x}^\nu.
\] (3.38)

We thus establish (Tarko and Fisher, 1975) the universality of the combination \( Q_2 \) defined in (2.53):
\[
Q_2 = (\xi_c)^{2-\eta}/\tilde{\Gamma}_c.
\] (3.39)

Finally, one can also consider the large-\( (\xi q) \) behaviour of \( \hat{G}(q, t, M) \). In this limit, relations (3.26), (3.30) and (2.34) can be utilized (Fisher and Aharony, 1973, 1974; Tarko and Fisher, 1975) to construct the universal
combination \(Q_3\), defined in (2.54), where

\[
Q_3 = \lim_{\tilde{y} \to \infty} \tilde{Z}(0, \tilde{y})\tilde{y}^{2-\eta}.
\] (3.40)

### 3.3 Two-scale-factor universality

Thus far the exponent and amplitude relations we have discussed have not involved the dimensionality of space \(d\), and they in fact hold in mean-field theory. We now make an additional assumption (see Widom, 1965; Kadanoff, 1966) that the singular part of the free energy in a correlation volume \(\xi^d\) approaches a constant at the critical point:

\[
\lim_{t \to 0^+} [\xi^d(t, H = 0) f_s(t, H = 0)] = \text{const.}
\] (3.41)

Obviously, this yields the hyperscaling relation

\[
2 - \alpha = dv.
\] (3.42)

A similar assumption for the interfacial free energy (per \(k_B T\) in a correlation area \(\xi^{d-1}\), i.e. assuming a constant limiting value of \(\xi^{d-1}\Sigma\), yields the hyperscaling relation for the surface tension exponent \(\mu\):

\[
\mu = (d - 1)v.
\] (3.43)

The associated hypothesis of two-scale factor universality or hyperuniversality (Stauffer et al., 1972) is the assumption that the constant in (3.41) is universal, see relation (1.14). Since the singular part of the specific heat is given by

\[
C_s = -\frac{\partial^2 f_s}{\partial t^2},
\] (3.44)

the universality of the combination \(R^+_\xi\), defined in (2.50), follows. A similar argument for the surface tension establishes the universality of the combinations \(R_{\sigma,\xi}^+\) and \(R_{\sigma,\lambda}^+\), defined in (2.58)–(2.59). The finite-size scaling equivalent of two-scale-factor universality (Privman and Fisher, 1984) is that the singular part of the free energy in volume \(L^d\), in a finite system at \(T_c\), is universal (see relation (1.8)).

Two-scale-factor universality can be related to the property that the unsubtracted correlation function scales in the same way as the subtracted (connected) correlation function. Taking \(n = 1\) for simplicity, the unsubtracted correlation function is defined by

\[
U(r, t, M) = \langle \psi(r)\psi(0) \rangle \quad (n = 1),
\] (3.45)
compare (2.9), and its Fourier transform has an additional delta-function contribution:

\[ \hat{U}(q, t, M) = \hat{G}(q, t, M) + (2\pi)^d M^2 \delta(q). \]  

(3.46)

Let us try to rearrange the added term to conform with the scaling form similar to (3.26):

\[ M^2 \delta(q) = \left[ t^{2\beta}(t M^{-1/\beta})^{-2\beta} \right] \left[ t^{-d\nu} \delta(t^{-\nu} q) \right] = t^{2\beta - d\nu} x^{-2\beta} \delta(y). \]  

(3.47)

Already at this level we get the exponent relation \(-\gamma = 2\beta - d\nu\) (see (3.26)), which is equivalent to (3.42). (Recall the scaling relation \(\alpha + 2\beta + \gamma = 2\).) However, we can also scale out the amplitudes:

\[ t^\gamma M^2 \delta(q) = x^{-2\beta} \delta(y) = (\Gamma^{-1} x_0^{-2\beta} \xi_0^d) [\Gamma(x/x_0)^{-2\beta} \delta(x_0^d y)]. \]  

(3.48)

In order to have a universal scaling function (see (3.30)) we must assume that the combination

\[ \Gamma^{-1} x_0^{-2\beta} \xi_0^d = \Gamma^{-1} B^2 \xi_0^d \]  

(3.49)

is universal, where we used (3.5). However, \(\Gamma^{-1} B^2\) is universally proportional to \(A\) (see (2.48)) on scaling grounds alone. Thus, the universality of \(A \xi_0^d\) follows.

We now turn to the case \(n > 1, t < 0\), and consider the transverse correlation function \(\hat{G}_T\) given by (2.15). Thus, we take \(H = 0\), although the extension of the arguments below for small non-zero \(H\) is possible (see (2.19)). We first note that the critical behaviour of the stiffness constant \(\rho_s\) is given by (Halperin and Hohenberg, 1969; Fisher et al., 1973)

\[ \rho_s \sim (-t)^s, \]  

(3.50)

where the mean-field value is \(s(d \geq 4) = 1\), while for \(d \leq 4\), the exponent \(s\) is given by the hyperscaling relation (Josephson, 1966)

\[ s = (d-2)\nu. \]  

(3.51)

Thus, we can assume that \(v_T = \nu\) in (2.28) provided \(d \leq 4\) (see 2.17). Restricting our consideration to \(d \leq 4\), let us put (2.15) in the general scaling form (3.26):

\[ \hat{G}_T(q, t, M) \approx M^2 \xi_T^{d-2} q^{-2} \approx B^2 t^{2\beta} (\xi_0^T t^{-\nu})^{d-2} q^{-2} \]

\[ = t^{-\nu}(\Gamma^{-1} B^2 \xi_0^d)(\xi_0^T/\xi_0)^{d-2}(\xi_0 t^{-\nu} q)^{-2}. \]  

(3.52)

Since the product \((\Gamma^{-1} B^2 \xi_0^d)\) encountered in relation (3.49) is universal, while the combination

\[ t^\gamma \hat{G}_T/\Gamma \approx \tilde{Z}(-1, y) = \tilde{Z}(-1, \xi_0 t^{-\nu} q), \]  

(3.53)
is just the universal scaling function, we conclude that the ratio $\xi^T_0/\xi_0$ must be universal as well. Note that (2.15) is a small-$q$ relation. The correct condition is, in fact, $\xi_T q \ll 1$. Thus relations (3.52)–(3.53) yield the scaling function $\tilde{Z}(-1, y)$ for small $y$ only. In the case of superfluid helium, it is customary to define the hyperuniversal combination (e.g. Ferer, 1974)

$$R^T_\xi = (A')^{1/d} \xi^T_0 = R^+_\xi (A'/A)^{1/d} (\xi^T_0/\xi_0),$$

(3.54)
since it can be determined directly from experiments below $T_c$; see (2.17) et seq.

### 3.4 Crossover scaling

The general term crossover scaling is customarily used for scaling forms appropriate for multicritical points, where lines of different phase transitions meet. There are a variety of multicritical points (e.g. bicritical, tricritical, tetracritical, etc.), the classification of which constitutes an important part of the scaling theory (e.g. Fisher, 1974a; Aharony, 1976a, 1983). Typically, several RG-relevant scaling fields are needed to describe the behaviour at a multicritical point, while the critical behaviour at the meeting phase transition lines manifests itself as the singularities of the multicritical scaling function. Universal amplitude combinations can then be constructed, involving amplitudes associated with, for example, the shape of the critical lines meeting at the multicritical point. In order to illustrate the above statements, we consider here the case of bicriticality (Fisher et al., 1980; Fisher and Chen, 1982).

A prototype system with a bicritical point at $(T_b, H_b)$ is a weakly anisotropic $n$-vector-spin antiferromagnet, with the magnetic field $H$ parallel to the dominant anisotropy axis (Fisher and Nelson, 1974). With the notation

$$\bar{h}_b = (H^2 - H_b^2)/H_b^2 \quad \text{and} \quad \bar{t}_b = (T - T_b)/T_b,$$

(3.55)

the appropriate scaling fields are given by

$$h_b = \bar{h}_b - \bar{c}_h \bar{t}_b,$$

(3.56)

$$t_b = \bar{t}_b + \bar{c}_t \bar{h}_b,$$

(3.57)

where the coefficients $\bar{c}_h$ and $\bar{c}_t$ are typically small and positive. Three phase transition lines meet at the bicritical point. For $t_b < 0$, there is a first-order transition line, at $h_b \approx 0$ (for small $|h_b|$). For $t_b > 0$, two lines of critical points are present, at $h_{\pm}(t_b)$, where typically $h_+(t_b) > 0$, while $h_-(t_b) < 0$. We consider first the scaling behaviour in the regime of two critical lines, i.e. we take $t_b > 0$.  

Consider the critical behaviour of the ordering susceptibility (e.g. Fisher and Chen, 1982),
\[ \chi(T, H) \approx \kappa t_b^{-\gamma_b} \tilde{B}(\tilde{k} h_b / t_b^\phi), \]  
(3.58)
as \( t_b, h_b \to 0 \), where \( \phi \) is customarily termed the crossover exponent. Here
the scaling function \( \tilde{B}(z) \) is universal, since we allowed for the non-universal scale factors, \( \tilde{k} \) and \( \kappa \). In order to describe the two critical lines, with the
appropriate susceptibility exponents \( \gamma_\pm \), we assume singularities in \( \tilde{B}(z) \) at
\[ z = z_+ > 0 \quad \text{and} \quad z = -z_- < 0. \]  
(3.59)
Thus, for
\[ -z_- < z < z_+, \]  
(3.60)
we have
\[ \tilde{B}(z) = b_+(z_+ - z)^{-\gamma_+} + b_-(z + z_-)^{-\gamma_-} + \tilde{B}_0(z). \]  
(3.61)
Here the amplitudes \( b_\pm \), the values of \( z_\pm \), and the less singular "background"
term \( \tilde{B}_0(z) \) are all universal. (The behaviour of \( \tilde{B}(z) \) outside the range (3.60)
can also be defined straightforwardly. However, we omit the details here.)

The above scaling formulation obviously corresponds to critical lines at
\[ h_+(t_b) \approx k^{-1} z_+ t_b^\phi, \]  
(3.62)
\[ h_-(t_b) \approx -\tilde{k}^{-1} z_- t_b^\phi, \]  
(3.63)
for small \( t_b \). The above relations predict the shape of the critical lines near
the bicritical point. Associated with (3.62)–(3.63), there is a universal amplitude ratio for the critical-line shapes:
\[ Q_b = z_- / z_+. \]  
(3.64)

Note that for fixed small \( t_b \) (\( > 0 \)), the critical behaviours at the two critical
lines, i.e. for \( h_b \to [h_+(t_b)]^- \) or \( h_b \to [h_-(t_b)]^+ \), are given by
\[ \chi \approx \kappa b_+ k^{-\gamma_+} t_b^{-\gamma_b} (h_+ - h_b)^{-\gamma_+} \quad \text{and} \quad \kappa b_+ \tilde{k}^{-\gamma_-} t_b^{-\gamma_b} (h_b - h_-)^{-\gamma_-}. \]  
(3.65)
(As mentioned, the behaviour for \( h_b \to (h_+)^+ \) and \( h_b \to (h_-)^- \) can also be
analysed similarly.)

The singular free energy has a scaling form analogous to (3.58), but with
exponent \( 2 - \alpha_b \), so that the specific heat along the spin flop line \( t_b < 0, \)
\( h_b = 0 \), contains the singular term \( (A_b / \alpha_b)(-t_b)^{-\alpha_b} \). The non-ordering uniaxial
magnetization, \( M_{n.o.} \), and susceptibility, \( \chi_{n.o.} \), are obtained as free-energy
derivatives with respect to \( h_b \). Their critical behaviour along the spin flop
line is given by \( B_{n.o.}(-t_b)\beta \) and \( \Gamma'_{n.o.}(-t_b)^{-\gamma} \), respectively, where the exponents are given by \( \beta = 2 - \alpha - \phi \) and \( \gamma = 2\phi + \alpha - 2 \) (Fisher and Nelson, 1974). Thus, the combination
\[
\bar{R}^{(n.o.)}_{c} = A'_b \Gamma'_{n.o.}/B^2_{n.o.},
\]
is universal. One can also consider behaviour at \( t_b = 0 \), and identify many new universal amplitude ratios, in analogy with (2.45)–(2.54). Note that the above discussion applies not only to spin flop bicritical points. “Non-ordering” quantities can be considered for other bicritical points as well, e.g. the strain non-ordering parameters near the displacive structural bicritical point in perovskites (Aharony and Bruce, 1974), etc.

Scaling relations of the type (3.58) apply for many other multicritical points, whenever there occurs a crossover to a new critical behaviour. Examples include disorder in the exchange coefficients, when \( \phi = \alpha \) (Harris, 1974; Aharony, 1976a), random-ordering fields without random exchange, when \( \phi = \gamma \) (Fishman and Aharony, 1979; Shapir and Aharony, 1981), random-ordering fields in disordered Ising systems, when \( \phi \approx 1.1\gamma \) in 3d (Aharony, 1986a). Another widely studied case concerns the zero-temperature percolation threshold of dilute magnets, where the temperature is varied in addition to the concentration (for recent reviews see, for example, Aharony (1986b) and Adler et al. (1990b)). The above references contain some theoretical predictions for the associated amplitude ratios, a more detailed review of which will not be attempted here.

### 3.5 Scaling for dynamics

The scaling assumption for the time-dependent correlations may be written as (Halperin and Hohenberg, 1969)
\[
\hat{G}(q, t, M; \omega) = \hat{G}(q, t, M)(2\pi q^{-\lambda}) W(tM^{-\gamma}, t^{-\nu}q, q^{-\zeta}\omega),
\]
in terms of the dynamical exponent \( \zeta \) introduced in (2.72). From the scaling assumption (3.67) we immediately obtain the dynamical scaling law
\[
x_D = x_\lambda + \gamma = \nu(z - 2),
\]
where \( x_D \) and \( x_\lambda \) are defined in (2.69) and (2.70).

The universality hypothesis for the scaling function \( W(x, y, w) \), with
\[
w \equiv q^{-\zeta}\omega,
\]
may be written, in analogy with (3.30), as
\[
\hat{W}(\bar{x}, \bar{y}, \bar{w}) = \hat{W}(x/x_0, \xi_0 y, \Omega^{-1}\xi w) = \Omega W(x, y, w).
\]
Then the sum rule (2.63) becomes
\[ \int_{-\infty}^{\infty} dw \, W(x, y, w) = \int_{-\infty}^{\infty} dw \, \tilde{W}(\tilde{x}, \tilde{y}, \tilde{w}) = 1. \] (3.71)

The characteristic frequency \( \tilde{Q}(x, y) \) is defined, according to (2.66), as
\[ \int_{-\tilde{Q}}^{+\tilde{Q}} d\tilde{w} \, \tilde{W}(\tilde{x}, \tilde{y}, \tilde{w}) = \frac{1}{2}. \] (3.72)

For \( M = 0 \) (i.e. \( \tilde{x} = \pm \infty \)), the above relations yield the scaling form
\[ \omega_\psi(q, t) \approx \Omega_\infty \, q^\frac{\xi}{z} \tilde{Q}_\pm(\xi q), \] (3.73)
where
\[ \tilde{Q}_\pm(\xi q) = \tilde{Q}(\pm \infty, \xi q) \quad \text{and} \quad \tilde{Q}_\pm(\infty) = 1. \] (3.74)

For a given type of dynamics the function \( \tilde{W}(\tilde{x}, \tilde{y}, \tilde{w}) \) is assumed to be universal (Hohenberg and Halperin, 1977), which implies the universality of the scaling functions for the characteristic frequency, \( \tilde{Q}_\pm(\tilde{y}) \). (Of course, when comparing different systems or models, one must make sure that the definition of the characteristic frequency, via (2.65) or (2.66), for instance, is the same in each case.) A pair of universal ratios that characterize the function \( \tilde{Q}(\tilde{y}) \) are
\[ \tilde{R}_\pm = \lim_{q \to 0, |t| \to 0} \left\{ (\xi q)^{p-\frac{z}{p}} \frac{\omega_\psi(q, t = 0)}{\omega_\psi(q \to 0, t)} \right\} = \lim_{\tilde{y} \to 0} \left[ \tilde{Q}(\tilde{y}) \tilde{y}^{\frac{z}{p} - \frac{p}{z}} \right]^{-1}, \] (3.75)
where \( p = 0, 1, 2, \ldots \) is determined by the hydrodynamic form \( \omega_\psi(q, \pm t) \sim q^p \xi^{p-\frac{z}{p}} \), of the characteristic frequency for \( q^\xi \ll 1 \).

In general, the dynamics introduces one new (universal) exponent \( z \), and one new non-universal amplitude \( \Omega_\infty \). In some cases these quantities are related to static exponents and amplitudes in a universal way (Halperin and Hohenberg, 1969; Hohenberg and Halperin, 1977).

It is frequently the case that the critical dynamics of \( \psi \) is determined by its coupling to some other mode, which we designate as \( \Psi \). One can define a dynamical correlation function, characteristic frequencies and a scaling function for \( \Psi \) in analogy with (2.61), (2.65) and (3.67) for \( \psi \). If we define
\[ \omega_\Psi(q) \approx \Omega_\Psi q^{\frac{z}{p}} \tilde{Q}_\Psi^\frac{\xi}{p}(q\xi), \] (3.76)
then dynamical scaling for \( \Psi \) implies (Ferrell et al., 1967; Halperin and Hohenberg, 1967, 1969)
\[ z_\Psi = z, \] (3.77)
and a corresponding universal relation for dynamical amplitudes \( \Omega_\infty \) and \( \Omega_\Psi \). Examples of such behaviour will be discussed in Section 6.7 below.
4 The renormalization group

4.1 Review of renormalization group theory

The RG theory of critical phenomena (e.g. Ma 1973, 1976; Fisher, 1974a; Wilson and Kogut, 1974; Aharony, 1976a; Brézin et al., 1976; Wallace, 1976; Wegner, 1976) has elucidated the mathematical mechanism for scaling and universality, and has provided a number of calculational tools for estimating universal properties. We return to calculational methods in Sections 5 and 6 below. Here we briefly survey the basic ideas of the RG approach.

In the RG transformation (Kadanoff, 1966; Wilson, 1971a, b), one increases the basic length scale by a factor \( b \), and one eliminates from the partition function the \( N(1-b^{-d}) \) degrees of freedom which correspond to the short-range fluctuations, of range between \( a \) and \( ba \), where \( a \) is a characteristic microscopic length scale (e.g. the lattice spacing). It is then convenient to rescale the lengths \((a \rightarrow a/b)\) and spins, so that the partition function becomes a trace over the remaining \( N/b^d \) degrees of freedom, of an effective Boltzmann factor \( \exp(-\mathcal{H}') \). The new "Hamiltonian" \( \mathcal{H}' \) is related to the original one via the RG transformation:

\[
\mathcal{H}' = \mathcal{R}(\mathcal{H}).
\]  

One next searches for fixed-point Hamiltonians, which satisfy

\[
\mathcal{H}^* = \mathcal{R}(\mathcal{H}^*).  
\]  

Since the length scales are changed by a factor \( b \), the correlation length \( \xi' \) which relates to \( \mathcal{H}' \) is given by

\[
\xi' = \xi/b. \tag{4.3}
\]

Similarly, the singular free-energy density is rescaled via

\[
f'_s = b^d f_s. \tag{4.4}
\]

By (4.3), \( \mathcal{H}^* \) corresponds to \( \xi = 0 \) or \( \xi = \infty \). The fixed points with \( \xi^* = \infty \) are identified with critical-point behaviour. In fact, different fixed-point Hamiltonians correspond to different universality classes.

We next consider small deviations from a given \( \mathcal{H}^* \):

\[
\mathcal{H} = \mathcal{H}^* + \sum_i g_i \bar{O}_i, \tag{4.5}
\]

where \( \bar{O}_i \) are "operators", i.e. they depend on the order parameters, order parameter gradients, etc., or generally on fluctuating quantities. The coefficients \( g_i \) depend on the coupling constants, including \( t, H, \) etc. Application of the
RG transformation now yields
\[
\tilde{\mathcal{H}}' = \tilde{\mathcal{H}}^* + \sum_i \tilde{g}_i \mathcal{A}(\bar{\mathcal{O}}_i).
\] (4.6)

Assuming that $\bar{\mathcal{O}}_i$ form a suitable basis set, we expect to have the linear representation
\[
\mathcal{A}(\bar{\mathcal{O}}_i) = \sum_j L_{ij} \bar{\mathcal{O}}_j,
\] (4.7)

which implies
\[
\tilde{g}_j' = \sum_i L_{ij} \tilde{g}_i.
\] (4.8)

Diagonalizing the matrix $L$, we can now find eigenoperators $\mathcal{O}_i$ such that
\[
\mathcal{A}(\mathcal{O}_i) = \Lambda_i \mathcal{O}_i.
\] (4.9)

In the diagonal representation, the appropriate coefficients $g_i$ are termed scaling fields. They transform according to
\[
g'_i = \Lambda_i g_i = b^{\lambda_i} g_i,
\] (4.10)

where the semigroup property of the RG,
\[
\mathcal{A}_b \mathcal{A}_{\tilde{b}} = \mathcal{A}_{b \tilde{b}},
\] (4.11)

implies that the $\lambda_i$ are independent of $b$.

The scaling field $g_i$ is called relevant if $\lambda_i > 0$, marginal if $\lambda_i = 0$ and irrelevant if $\lambda_i < 0$. All the Hamiltonians which differ from $\tilde{\mathcal{H}}^*$ only by irrelevant-field operators will "flow" towards $\tilde{\mathcal{H}}^*$ under the RG iterations. These Hamiltonians represent critical points in the universality class of $\tilde{\mathcal{H}}^*$. Combining (4.4) and (4.10), the scaling of the singular free energy becomes
\[
f_s(g_i) = b^{-d} f_s(b^{\lambda_i} g_i).
\] (4.12)

For the correlation length, we get
\[
\xi(g_i) = b^\gamma (b^{\lambda_i} g_i).
\] (4.13)

A usual critical point, in the ferromagnetic notation, is reached at $t = 0$ and $H = 0$, which is the only point with $\xi = \infty$. It transpires that $t$ and $H$ correspond to the relevant scaling fields (see (1.26)). For a simple critical point we also expect all the other fields to be irrelevant. Thus, we have
\[
g_1 \equiv g_t \sim c_t t, \quad \lambda_t > 0, c_t > 0, \quad (4.14)
\]
\[
g_2 \equiv g_h \sim c_h H, \quad \lambda_h > 0, c_h > 0, \quad (4.15)
\]
and

$$\lambda_i < 0 \quad \text{for} \quad i > 2.$$  \hspace{1cm} (4.16)

We now iterate the RG transformation (see (4.11)) until we get

$$b^\lambda g_t = \pm 1 \quad \text{for} \quad t \geq 0.$$  \hspace{1cm} (4.17)

For the free energy, (4.12) then yields

$$f_s = |g_t|^{d/\lambda} f_s(\pm 1, g_h |g_t|^{-\lambda_h/\lambda}; g_j |g_t|^{-\phi_j}) \quad (j > 2),$$  \hspace{1cm} (4.18)

where, by considering also the corresponding correlation length relation and using (4.14) and (4.15), we can identify the exponents

$$\lambda_i = 1/v, \quad \lambda_h = 1/v_c,$$

$$\lambda_h/\lambda_i = \beta \delta = \Delta = \beta + \gamma,$$  \hspace{1cm} (4.19)

$$\phi_j = \lambda_j/\lambda_i = v \lambda_j < 0 \quad (j > 2).$$  \hspace{1cm} (4.20)

Now, for $j > 2$, we have

$$g_j |g_t|^{\phi_j} \rightarrow 0 \quad \text{as} \quad t \rightarrow 0.$$  \hspace{1cm} (4.21)

Except for "dangerous" cases in which $f_s$ is non-analytic in the $g_j$ (Fisher, 1974b), we can write the asymptotic form

$$f_s \approx \overline{A} |t|^{d/\nu} f_s(\pm 1, \overline{B} |t|^{-\Delta}; 0, \ldots),$$  \hspace{1cm} (4.22)

where

$$\overline{A} \equiv c_t^{d/\nu}, \quad \overline{B} \equiv c_h c_t^{-\Delta}.$$  \hspace{1cm} (4.23)

A similar line of argument for the correlation length also yields

$$\xi \approx \overline{A}^{-1/d} |t|^{-\nu} (\pm 1, \overline{B} |t|^{-\Delta}; 0, \ldots).$$  \hspace{1cm} (4.24)

Since the only microscopic system parameters entering are the proportionality constants $c_t$ and $c_h$ in (4.14) and (4.15), relating the "physical" scaling variables to the RG scaling fields, the scaling results (4.23) and (4.25) entail two-scale factor universality. Specifically, the hyperscaling relation $d\nu = 2 - \alpha$ is explicit in (4.23), and there are only two non-universal coefficients, denoted $\overline{A}$ and $\overline{B}$ here, entering the free energy and correlation length scaling. A possible mechanism for breakdown of hyperscaling will be discussed in Section 4.4 below.

A straightforward extension of the above considerations also yields the correlation function scaling. Indeed, $r$ is rescaled into $r/b$, so that the appropriate scaling combination is $\overline{A}^{1/d} |t|^{\nu}$. Scaling for dynamics can be
also derived by the RG approach (e.g. Hohenberg and Halperin, 1977), with time rescaling according to \( \tau' = b^{-\delta} \tau \).

Unlike the usual critical point discussed above, a *multicritical point* has one additional relevant field, \( g_3 \), with \( \phi = \phi_3 = \lambda_3 / \lambda_4 > 0 \). Near such a point, (4.23) becomes

\[
f_i \approx A |t|^d f_i (\pm 1, \bar{B} |t|^{-\delta}, \bar{k} g_3 |t|^{-\phi}; 0, \ldots),
\]

which should be compared with (3.58), with \( H = 0, g_3 \sim h_b \). etc.

The above discussion illustrates an important conceptual point about universality of amplitude combinations as compared to the universality of critical exponents within the RG theory. Critical exponents emerge from the "local" properties of the RG flow in the immediate vicinity of each fixed point, and they can be calculated from the linearized recursion relations (4.7)–(4.11), etc. On the other hand, the scaling functions (of which the amplitudes are just particular values) are properties of the complete "global" (nonlinear) RG flow away from the fixed point under consideration, along "relevant" trajectories in the parameter space (leading to other fixed points or to infinity), along which \( f_i (\pm 1, \ldots), \xi (\pm 1, \ldots) \) on the right-hand side of (4.23), (4.25) are evaluated. This point is illustrated on a more technical level by the following considerations which also further clarify the emergence of two-scale-factor universality in the RG formalism.

The free-energy recursion relation (4.4) refers to the "singular part". However, it can be extended to the full free-energy density, to read

\[
f(\{b^l g_i\}) = b^d f(\{g_i\}),
\]

provided we include among the fields \( \{g_i\} \) also the one which corresponds to a constant, \( g_0 \). Such a constant is generated by the RG iterations even if it was not present in the initial interaction Hamiltonian. If the contribution to the constant due to the other fields at the \( l \)th iteration is \( \tilde{G}(l) \), and if the free-energy density calculated from the Hamiltonian without the constant at this stage is denoted \( \tilde{f}(\{g_i(l)\}) \), then one may write (Nauenberg and Nienhuis, 1974; Nelson and Rudnick, 1975)

\[
f = \sum_{m=0}^{l-1} b^{-dm} \tilde{G}(m) + b^{-dl} \tilde{f}(\{g_i(l)\}).
\]

Replacing \( b \) by \( e^{\delta l} \), with \( \delta l \to 0 \), and letting \( l \to \infty \), this becomes

\[
f = \int_{0}^{\infty} e^{-\delta l} \tilde{G}(l)(\delta l),
\]

where the integration element is denoted \((\delta l)\) to prevent confusion with the product \( dl \). Note that the kernel \( \tilde{G}(l) \) depends on \( g_i(l) = e^{\lambda l} g_i \). We now follow
the procedure of Hohenberg et al. (1976a; see Appendix A there) to establish
the validity of two-scale-factor universality by using (4.29). First, we note
that (4.3) implies that \( \tilde{\xi}(l) = e^{-l \tilde{\xi}} \), where \( \tilde{\xi} \) is the initial \( (l = 0) \) correlation
length. We then define \( \tilde{l} \) such that \( \tilde{\xi}(\tilde{l}) = 1 \), i.e. \( \tilde{\xi} = e^{\tilde{l}} \). We also choose \( \tilde{l} \) to
be a value of \( l \) sufficiently large so that for \( l \geq \tilde{l} \) all the “transient” irrelevant
variables \( g_l \) have died away and can be neglected. For simplicity, we restrict
our consideration to \( H = 0 \), \( T > T_c \), so that for the ordinary critical point
we are left only with \( g_t(l) \). We further assume that the initial \( T \) value is
sufficiently close to \( T_c \), so that \( \tilde{l} \gg \tilde{l} \). On the other hand, we also have
\[
g_t(l) = g_t(\tilde{l}) e^{-(\tilde{l} - l)\lambda_t},
\]
(4.30)
Thus, \( \bar{G}(l) \) is actually a function of \( (l - \tilde{l}) \), for \( l \gg \tilde{l} \), and one can argue
(Hohenberg et al., 1976a) that apart from an additive non-universal constant
(which does not contribute to the singular behaviour of the free energy and
is omitted here), \( \bar{G}(l) = \bar{\varphi}(l - \tilde{l}) \) is universal in the sense that the coefficients
in the expansion in powers of \( e^{-(l - \tilde{l})\lambda_t} \),
\[
\bar{G}(l) = \bar{\varphi}(l - \tilde{l}) = \bar{G}^* + \bar{\varphi}_1 e^{-(l - \tilde{l})\lambda_t} + \bar{\varphi}_2 e^{-2(l - \tilde{l})\lambda_t} + \bar{\varphi}(l - \tilde{l})
\]
(4.31)
(which we carried out explicitly only to two orders here), are universal
properties of the RG flow along the “relevant” trajectory.

We now divide the integral (4.29) into three parts, \( \int_0^\infty = \int_0^{\tilde{l}} + \int_{\tilde{l}}^{\infty} + \int_{\tilde{l}}^{\infty} \). The last integral is of the form
\[
\int_{\tilde{l}}^{\infty} \bar{G}(l) e^{-dl(\partial l)} = a_1 e^{-d\tilde{l}},
\]
(4.32)
where
\[
a_1 = \int_0^{\infty} \bar{\varphi}(\tau) e^{-d\tau(\partial \tau)}
\]
(4.33)
is universal. Similarly,
\[
\int_0^{\tilde{l}} \bar{G}(l) e^{-dl(\partial l)} = \bar{G}^*(e^{-d\tilde{l}} - e^{-d\tilde{l}})/d + \bar{\varphi}_1 [e^{-(d - \lambda_t)\tilde{l}} e^{-\tilde{\lambda}_t} - e^{-d\tilde{l}}]/(d - \lambda_t)
+ \bar{\varphi}_2 [e^{-(d - 2\lambda_t)\tilde{l}} e^{-2\tilde{\lambda}_t} e^{-d\tilde{l}}]/(d - 2\lambda_t) + a_2 e^{-d\tilde{l}},
\]
(4.34)
where
\[
a_2 = \int_{-\infty}^{0} \bar{\varphi}(\tau) e^{-d\tau(\partial \tau)} < \infty
\]
(4.35)
is again universal. The contribution of the first integral, from 0 to \( \tilde{l} \), is
non-universal, but analytic in \( t \), and thus does not contribute to the singular
free energy $f_\text{s}$. Similarly, the terms which are powers of $e^{-\lambda t}$ are regular because $e^{-\lambda t} = \xi^{d-\lambda} \propto t$. The only singular terms are those proportional to $e^{-d\lambda}$, and these yield (compare (1.14))

$$f_\text{s} = a_+ e^{-d\lambda} = a_+ \xi^{-d}, \quad (4.36)$$

where

$$a_+ = a_1 - \bar{C} \bar{\Phi}_1/(d - \lambda_1) - \bar{\Phi}_2/(d - 2\lambda_2) + a_2, \quad (4.37)$$

is universal. Thus, $(f_\text{s} \xi^d)_{\lambda \to 0}$ is universal. An alternative proof is given by Wegner (1976).

The above example illustrates our earlier statement that the universal amplitude ratio $R_\text{s}^+$ (see (2.50)), which is universally related to $a_+$ in (4.36), depends on the full RG trajectory from the critical fixed point ($l = \infty$) to the high-temperature disordered fixed point ($l = 0$). In contrast, the exponents $\nu$ and $v$ may be obtained by linearizing the recursion relations about the critical fixed point.

### 4.2 Confluent singularity corrections to scaling

The irrelevant scaling fields $g_j$ are functions of $t$, $H$ and other microscopic parameters. Near the critical point, we approximate $g_j(t, H; \ldots)$ by $g_j(0, 0; \ldots)$, which we will denote by $g_j$ for simplicity in this section. Usually, it is only necessary to consider the leading irrelevant field, say $g_1$, with the smallest exponent

$$|\phi_1| \equiv \theta \equiv |\lambda_1| \nu. \quad (4.38)$$

Assuming that the function $f_\text{s}(\pm 1, x; y_1, \ldots)$ in (4.18) is analytic in the $y$ variables representing irrelevant fields, we expand to obtain (Wegner, 1976)

$$f_\text{s} \approx \bar{A}|t|^d [\mathcal{F}_\pm (\bar{B}H|t|^{\Delta}) + \bar{C}|t|^{\theta} \mathcal{G}_\pm (\bar{B}H|t|^{-\Delta}) + O(|t|^{2\theta}) + \ldots]. \quad (4.39)$$

Here the scaling functions $\mathcal{F}$ and $\mathcal{G}$ are universal, while the new non-universal scale $\bar{C}$, associated with the irrelevant-variable correction-to-scaling term of relative order $|t|^\theta$, is

$$\bar{C} \equiv g_1 e_1^\theta. \quad (4.40)$$

A discussion of several irrelevant-variable corrections can be formulated along the same lines (see, for example, a review by Adler et al. (1983)). For the correlation length, we get a similar relation:

$$\xi \approx \bar{A}^{-1/d}|t|^{-\nu} [\mathcal{F}_\pm (\bar{B}H|t|^{-\Delta}) + \bar{C}|t|^{\theta} \mathcal{G}_\pm (\bar{B}H|t|^{-\Delta}) + \ldots]. \quad (4.41)$$
If one now generates the field derivatives of the free energy, i.e. $M, \chi, \text{etc.}$, and considers their critical behaviour at $H = 0$, then it is obvious that the coefficients $a_i$ in (2.60) will be proportional to $\tilde{C}$, with a universal proportionality factor (Aharony and Ahlers, 1980). Thus the ratios of any two coefficients (including those for $\xi$) from the set $a_i, a'_i$, are universal.

One can also work out the form of the correction terms for $t = 0$, with varying $H$ and $M$. For example,

$$H = D_e M |M|^{\delta-1}(1 + \tilde{a}_H |M|^\beta + \ldots).$$

Universal combinations involving $t = 0$ and $H = 0$ amplitudes can then be formed. One such combination is

$$(B^{0|\beta} \tilde{a}_H)/a_i,$$

where $B$ is the amplitude defined by (2.26).

### 4.3 Nonlinear scaling fields and analytic corrections to scaling

Another source of corrections to scaling are the higher-order contributions neglected in (4.14) and (4.15). Specifically, $g_t$ and $g_h/H$ are analytic functions of $t$ and $H^2$ (e.g. Wegner, 1976). In general, one must consider both relevant and irrelevant scaling fields, with their full $t$ and $H$ dependence. However, in order to study the leading analytic correction terms, $e_i t$ in (2.60), it suffices to put

$$g_t = c_t t(1 + b_t t) + \ldots,$$

$$g_h = c_h H(1 + b_h H) + \ldots,$$

in the leading scaling form

$$f_s \approx |g_t|^{d_\nu} \tilde{\mathcal{F}}_\pm (g_h |g_t|^{-\Delta}).$$

(see Aharony and Fisher, 1983). Expanding various quantities in powers of $t$ and $H$, and considering for simplicity the $H = 0$ critical behaviour, we have

$$f_s \approx \tilde{A} \tilde{\mathcal{F}}_\pm (0)|t|^{d_\nu}(1 + e_f t) + \ldots,$$

$$M \approx B |t|^\beta (1 + e_M t) + \ldots \quad (t < 0),$$

$$\chi \approx \Gamma^{(t)} |t|^{-\gamma}(1 + e_\chi t) + \ldots,$$

where $\Gamma^{(t)}$ denotes $\Gamma$ or $\Gamma'$ for $t \gtrless 0$. Here the amplitudes of the correction terms are given by

$$e_f = d_\nu b_t,$$

$$e_M = b_h + \beta b_t,$$

$$e_\chi = 2b_h - \gamma b_t.$$
The correction coefficients (4.50)–(4.52) are therefore related *universally* (Aharony and Fisher, 1983):

\[ e_f - 2e_M + e_x = 0. \]  \hspace{1cm} (4.53)

### 4.4 Hyperscaling and its breakdown in mean-field theory

As described in Section 4.1, a straightforward use of the RG ideas yields not only scaling, but also hyperscaling and two-scale-factor universality. This point of view has been advocated by, for example, Privman and Fisher (1984). More detailed and careful arguments for hyperscaling, etc., have been developed by, for example, Aharony (1974a), Hohenberg *et al.* (1976a) (see derivation leading to (4.36) in Section 4.1) and Wegner (1976).

An explicit construction of the RG and its fixed points can be carried out by using perturbation theory near four dimensions, in the parameter \( \varepsilon = 4 - d \) (Wilson and Fisher, 1972; Fisher, 1974a; Wilson and Kogut, 1974; Aharony, 1976a; Brézin *et al.*, 1976; Ma, 1976; Nelson and Rudnick, 1976). Results for \( d < 4 \) will be discussed in Section 6 below. Here we briefly address the issue of the breakdown of hyperscaling for \( d > 4 \), and also quote some results for \( d = 4 \).

For \( n \)-component vector spin models, the Landau–Ginzburg–Wilson effective Hamiltonian takes the form

\[
\mathcal{H} = \mathcal{H} / k_B T = \int d^d x \left[ \frac{1}{2} (\nabla s)^2 + \frac{1}{2} r |s|^2 + u |s|^4 - H \cdot s \right],
\]  \hspace{1cm} (4.54)

with \( r \sim t \), where higher-order terms are ignored, since they turn out to be highly irrelevant. For \( d > 4 \), the critical behaviour is controlled by the Gaussian fixed point, and one finds

\[ \lambda_t = 2, \quad \lambda_h = \frac{1}{2} (d + 2), \]  \hspace{1cm} (4.55)

while the leading irrelevant field is

\[ g_u \propto u \quad \text{with} \quad \lambda_u = 4 - d < 0. \]  \hspace{1cm} (4.56)

However, \( u \) is a *dangerous irrelevant variable* (Fisher, 1974b) in the sense that for the range \( u \leq 0 \), which includes the fixed-point value \( u^* = 0 \), the Hamiltonian (4.54) develops an instability for \( r \leq 0 \). If one modifies the spin rescaling of the RG transformation is such a way that the coefficient of \( |s|^4 \) remains constant (and positive), while the coefficient of the \( (\nabla s)^2 \) term is rescaled, then this coefficient is found to rescale rapidly to \( +\infty \) as \( b \) grows (e.g. see Aharony, 1983). This justifies the classical saddle point approximation (see Privman and Fisher, 1983), which shows that \( u \) enters the free energy.
in the form
\[ f_{\text{mean field}} = u^{-1/2} f(t, hu^{1/2}). \] (4.57)

Thus, the free energy is not analytic in the irrelevant variable \( u \). Further detailed discussion on how this non-analyticity leads to the mean-field scaling and to the violation of hyperscaling and hyperuniversality for both bulk and finite-size systems can be found, for example, in Privman and Fisher (1983) and Binder et al. (1985).

At the upper marginal dimensionality, \( d = 4 \) for \( n \)-vector spin models, hyperscaling and hyperuniversality are broken at the level of multiplicative logarithmic corrections to the leading power-law critical behaviour with the mean-field exponents (see Larkin and Khmel'nitzkii, 1969; Aharony and Halperin, 1975). For instance, the latter authors found the modification of (4.36),

\[ \xi^4 \xi S \approx \frac{n(n + 8)}{64\pi^2(4 - n)} \ln t \quad \text{or} \quad \frac{n + 8}{64\pi^2(4 - n)} \ln(-t), \] (4.58)

for \( t \to 0^+ \) and \( 0^- \), respectively. The coefficients of the logarithms in (4.58) are universal. Similarly, for the dipolar Ising model in \( d = 3 \) they found

\[ \xi^2 \xi S \approx \frac{3}{64\pi} \ln|t|, \] (4.59)

where \( \xi = \sqrt{\kappa_0 \xi^2}; \xi \) and \( \kappa_0 \) are extracted from the Fourier-transformed correlation function (or the structure factor):

\[ \hat{\chi}(q, t) = \chi(t) \left\{ 1 + \xi^2 \left[ q^2 + \kappa_0 \left( \frac{q_z}{q} \right)^2 \right] \right\}^{-1}. \] (4.60)

### 4.5 Background terms in the specific heat

The singular part of the specific heat is given by

\[ C_s(t) = \frac{A}{\alpha} t^{-\alpha} (1 + a_c t^\theta + e_c t + \ldots), \] (4.61)

for \( t > 0 \), with the appropriate modifications for \( t < 0 \). The "conventional wisdom" is that the non-singular background,

\[ C_{\text{ns}}(t) = C(t) - C_s(t), \] (4.62)

is not related to universal quantities. However, Bagnuls and Bervillier (1986a, and references quoted therein) have argued that within certain renormalization
schemes one can identify unambiguously a “critical” contribution \( C_{cr} \) (i.e. \( C_B = C_{ns}(0) = C_{cr} + \Delta C_B \)) such that the amplitude combination

\[
A|a_c|^{|x|}/(xC_{cr})
\]

is universal. This proposition is incorrect, as has been shown most convincingly by Nicoll and Albright (1986).

5 Methods of calculating bulk critical-point amplitudes

5.1 General considerations

In this section we outline the different methods which have been used to calculate universal amplitude combinations. We will briefly comment on the relative advantages and disadvantages of the different methods, as well as their respective limitations. Results for specific models will be given in Section 6. We begin by commenting on two general principles which yield information on 2d amplitudes, namely conformal invariance and duality. We also discuss certain additional topics of interest in connection with amplitude universality.

5.1.1 Conformal invariance

As mentioned in Section 1, conformal invariance, reviewed by Cardy (1987), provides exact information on the 2d critical exponents (Belavin et al., 1984; Friedan et al., 1984), and on finite-size amplitudes (Section 10). However, results for bulk amplitudes are limited at the present time to relation (1.17) and its equivalents along other thermodynamic paths (Cardy, 1988a), and more recent results (Zamolodchikov, 1989) on ratios of amplitudes for certain different definitions of correlation lengths (see Henkel’s (1990) review for further details and some numerical results).

5.1.2 Duality

The property of self-duality, i.e. transformations relating thermodynamic and/or correlation properties for \( T > T_c \) to those for \( T < T_c \), holds for certain spin models on some 2d lattices (e.g. Ising, Potts), as well as for percolation. A recent discussion of the implications of duality for critical amplitudes has been given by Kaufman and Andelman (1984), who considered \( q \)-state Potts models on the square lattice, for general \( 0 \leq q \leq 4 \). Specifically, they argue that, for general \( q \), self-duality implies

\[
A/A' = 1.
\]
5.1.3 "Renormalized coupling constant" amplitude combination

Let us consider the leading nonlinear susceptibility which is finite as \( H \rightarrow 0 \) for \( t > 0 \), i.e.

\[
\chi^{(nl)} \equiv \frac{\partial^3 M}{\partial H^3} \approx \Gamma^{(nl)} t^{-2\beta - 3\gamma}. \tag{5.2}
\]

Taking \( H = 0 \) and \( t > 0 \) for simplicity, we define the combination (Baker, 1977)

\[
g_\ast(t) = -\frac{\chi^{(nl)}}{\xi d \lambda^2} \sim t^{d-2-(2-\alpha)}, \tag{5.3}
\]

which is customarily termed "the renormalized coupling constant" in the literature. If hyperscaling holds, \( g_\ast(t) \) is asymptotically constant:

\[
g_\ast(t) \approx g_\ast(0)(1 + a_\alpha t^\alpha + \ldots) \quad (t > 0), \tag{5.4}
\]

where

\[
g_\ast(0) = -\frac{\Gamma^{(nl)}}{(\xi_0^d \Gamma^2)} \tag{5.5}
\]

is a hyperuniversal amplitude combination. The quantity \( g_\ast(t) \), and its finite-size counterpart considered in Section 10.1 below, have been extensively studied by series analyses and Monte Carlo (MC) methods.

5.1.4 Amplitudes for polymer solutions and mixtures

Phase separation in polymer solutions is typically an Ising-universality critical point. The polymerization index, \( N \), of the polymer chains (i.e. the number of monomers in a chain) is therefore a non-universal microscopic parameter from the critical behaviour point of view and must cancel out in universal amplitude combinations. Relevant experimental results will be reviewed in Section 7. Based on Flory-type scaling ideas advanced by De Gennes (1968, 1979), Widom (1988) concludes that

\[
\xi_0 \propto N^{(1-v)/2} \tag{5.6}
\]

and

\[
\sigma_0 \propto N^{(u-d+1)/2}, \tag{5.7}
\]

for polymer solutions, so that \( R_{\sigma \xi} \propto N^0 \), by (2.57) and (2.58). Assuming that \( R_C \) and \( R_\xi^+ \propto N^0 \), we then expect

\[
A \propto N^{(v-1)d/2}, \tag{5.8}
\]

\[
\Gamma/B^2 \propto N^{(1-v)d/2}. \tag{5.9}
\]
For a mixture of two polymer species with comparable \( N_2 \approx N_1 \equiv N \), Sariban and Binder (1987) proposed different relations in 3d, namely,

\[
B \propto N^{\beta-1/2} \quad \text{and} \quad \Gamma \propto N^{2-\gamma} \quad (d = 3),
\]

as well as results for certain other amplitudes, which follow from (5.10).

5.1.5 Cluster shape ratios

Percolation, lattice animal and random walk models can be formulated as models of statistical properties of \( N \)-site or bond clusters. The shape of \( N \)-element clusters is then characterized by \( d \)-dimensional tensors, such as the moment-of-inertia tensor (Privman and Barma, 1984) or the squared-radius-of-gyration tensor, \( G_{ij} \) (e.g. Aronovitz and Stephen, 1987). Let \( \lambda_1 > \ldots > \lambda_d \) denote the eigenvalues of \( G_{ij} \). Family et al. (1985) proposed to use

\[
\mathcal{A}_d \equiv \lim_{N \to \infty} \left\langle \frac{\lambda_d}{\lambda_1} \right\rangle
\]

as a (universal) measure of cluster shape anisotropy. Aronovitz and Nelson (1987) and Aronovitz and Stephen (1987) introduced instead certain invariants of \( G_{ij} \). Thus, they define (universal) anisotropy measures

\[
\mathcal{J}_d \equiv \lim_{N \to \infty} \frac{d}{d-1} \frac{\left\langle \text{Tr}\left\{ \frac{G - \text{Tr}(G)/d}{d} \right\}^2 \right\rangle}{\left\langle \left[ \text{Tr}(G) \right]^2 \right\rangle},
\]

\[
\mathcal{J}_d \equiv \frac{d^2}{(d-1)(d-2)} \lim_{N \to \infty} \frac{\left\langle \text{Tr}\left\{ \frac{G - \text{Tr}(G)/d}{d} \right\}^3 \right\rangle}{\left\langle \left[ \text{Tr}(G) \right]^3 \right\rangle},
\]

where \( \mathcal{J}_{ij} = \delta_{ij} \) is the unit tensor. Here the average \( \langle \ldots \rangle \) is taken over all \( N \)-element clusters, with their appropriate weight factors. These measures satisfy

\[
0 \leq \mathcal{A}_d, \mathcal{D}_d \leq 1,
\]

\[
-(d-1)^{-3} \leq \mathcal{J}_d \leq 1.
\]

For spherical symmetry, \( \mathcal{A}_d = 1, \mathcal{D}_d = 0, \mathcal{J}_d = 0 \). Numerical estimates of these and certain other anisotropy parameters will be described in Sections 6.5.2 and 9.1. Finally, note that \( \mathcal{J}_2 \) vanishes identically (Straley and Stephen, 1987).

5.2 Exactly solvable models

Given an exact solution, it is in general easy to extract the relevant universal amplitude ratios. The models we consider in detail in Section 6 are mean-field
theory (i.e. infinite-range interactions), the spherical model and the 2d Ising model. Several other 2d lattice spin models have been solved exactly (see Baxter’s (1982) book).

Free-energy amplitudes for certain hierarchical models have been calculated by Derrida et al. (1984), Kaufman (1984) and Kaufman and Andelman (1984). For the ratio, however, (5.1) generally applies.

5.3 Series expansions

High-temperature series have traditionally been the most accurate way to obtain quantitative information on critical behaviour (see reviews in Domb and Green, 1974). Many amplitude ratios, however, require information below $T_c$, and low-temperature series are less reliable; moreover, long series are only available for scalar spin systems (e.g. Ising ($n = 1$), Potts, etc.), so information on amplitude ratios is generally not as good as for exponents.

The “traditional” series analyses studies in 3d have led to exponent estimates which slightly violate hyperscaling relations. Indeed, the extrapolation methods involved systematic errors (see below) of roughly 1%. One might therefore expect the accuracy of the amplitudes to be of the order of a few per cent at best. Indeed, the values of universal combinations calculated on different lattices have a spread of about 5%.

Following the work by Nickel (1982) on extended high-temperature series for the body-centred cubic (BCC) 3d Ising model, new powerful series analysis methods have been developed (Adler et al., 1982; Chen et al., 1982), which properly account for the leading correction-to-scaling terms. New critical-exponent estimates are consistent with various RG results, and therefore satisfy hyperscaling. Recent series analysis studies, allowing for corrections to scaling, have been focused mostly on critical-exponent estimates. More careful methods for estimating amplitudes remain to be developed and applied (see, for example, a recent work by Meir (1987), and also Liu and Fisher (1989)).

5.4 Monte Carlo and transfer matrix methods

Both the MC method, reviewed by, for example, Binder (1985), and the transfer matrix approach, reviewed by, for example, Nightingale (1982), are naturally suited for the estimation of finite-size amplitudes. In MC studies, the full scaling functions are typically obtained.

Only a limited number of transfer matrix studies estimate bulk amplitudes directly (e.g. see Nightingale, 1976; Stella et al., 1987). On the other hand, several MC estimates of bulk amplitudes have been reported in the literature, for thermodynamic quantities, surface tension and the renormalized coupling
constant. We survey some results in Section 6. Due to the development of large-scale computing and general flexibility of the MC method, it has become a major theoretical tool of growing importance.

5.5 Renormalization group techniques

As described in Section 4, the RG theory confirms the scaling and universality hypotheses. It also provides a number of calculational tools for obtaining universal amplitude combinations.

5.5.1 The $\varepsilon$-expansion

This is an exact expansion away from mean-field theory, in $\varepsilon = 4 - d$ (Wilson and Fisher, 1972), of which two or three terms have been calculated for most critical quantities (and up to five terms in some cases, see Section 6.2). The advantage of the method is its versatility. It is equally applicable for $T \gg T_c$, and provides a reasonable estimate for nearly any quantity of interest. In fact, the full universal scaling functions, e.g. $\tilde{h}(\tilde{x})$, $\tilde{Z}(\tilde{x}, \tilde{y})$, can be calculated by this method. The limitation is, of course, that extrapolation to $\varepsilon = 1$ is quite uncontrolled, so that it is impossible to estimate the errors reliably. In general, it appears that exponents are more accurately calculated than amplitudes, and a typical error in the latter is $\pm 30-50\%$, or greater. Dohm (1985) has attributed the difference to geometrical factors such as $K_d$ (see (6.46) below) entering for amplitudes but not for exponents.

5.5.2 The $(1/n)$-expansion

This is another exact expansion (e.g. Ma, 1973), the zeroth order being the spherical model which corresponds to the $n \to \infty$ limit of the $n$-vector model. In general, a smaller number of terms are known than for the $\varepsilon$-expansion, and the method is primarily of theoretical interest.

5.5.3 Field-theoretic perturbation theory for $d = 3$

Exponents and amplitudes can also be calculated directly in three dimensions, using perturbation techniques proposed by Parisi (1980) (see, for example, Bagnuls and Bervillier (1985), who also refer to earlier work), results of which will be surveyed in Section 6. More terms are known in this perturbation theory than for the $\varepsilon$-expansion, so that the accuracy of the method appears to be considerably greater: roughly $0.5\%$ for amplitudes.

An important recent achievement (especially for amplitude ratio estimations) has been the development of the variants appropriate for $T < T_c$, by Bagnuls et al. (1987), Schloms and Dohm (1990), Krause et al. (1990).
5.5.4 Other renormalization group methods

The MC RG method (e.g. Swendsen, 1982), as well as methods relying on Wilson’s (1971a,b) exact functional RG equation (e.g. Newman et al., 1984), have been primarily used thus far to calculate critical exponents. The same is true for the real-space RG method which, however, has sometimes been used also to estimate amplitudes (e.g. see recent works by Nightingale and Indekeu (1985) and by Caride and Tsallis (1987), and references therein). Generally, real-space results are at best semiquantitative.

5.6 Dynamics

For dynamics one can again develop an $\varepsilon$ expansion and calculate all universal quantities at least to lowest-non-trivial order. In particular, the full universal function $\tilde{W}(\vec{x}, \vec{y}, \vec{w})$ is in principle calculable for both the order parameter and any additional operator which is dynamically coupled to it (see Hohenberg and Halperin, 1977; and Section 6 below).

A formalism which relies explicitly on the dynamical mode coupling near $T_c$ had been developed before the RG theory (Fixman, 1962; Kadanoff and Swift, 1968; Kawasaki, 1970). This method can operate directly in 3d, and it may be looked upon as a precursor of the field-theoretic perturbation theory in $d = 3$, mentioned in the preceding section. Although the perturbation series can be made self-consistent, it has not so far proved possible to evaluate more than one or two terms, so the method is somewhat uncontrolled.

The dynamical RG can also be formulated directly in $d = 3$, thus providing a somewhat more systematic approach than mode-coupling theory. The method has been applied primarily to the superfluid transition in $^4$He (Ahlers et al., 1982; Dohm and Folk, 1982a).

6 Numerical results for selected models

6.1 Mean-field, two-dimensional Ising, and spherical models: exact results

6.1.1 Mean-field theory

In the presentation below, we start from the following form of the Ising model interaction energy:

$$E = - \sum_{\text{pairs}} J(r_i - r_j)\sigma_i\sigma_j - \mu_0 H \sum_{\text{spins}} \sigma_i,$$

(6.1)
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where \( \sigma_i = \pm 1 \) is a dimensionless spin variable at lattice site \( r_i \), and \( \mu_0 \) is the magnetic moment per spin. Turning to dimensionless variables, we measure \( E \) and \( J \) in units of \( k_B T \), introduce a dimensionless magnetic field \( H \), and set the lattice spacing equal to unity, etc., as described in Section 2. The exchange couplings \( J(r_i - r_j) \) are positive for ferromagnetic interactions and, for the nearest-neighbour case, for instance, one chooses \( J(r_i - r_j) = 0 \) for all the distant-neighbour pairs \( ij \). For the dimensionless correlation function

\[
G(r_i - r_j) = \langle \sigma_i \sigma_j \rangle - \langle \sigma_i \rangle \langle \sigma_j \rangle, \tag{6.2}
\]

the following expression is obtained in the standard Ising mean-field theory (Brout, 1965; Tarko and Fisher, 1975):

\[
\hat{G}(q, t, M) = \frac{1 - m^2}{1 - (1 - m^2) \hat{J}(q)}, \quad m \equiv (1 + t)M. \tag{6.3}
\]

The Fourier transform here is defined for convenience as

\[
\hat{J}(q) = C_N \int d^d r \, J(r) e^{i \cdot q \cdot r}, \tag{6.4}
\]

etc., where the extra factor \( C_N \) is just the inverse unit cell volume in this case (compare Section 2). For isotropic short-range interactions (i.e. \( J(r) \) decaying as \( |r|^{-(d+2)} \) or faster), one can generally write, for small \( q \),

\[
\hat{J}(q) = \hat{J}(0)\left[1 - \rho^2 q^2 + o(q^2)\right], \tag{6.5}
\]

where \( \hat{J}(0) \) and the length \( \rho \) depend on the details of \( J(r) \). In the mean-field approximation (e.g. Tarko and Fisher, 1975), we have (in dimensioned units)

\[
T_c = \hat{J}(0)/k_B, \tag{6.6}
\]

while the asymptotic equation of state near \( T_c \) takes the form

\[
H \approx M(t + \frac{1}{3}M^2). \tag{6.7}
\]

Thus one easily identifies the “classical” Landau (1937) mean-field exponents \( \beta = \frac{1}{3}, \gamma = 1, \delta = 3 \), etc. Also, in the critical region one finds

\[
\chi \approx (t + M^2)^{-1}, \tag{6.8}
\]

\[
\xi^2 \approx \rho^2/(t + M^2), \tag{6.9}
\]
so that \( v = v' = \frac{1}{2}, v_c = \frac{1}{3}, \eta = 0, \) etc. The mean-field values of several universal amplitude ratios can then be evaluated:

\[
\Gamma / \Gamma' = 2, \tag{6.10}
\]
\[
R_x = 1, \tag{6.11}
\]
\[
\xi_0 / \xi'_0 = \sqrt{2}, \tag{6.12}
\]
\[
Q_2 = 1, \tag{6.13}
\]
\[
Q_3 = 1. \tag{6.14}
\]

Since \( \alpha = 0 \) in the mean-field theory, with a discontinuous specific heat, it is meaningless to discuss the ratios which involve \( A = A' = 0 \). In fact, the specific heat actually vanishes above \( T_c \). However, the discontinuity in the specific heat is universally related to other amplitudes, via

\[
|\Delta C| \Gamma B^{-2} = \frac{1}{2}. \tag{6.15}
\]

Two-scale-factor universality is a result of fluctuations and therefore it does not hold in the mean-field theory. Finally, note that most of the above results also apply for \( n > 1 \), except for the form of the correlations below \( T_c \), as mentioned in Sections 2 and 3.

6.1.2 Two-dimensional Ising model

The original solution by Onsager (1944) of the square lattice Ising model with unequal couplings

\[
J(\hat{x}) = J_x, \quad J(\hat{y}) = J_y, \quad J(|r_i - r_j| > 1) \equiv 0, \tag{6.16}
\]

and \( H \equiv 0 \), yielded expressions for the specific heat \( (\alpha = 0(\log)) \), correlation length and surface tension \( (\mu = v = 1) \). Further calculations by Kaufman and Onsager (1949), Fisher (1959), Montroll \textit{et al.} (1963), Wu \textit{et al.} (1976) and others have yielded results for the \( H = 0 \) correlation functions \( (\eta = \frac{1}{4}) \), extensions to lattices other than square, etc. Numerous reviews on the exact results for the 2d Ising models include, for instance, Domb (1960a,b), and two books by Baxter (1982) and McCoy and Wu (1973), respectively. Finally, the spontaneous magnetization \( (\beta = \frac{1}{3}) \) has been derived by Yang (1952), while Barouch \textit{et al.} (1973) calculated the amplitudes for the critical behaviour of the zero-field susceptibility \( (\gamma = \frac{2}{3}) \).

Exact values of several amplitude ratios involving \( H = 0 \) thermodynamic and correlation quantities can be calculated. We consider the isotropic case, \( J_x = J_y = K \). Indeed, lattice anisotropy is an RG-marginal perturbation. Thus, the amplitude ratios are not necessarily universal as functions of \( J_x/J_y \). (It turns out that in some cases the non-universality of this sort in 2d, can be
repaired by a space rescaling (see Nightingale and Blöte, 1983). We will not discuss this issue here.) In terms of Baxter’s elliptic modulus,

\[
\hat{k} = [\sinh(2K)]^{-2},
\]

(6.17)

the critical point is given by \( \hat{k}(T_c) = 1 \), so that

\[
\hat{k} = 1 + [2\sqrt{2} \ln(1 + \sqrt{2})] t + O(t^2).
\]

(6.18)

The magnetization \( m \equiv (1 + t)M \) takes a particularly simple form as a function of \( \hat{k} \):

\[
m_{H=0} = (1 - \hat{k}^2)^{1/8}.
\]

(6.19)

We also quote the following results:

\[
\xi(T > T_c) = -(2K + \ln \tanh K)^{-1} \approx t^{-1}/[2 \ln(1 + \sqrt{2})], \quad (6.20)
\]

\[
\xi(T < T_c) = \frac{1}{2}(2K + \ln \tanh K)^{-1} \approx (-t)^{-1}/[4 \ln(1 + \sqrt{2})], \quad (6.21)
\]

\[
\Sigma = 2K + \ln \tanh K \approx [2 \ln(1 + \sqrt{2})](t).
\]

(6.22)

The results (6.20) and (6.21) are for the so-called true correlation lengths (t.c.l.) defined via the exponential decay of the correlation function.

Detailed expressions for the free energy, susceptibility and correlations involve various mathematical functions: we only list several results near \( T_c \). For the susceptibility and correlations, we have

\[
\bar{\chi}(T > T_c) \approx (0.96258 \ldots) t^{-7/4} [1 + (0.82790 \ldots)t], \quad (6.23)
\]

\[
\bar{\chi}(T < T_c) \approx (0.025536 \ldots)(-t)^{-7/4} [1 + (0.82790 \ldots)t], \quad (6.24)
\]

where \( \bar{\chi} \equiv (1 + t)\chi \), and

\[
\hat{G}(q \ll 1, t = 0) \approx (0.75612 \ldots)q^{-7/4}, \quad (6.25)
\]

where we use the dimensionless definitions as in (6.2)–(6.4). The details of the numerical values can be found in Fisher and Burford (1967) and Barouch et al. (1973). For the magnetization and for the singular part of the free energy the behaviour near \( T_c \) is given by

\[
m \approx 2^{5/16} [\ln(1 + \sqrt{2})]^{1/8} (1 + \frac{9\sqrt{2} \ln(1 + \sqrt{2}) - 4}{32} t)^{1/8},
\]

(6.26)

\[
f_s \approx \pi^{-1} [\ln(1 + \sqrt{2})]^{2}(t^2 \ln t) \left(1 + \frac{\sqrt{2} \ln(1 + \sqrt{2}) - 4}{2} t\right). \quad (6.27)
\]

In relations (6.23), (6.24) and (6.26), (6.27) we included the leading analytic correction terms. Indeed, Aharony and Fisher (1980) have argued that there
are no "irrelevant-variable" corrections to scaling in the thermodynamic quantities of the isotropic 2d Ising model.

Relations (6.19)–(6.27) allow verification of the sum rule (4.53) and calculation of several universal ratios (with $A = A'$ defined as in (3.15)):

$$\Gamma'/\Gamma' = 37.69 \ldots ,$$

$$R_c = 0.3185699 \ldots ,$$

$$\xi_0'/\xi_0 = 2 \quad \text{(t.c.l.)},$$

$$R_\xi^+ = 1/\sqrt{2\pi} = 0.3989 \ldots \quad \text{(t.c.l.)},$$

$$Q_3 = 0.291 \ldots \quad \text{(t.c.l.)},$$

$$R_{\sigma\xi} = 1 \quad \text{(t.c.l.)},$$

$$R_{\sigma A} = 1/\sqrt{2\pi} = 0.3989 \ldots \quad \text{(6.34)}$$

Some of these ratios are available for other lattices, notably triangular (e.g. Stauffer et al., 1972), so that their universality can be confirmed. Certain amplitudes are also known for the antiferromagnetic Ising model (e.g. Fisher, 1960a, b; Kong et al., 1986; Kaufman, 1987). Note that the above amplitude combinations involve the amplitudes of the t.c.l. ("true") instead of the "second-moment" correlation lengths (see also Section 6.2 below).

6.1.3 Spherical model

The spherical model (Berlin and Kac, 1952; reviewed by Joyce, 1972) replaces the condition $\sigma_i^2 = 1$ in (6.1) by

$$\sum_i \sigma_i^2 = V/a^d \quad \text{or} \quad \sum_i \langle \sigma_i^2 \rangle = V/a^d,$$

which are equivalent as $V \to \infty$. This model is of interest since it is equivalent to the $n \to \infty$ limit of the $n$-component spin model (Stanley, 1968). The short-range spherical model shares many features of the $1 < n < \infty$ spin models, including soft modes at the phase boundary ($H = 0$, $T < T_c$), upper critical dimension $d = 4$, above which the exponents become mean field, lower critical dimension $d = 2$, at and below which there is no ferromagnetic critical point at non-zero $T$, and variation of exponents with dimensionality for $2 < d < 4$, consistent with scaling and hyperscaling. Results are also available for the long-range spherical model (Joyce, 1972; Fisher and Privman, 1986). To avoid unilluminating mathematical complications, we
restrict our consideration here to the short-range case, with $3 \leq d < 4$, i.e.

$$0 < \varepsilon \leq 1 \quad \text{where} \quad \varepsilon = 4 - d. \quad (6.36)$$

In the notation of Section 3, the universal equation of state for the spherical model is (Joyce, 1972)

$$\tilde{h}(\tilde{x}) = (1 + \tilde{x})^\gamma,$$  

where

$$\gamma = \frac{2}{2 - \varepsilon}, \quad \beta = \frac{1}{2} \quad \text{(for} \ 2 < d < 4), \quad (6.38)$$

with other exponents obtained by scaling relations. In particular,

$$\alpha = -\frac{\varepsilon}{2 - \varepsilon}, \quad (6.39)$$

so that relations (3.9), (3.10), etc. (see Section 3), must be replaced by the appropriate results for $-1 \leq \alpha < 0$ (see Appendix C of Barmatz et al. (1975)). For $3 < d < 4$, one obtains

$$\tilde{A} = \frac{\varepsilon}{(2 - \varepsilon)^2}, \quad \tilde{A}' = 0. \quad (6.40)$$

By (3.11), one can also calculate

$$\tilde{\Gamma} = 1, \quad (6.41)$$

so that

$$(R_C)_{\text{spherical}} = \frac{\varepsilon}{(2 - \varepsilon)^2} = (R_C/n)_{n \to \infty}, \quad R_\chi = 1. \quad (6.42)$$

Note that the susceptibility is infinite below $T_c$. We indicate the limiting correspondence between the spherical model and large-$n$ amplitudes, whenever factors of $n$ are needed.

For $d = 3 (\alpha = -1)$, the calculations must be appropriately modified. The equation of state becomes analytic in $\tilde{x}$ ($\gamma = 2$), and the singular part of the free energy mixes with the analytic background. This leads to a certain arbitrariness in the definitions of $\tilde{A}$ and $\tilde{A}'$. The ambiguity is removed if one considers the large-$n$ limit of the $n$-vector model. Indeed, to the leading order in $1/n$, Abe and Hikami (1975) find, for $3 < d < 4$,

$$\frac{A}{A'} = \frac{n^{2d(3-d)/(d-2)}}{\Gamma((4-d)/(d-2))\Gamma(2(d-3)/(d-2))\Gamma((d-1)/2)} \left[ \frac{\Gamma(\frac{3}{2})\Gamma(d/2)}{\Gamma(d/2)} \right]^{d/(d-2)} \quad (6.43)$$
At $d = 3$, however, they get

\[ \frac{A}{A'} = \frac{\pi^2}{4} - 1 \approx 1.467, \]  

(6.44)

which is not proportional to $n$.

Two-scale-factor universality has been checked for the spherical model by Gerber (1975), who finds

\[ (R_\xi^+)^d_{\text{spherical}} = \frac{K_d \pi \varepsilon}{2(2-\varepsilon)^2 \sin(\pi \varepsilon/2)} = [\langle R_\xi^+ \rangle^d/n]_{n \to \infty}, \]  

(6.45)

\[ K_d^{-1} = 2^{d-1} \pi^{d/2} \Gamma\left(\frac{d}{2}\right). \]  

(6.46)

Similarly, in the $n \to \infty$ limit, Hohenberg et al. (1976a) find

\[ \left(\frac{\xi^+}{\xi}\right)^{d-2} = \frac{\pi K_d n}{2 \sin(\pi \varepsilon/2)}. \]  

(6.47)

The scaling of the correlation function in the large-$n$ limit was calculated by Aharony (1974b), who finds

\[ Q_3 = 1. \]  

(6.48)

Further results in the large-$n$ limit, including those for correction-to-scaling amplitude ratios, will be summarized in Section 6.2.2 below, which is devoted to the $(1/n)$-expansion.

### 6.2 $n$-Vector models: survey

#### 6.2.1 $\varepsilon$-Expansion results

Although the $\varepsilon$-expansion (Wilson and Fisher, 1972) provides in some cases only qualitative numerical estimates of amplitude ratios, it has attracted much attention. Indeed, this is probably the only method to study systematically deviations from the mean-field results as a function of $\varepsilon \equiv 4 - d$, $n$, and other system characteristics. The ratio $A/A'$ up to $O(\varepsilon^2)$ is given as

\[ \frac{A}{A'} \approx \frac{n}{4} 2^2 \left\{ 1 + \varepsilon + \frac{\varepsilon^2}{2(n + 8)^2} [3n^2 + 26n + 100] + (4 - n)(n - 1)\zeta(2) - 6(5n + 22)\zeta(3) - 9(4 - n)\lambda \right\}, \]  

(6.49)

by Okabe and Ideura (1981; see also Bervillier, 1986). Here $\zeta(2) \approx 1.6449$, $\zeta(3) \approx 1.2021$, and $\lambda \approx 1.1719$ (see Bervillier (1986) and Nicoll and Albright...
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(1985) for details), while the exponent $\alpha$ can in turn be expanded as

$$\alpha = \frac{4 - n}{2(n + 8)} e^{- \frac{(n + 2)(n^2 + 30n + 56)}{4(n + 8)^3} e^2 + ...}$$  \hspace{1cm} (6.50)

Typical $A/A'$ estimates in 3d, for $n = 1, 2, 3$ are, respectively, $0.524 \pm 0.010, 1.029 \pm 0.013, 1.521 \pm 0.022$ (Bervillier, 1986).

The ratio $\Gamma/\Gamma'$ is defined only for $n = 1$. Nicoll and Albright (1985) derived the expansion

$$\frac{\Gamma}{\Gamma'} = 2^{\gamma - 1} \left( \frac{\gamma}{\beta} \right) \left[ 1 + \frac{e^3}{36} \left( \zeta(3) + \frac{6\lambda + 1}{4} \right) + ... \right],$$  \hspace{1cm} (6.51)

and proposed the estimate $\Gamma/\Gamma' \simeq 4.9$, in 3d. The $O(e^3)$ results for the exponents $\gamma$ and $\eta$ (with $\beta \equiv (d - 2 + \eta)/[2(2 - \eta)]$), are given in, for example, the review by Brézin et al. (1976) (see equations (9.16) and (9.18) there).

For general $n$, the universal ratio $R_x \equiv \tilde{R}$ (see (3.11), (3.21)) is given by Abe and Hikami (1977), and Abe and Masutani (1978), to $O(e^2)$, as

$$R_x = 1 + \frac{3e}{2(n + 8)} \ln \frac{27}{4} + \left[ \frac{e}{2(n + 8)^2} \right]^2 \left[ \frac{9}{2} \left( \ln \frac{27}{4} \right)^2 \right.$$

$$\left. + \frac{188 + 38n - n^2}{n + 8} \ln \frac{27}{4} + 2(n - 1) B \right] + O(e^3),$$  \hspace{1cm} (6.52)

where $B \simeq -8.8224$. This expansion was derived by Aharony and Hohenberg (1976), but with a slightly inaccurate $B$ value. They estimate $R_x = 1.6 \pm 0.1$ for $n = 1$ (in 3d), while Abe and Masutani (1978) obtain $R_x \simeq 1.33$ for $n = 3$. According to Brézin et al. (1974), the $n = 1$ result can be written concisely as

$$R_x = 3^{-\delta}(2^{1-2\beta}/27)^{(1-\delta)/2} + O(e^3),$$  \hspace{1cm} (6.53)

where we used the identity (Aharony and Hohenberg, 1976)

$$R_x \equiv Q_1^{-\delta},$$  \hspace{1cm} (6.54)

for the universal amplitude combination $Q_1$ defined originally by Tarko and Fisher (1975). Brézin et al. (1974) also calculate the $n = 1$ expansions

$$Q_2 \simeq 1 + e/18 + (23 - 6I)e^2/486,$$  \hspace{1cm} (6.55)

$$Q_3 \simeq 1 + (4I - 15)e^2/648,$$  \hspace{1cm} (6.56)
\( I \approx -2.3439 \). They suggest the estimates \( Q_2 \approx 1.13 \) and \( Q_3 \approx 0.96 \), in 3d.

For \( Q_3 \), Bray (1976) derived also the \( O(\varepsilon^3) \) general-\( n \) expansion, which is too cumbersome to reproduce here. His estimates are \( Q_3 \approx 0.922 \) (\( n = 1 \)) and \( Q_3 \approx 0.919 \) (\( n = 3 \)).

For the ratio \( R_C \), Aharony and Hohenberg (1976) derived the expression

\[
R_C \approx \frac{n \varepsilon^2}{n + 8} \left\{ 1 + \varepsilon \left[ 1 - \frac{30}{(n + 8)^2} \right] \right\},
\]

and estimated \( R_C \approx 0.066 \) and 0.17, for \( n = 1 \) and 3, respectively (in 3d).

For the ratio \( R_C \), Aharony and Hohenberg (1976) derived the expression

\[
R_C \approx \frac{nK_d}{4} \left( 1 + \frac{n - 1}{n + 8} \varepsilon + c_1 \varepsilon^2 \right) + O(\varepsilon^3),
\]

and estimated \( (\xi_0/\xi'_0)_3 \approx 1.91 \). Bervillier (1976) derived the \( \varepsilon \)-expansions for \( \xi_0/\xi'_0 \) and \( R_T^+ \), in the form

\[
\left( \frac{\xi_0}{\xi'_0} \right)^d = 2^n \left[ 1 + \frac{5\varepsilon}{24} + \frac{\varepsilon^2}{432} \left( \frac{295}{24} + 2I \right) \right] + O(\varepsilon^3),
\]

and estimate \( (\xi_0/\xi'_0)_{3d} \approx 1.91 \). Bervillier’s (1976) results for the coefficients \( c_1 \) and \( c_2 \) in (6.59) and (6.60) are apparently inaccurate (Okabe and Ideura, 1981), although numerical differences are not significant for \( n = 1, 2, 3 \). Okabe and Ideura (1981) report the value

\[
c_2 = \frac{1}{(n + 8)^2} \left[ \frac{21n^3 + 260n^2 + 2144n + 4064}{8(n + 8)^2} + \frac{(n^2 + 6n - 14)^2}{24} \right.
\]

\[
+ \frac{3(n + 14)\lambda + 12(5n + 22)\zeta(3)}{n + 8} \right].
\]

An amplitude combination similar to (2.58) for the surface tension has been calculated by Brézin and Feng (1984). They report the expansion for the hyperuniversal combination (compare with \( R_{\sigma\xi} \) in (2.58))

\[
\omega_W = \left[ 4 \pi \sigma_0 (\xi'_0)^{d-1} \right]^{-1}
\]

\[
= \frac{2\pi \varepsilon}{3} \left\{ 1 - \varepsilon \left[ \frac{47}{54} + \frac{1}{2} \ln(4\pi) - \frac{1}{2} c_\xi - \frac{5\pi \sqrt{3}}{18} \right] \right\} + O(\varepsilon^3), \quad (6.60)
\]
where \( \gamma = 0.5772 \ldots \) is Euler’s constant, and estimate \( \omega_\infty \simeq 1.5 \) in 3d. The combination \( \omega_\infty \) was introduced by Brézin et al. (1983) in their studies of the critical wetting transition near \( d = 3 \).

Some irrelevant-variable correction to scaling amplitude ratios have been obtained in the \( \varepsilon \)-expansion. Aharony and Ahlers (1980) noted that, to leading order in \( \varepsilon \), these ratios are equal to the ratios of the deviations of the corresponding critical exponents from their respective mean-field values, e.g. \( a_M/a_\chi \simeq (\beta - \frac{1}{2})/(\gamma - 1) \). In many 3d cases this approximation is accurate to within 30\%. Chang and Houghton (1980a,b) derived the \( O(\varepsilon^2) \) expansion for \( a_C, a'_C, a_M, a_\chi, a'_\chi, \) and \( a_\xi \). The resulting formulas are rather cumbersome, and we only reproduce the ratios for which numerical Padé estimates exist (Chang and Houghton, 1980a,b), in 3d:

\[
a_{\xi}/a_\chi = \frac{1}{2}(1 + 0.11 \varepsilon + 0.19 \varepsilon^2 + \ldots) \simeq 0.65, \quad n = 1, \quad (6.63)
\]

\[
a_{\xi}/a_\chi = \frac{1}{2}(1 + 0.1 \varepsilon + 0.16 \varepsilon^2 + \ldots) \simeq 0.63, \quad n = 2, \quad (6.64)
\]

\[
a_M/a_\chi = (1 - 0.65 \varepsilon + 2.23 \varepsilon^2 + \ldots) \simeq 0.85, \quad n = 1, \quad (6.65)
\]

\[
\frac{a_C}{a'_C} = (1 + 0.9 \varepsilon - 3.84 \varepsilon^2 + \ldots) \simeq 1.17, \quad n = 2. \quad (6.66)
\]

The full \( n \)-dependent expressions are given by Chang and Houghton (1980a,b). These authors (Chang and Houghton, 1981) also calculated the \( O(\varepsilon) \) expansion for \( a_{\rho_s} \) and obtained the ratio

\[
\frac{a'_C}{a_{\rho_s}} = \frac{4 - n}{6} \left[ 1 + \frac{9n^3 + 46n^2 - 32n - 32}{4(n + 8)(n + 2)(n - 4)} \varepsilon + \ldots \right], \quad (6.67)
\]

which is of interest in superfluids (\( n = 2 \)), where the stiffness constant \( \rho_s \) is actually the superfluid density. Evaluation at \( d = 3 \) (\( \varepsilon = 1 \)) yields \( a'_C/a_{\rho_s} \simeq \frac{1}{6} \) (\( n = 2 \)). High-order perturbative 3d results (Schloms and Dohm, 1990) indicate, however, that the value is actually negative; see Section 6.2.3 below. Finally, Chang and Rehr (1983) derived the \( O(\varepsilon) \) result for \( a_{\chi^{\text{on}}} \), where the nonlinear susceptibility \( \chi^{(\text{nl})} \) enters the definition of the renormalized coupling constant \( g_* \) (see (5.2)–(5.4)). Specific results include, for example,

\[
a_g/a_\chi = -3(1 - \frac{7}{27} \varepsilon + \ldots) \simeq -2.22, \quad n = 1, \quad d = 3. \quad (6.68)
\]

\[
a_g/a_\xi = -6(1 - \frac{10}{27} \varepsilon + \ldots) \simeq -3.78, \quad n = 1, \quad d = 3. \quad (6.69)
\]

### 6.2.2 Results in the large-\( n \) limit

Abe and Hikami (1975) derived the leading order in \( 1/n \) results for the ratio \( A/A' \), for \( 3 < d < 4 \) and at \( d = 3 \) (see (6.43) and (6.44)), which is discontinuous
as \( d \to 3 \). Indeed, discontinuities, etc., are expected at

\[
d = 2 + \frac{2}{m}, \quad m = 2, 3, 4, \ldots, \quad (6.70)
\]

since, for the spherical model, \( \alpha = 1 - m \) is then a negative integer.

The value of \( R_x \) is known to \( O(1/n^2) \) (see Oku and Okabe, 1979). It shows no special features as \( d \) is varied in \( 2 < d < 4 \). The expressions for the coefficients are, however, rather cumbersome. Thus, we only quote the \( d = 3 \) result:

\[
R_x \simeq 1 - \frac{1.912}{n} + \frac{9.22}{n^2} + \ldots \quad (6.71)
\]

Oku and Okabe (1979) estimate \( R_x = 1.20 \pm 0.15, 1.00 \pm 0.08 \) and \( 0.89 \pm 0.005 \), for \( n = 3, 4 \) and 10, respectively. By using additional input from series analysis, they also obtain the “mixed” estimate \( R_x \simeq 1.37 \) for \( n = 2 \).

The universal ratio \( \xi_0/(n\xi_0^T) \) has been calculated to \( O(1/n) \) by Okabe and Ideura (1981). Again, we only quote their \( d = 3 \) result (see also (6.47)), which they presented in the form

\[
\frac{\xi_0}{\xi_0^T} \simeq \frac{2^v}{8\pi n} \left( 1 + \frac{0.2142}{n} + \ldots \right). \quad (6.72)
\]

By combining this with the series analysis values for \( v \), Okabe and Ideura (1981) estimate \( \xi_0/\xi_0^T \simeq 0.140 \) and 0.208, for \( n = 2 \) and 3, respectively. We also note that the exponent \( v \) entering Okabe and Ideura’s (1981) expressions can be expanded as

\[
v = \frac{1}{d - 2} - \frac{4\Gamma(d) \sin[\pi(d - 2)/2]}{n(d - 2)\pi[\Gamma(d/2)]^2} + O\left( \frac{1}{n^2} \right). \quad (6.73)
\]

Sompolinsky and Aharony (1981) obtained the leading large-\( n \) behaviour of various correction-to-scaling amplitudes \( a_l \) (see (2.60)). Some amplitudes have singularities as

\[
d \to \tilde{d}_l = 2 + \frac{4}{l}, \quad l = 3, 4, 5, \ldots. \quad (6.74)
\]

Specifically, \( a_C \) and \( a_x \) are discontinuous at \( \tilde{d}_k \) and \( \tilde{d}_{2k} \) \((k \geq 2)\), respectively, while \( a_C' \) diverges at \( \tilde{d}_{2k+1} \) \((k \geq 1)\) but is continuous at \( \tilde{d}_{2k} \). Sompolinsky and Aharony (1981) further found that \( a_M = a_{p_s} = 0 \) in lowest order in \( 1/n \). For other amplitudes, they reported the following universal ratios to
leading order:

\[ a_{z}/a_{c} = \frac{1}{2}, \quad \text{(6.75)} \]

\[ a_{c}/a_{x} = \frac{d}{2 - 3} \quad (d \neq \tilde{d}), \quad \text{(6.76)} \]

\[
\frac{a_{c}}{a_{x}} = -\frac{3}{2} \left[ 1 + \frac{8(\frac{14}{3} - 64/\pi^{2})}{7(\pi^{2} - 4)(1 + 8/\pi^{2})} \right] \approx -1.2 \quad (d = 3), \quad \text{(6.77)}
\]

\[
\frac{a_{c}}{a_{c}} = \left( \frac{4}{d - 2} \right)^{(d - 4)/(d - 2)} \left[ B\left(\frac{d - 2}{2}, \frac{d - 2}{2}\right) \right]^{2/(d - 2)} \left[ B\left(\frac{4 - d}{d - 2}, \frac{2d - 6}{d - 2}\right) \right] \\
\times \left[ 3B\left(\frac{6 - d}{d - 2}, \frac{3d - 10}{d - 2}\right) + 2B\left(\frac{8 - 2d}{d - 2}, \frac{4d - 12}{d - 2}\right) \right]^{-1}, \quad 1.0 < d < 4,
\quad \text{(6.78)}
\]

where \( B(x, y) \equiv \Gamma(x)\Gamma(y)/\Gamma(x + y) \), as usual, and

\[
\frac{a_{c}}{a_{c}} = -\frac{\pi^{2}}{8} \left[ 1 + \frac{8}{\pi^{2}} \right] \left[ 1 + \frac{8(\frac{14}{3} - 64/\pi^{2})}{7(\pi^{2} - 4)(1 + 8/\pi^{2})} \right] \approx -1.8, \quad d = 3.
\quad \text{(6.79)}
\]

Sompolinsky and Aharony (1981) also calculated some correction-to-scaling amplitudes for \( H \neq 0 \), at \( T_{c} \). The singular behaviour at \( t = 0 \) (relations (2.30)–(2.32)), can be extended as follows (assuming \( M, H > 0 \) for brevity):

\[
H \approx D_{c}M^{\delta}(1 + \tilde{a}_{H}M^{\Theta} + \ldots), \quad \text{(6.80)}
\]

\[
\chi \propto M^{1-\delta}(1 + \tilde{a}_{x}M^{\Theta} + \ldots), \quad \text{(6.81)}
\]

\[
C_{s} \propto M^{2/\beta}(1 + \tilde{a}_{c}M^{\Theta} + \ldots). \quad \text{(6.82)}
\]

Note that the leading correction-to-scaling exponent here is given simply by

\[
\Theta \equiv \theta/\beta, \quad \text{(6.83)}
\]

where \( \theta = (4 - d)/(d - 2) \) in the large-\( n \) limit, while \( \beta = \frac{1}{2} \). Sompolinsky and Aharony (1981) get

\[
\tilde{a}_{x}/\tilde{a}_{H} = -(10 - d)/(d + 2), \quad \text{(6.84)}
\]

\[
\tilde{a}_{c}/\tilde{a}_{H} = (6 - d)/2. \quad \text{(6.85)}
\]

### 6.2.3 Field-theoretic expansions for \( d = 3 \)

Bervillier and Godrèche (1980) estimated the amplitude combination \( R_{\xi}^{+} \) by resummation of the 3d perturbation expansions of Nickel et al. (1977). They propose the values \( R_{\xi}^{+} = 0.2699 \pm 0.0008, 0.3597 \pm 0.0010, 0.4319 \pm 0.0017, \)
for \( n = 1, 2, 3 \), respectively, and also an empirical relation,

\[ R^+ = \nu(n/4\pi)^{1/3}, \quad (6.86) \]

which holds with good accuracy, in 3d. Improved results by Bagnuls and Bervillier (1985) yield new values and error estimates,

\[ R^+ = 0.2700 \pm 0.0007, \quad 0.3606 \pm 0.0020, \quad 0.4347 \pm 0.0020 \]  

(for \( n = 1, 2, 3 \)).

For other leading amplitude combinations, Bagnuls et al. (1984) proposed

\[ A/A' = 0.465 \pm 0.007, \quad \Gamma/\Gamma' = 5.12 \pm 0.49, \quad R_C = 0.052 \pm 0.026, \]  

for \( n = 1 \) (in 3d). Bagnuls et al. (1987) suggest the improved estimates

\[ A/A' = 0.541 \pm 0.014, \quad \Gamma/\Gamma' = 4.77 \pm 0.30, \quad R_C = 0.0594 \pm 0.0011. \]

Baker (1977) pointed out that the results by Baker et al. (1976) yield the estimate of the 3d Ising renormalized coupling constant,

\[ g^*(0) = 23.84 \pm 0.02. \]

Bagnuls and Bervillier, 1985 and Bagnuls et al., 1987 reported estimates of correction-to-scaling amplitude ratios. Above \( T_c \), one has \( a_x/a_x = 0.64, \quad 0.615 \pm 0.005, \quad 0.60 \pm 0.01, \) and \( a_c/a_x = 8.6 \pm 0.2, \quad 5.95 \pm 0.15, \quad 4.6, \) for \( n = 1, 2, 3 \), respectively. (Earlier estimates of these ratios were obtained by Bagnuls and Bervillier (1981).) Below \( T_c \), the \( n = 1 \) results are \( a_c/a_c' = 0.96 \pm 0.25, \quad a_x/a_x' = 0.315 \pm 0.031, \quad a_M/a_M' = 0.90 \pm 0.21. \) Bagnuls and Bervillier (1986b) obtained estimates of the universal ratio \( a_x/a_x'. \) For \( n = 1, 2, 3 \), respectively, their calculations lead to the values \(-2.85 \pm 0.06, \quad -2.08 \pm 0.05, \quad -1.65 \pm 0.04. \)

Very recently, high-order perturbative 3d results (Krause et al., 1990, Schloms and Dohm, 1990) were obtained for \( n > 1, \quad T < T_c. \) Münster (1990) estimated \( R_{\sigma\zeta}(n = 1). \) The numerical values are included in tables below.

### 6.2.4 Series analysis and Monte Carlo results

Recent work by Liu and Fisher (1989) yielded a new series estimate,

\[ A/A' = 0.523 \pm 0.009 \]  

\( (n = 1, 3d) \) which supersedes earlier estimates (e.g. Aharony and Hohenberg, 1976). (Marinari (1984) gave an MC estimate for the complex-\( t \) plane angle \( \phi \) entering (1.37) of \( 55.3 \pm 1.5^\circ \), which, given \( \alpha \), yields \( A/A' = F_+/F_- = 0.45 \pm 0.07. \) ) For \( n = 2 \) and 3, direct-series methods are not available and the most reliable series estimates of \( A/A' \) seem to be 1.08 and 1.52, respectively. These values have been obtained via the approximate empirical small-\( \alpha \) relation \( A/A' \approx 1 - 4\alpha \) (Hohenberg et al., 1976a), which corresponds to assuming \( P(\alpha) \approx 4 \) in (1.34)–(1.36).

The ratios \( \Gamma/\Gamma' = 4.95 \pm 0.15 \) and \( \xi_0/\xi_0' = 1.96 \pm 0.01 \) (\( d = 3; \) Liu and Fisher, 1989) are defined only for \( n = 1 \). Note that all the ratios involving correlation length amplitudes in 3d, assume the “second-moment” definition of \( \zeta \). However, exact 2d Ising \( (n = 1) \) results quoted in Section 6.1 used the “true correlation length” definition. Tarko and Fisher (1975) proposed the value \( \xi_0/\xi_0' = 3.22 \pm 0.08 \) for the “second-moment” ratio in 2d. In our
preceding survey of the field-theoretic and large-\( n \) results, we considered only the 3d estimates of universal amplitude combinations. The reason is simply that these methods are either inapplicable (like large-\( n \)) or unreliable (\( \varepsilon \)-expansion) for \( d = 2 \). Recall that the \( n = 1, 2, 3, \ldots \) vector models have a ferromagnetic transition only for \( n = 1 \), in 2d. All the numerical results for the 2d Ising models given below will assume the “second-moment” correlation length definition.

For the amplitude combination \( R_x \), Aharony and Hohenberg (1976) propose the estimates \( 1.75 \) (\( n = 1, 3d \)), \( 1.23 \) (\( n = 3, 3d \)) and \( 6.78 \) (\( n = 1, 2d \)), based on the series analyses by Tarko and Fisher (1975), Milošević and Stanley (1972a,b) and Ferer et al. (1971b). (The MC estimate by Binder and Müller-Krumbhaar (1973), \( R_x \approx 1.73 \) (\( n = 3, 3d \)), is less consistent with other methods.)

We next turn to the amplitude combinations \( R_c \) and \( R_c^+ \). Recent results by Liu and Fisher (1989) for \( n = 1 \) in 3d, are \( R_c = 0.0581 \pm 0.0010 \) and \( R_c^+ = 0.2659 \pm 0.0007 \). Note that all the five amplitude combination estimates by Liu and Fisher (1989), quoted in this paragraph and before, have been obtained for three different lattices. Thus, their results as quoted, as well as many other series analysis results described in this section, are representative ranges. The lattice-to-lattice consistency of the universal and hyperuniversal combinations is generally well within the error bars shown and for the best studied case of the Ising models (\( n = 1 \)) is typically less than 1 \%. (Universality of a combination equivalent to \( R_c^+ \) with the “true” correlation length was studied in 3d (\( n = 1 \)) as well (see Ferer and Wortis, 1972).) For \( n > 1 \), Aharony and Hohenberg (1976) estimated \( R_c \approx 0.165 \) (\( n = 3, 3d \)), while Ferer et al. (1973) studied a combination equivalent to \( R_c^+ \) for models in the \( n = 2 \) universality class (3d); their estimates span the range \( R_c^+ = 0.35 \pm 0.01 \) (\( n = 2, 3d \)). Regarding the remaining “leading” amplitude combinations, Tarko and Fisher (1975) estimate \( Q_2 \) and \( Q_3 \) for the Ising case (\( n = 1 \)): \( Q_2 = 2.88 \pm 0.02 \) and \( 1.21 \pm 0.04 \), for 2d and 3d, respectively, while \( Q_3 = 0.4128 \pm 0.0001 \) (2d) and \( Q_3 = 0.90 \pm 0.01 \) (3d). Ritchie and Fisher (1972) estimate \( Q_3 = 0.91 \pm 0.02 \) for \( n = 3 \) (in 3d).

Recently, Meir (1987) proposed a series estimate (\( n = 1, 3d \)),

\[
A' \Gamma' / B^2 \equiv R_c / [(A/A')(\Gamma/\Gamma')] = 0.025 \pm 0.001, \tag{6.87}
\]

which should be compared with the result \( 0.022 \pm 0.002 \) derived by combining three estimates by Liu and Fisher (1989) (listed above). Another “composite” universal amplitude combination has been studied by Kikuchi and Okabe (1985a,b), by the MC method (\( n = 1, 3d \)). It can be reduced to

\[
\frac{\xi_0}{\tilde{\xi}_0} \left( \frac{Q_2}{\delta R_x} \right)^{\nu/\lambda} \approx 0.77. \tag{6.88}
\]
The same combination can be determined by using the appropriate values of \(Q_2\), etc., from series analyses, as listed above, with the accepted exponent values (\(\gamma \approx 1.24, v \approx 0.63\), etc.): one finds the result \(\approx 0.73\). Ferer et al. (1971a) considered the general-power moment definitions of the correlation length for the 3d Ising case \((n = 1)\). Thus, they define the (unnormalized) moments

\[
\Xi_p = \sum_{r \neq 0} |r|^p G(r),
\]

(6.89)

and check the universality of the combinations

\[
R_{pq} \equiv \Xi_p \Xi_q / (\Xi_{(p+q)/2})^2
\]

(6.90)

by series analysis for the face centred cubic (FCC), BCC and simple cubic (SC) lattices, for various positive and negative \(p\) and \(q\) (including fractional values). The consistency in the values ranges from a few per cent, to a fraction of a per cent, depending on the choice of \(p\) and \(q\) (see Ferer et al., 1971a). The universality of \(R_{pq}\) was further investigated by Tarko and Fisher (1975) and Ritchie and Essam (1975). The latter authors also calculated certain amplitudes which combine to confirm Tarko and Fisher’s (1975) estimate of \(Q_2\) (in 3d). Certain other universal amplitude ratios for 3d Ising universality class systems have been studied by Saul et al. (1974). Generally, many numerical estimates of individual amplitudes for the 2d and 3d Ising models have been reported, both in the recent series analysis literature and in the pre-RG studies (see reviews in Domb and Green (1974)). As an example, Essam and Hunter (1968) estimated the amplitudes for the first six field- \((H-)\) derivatives of the magnetization below and above \(T_c\), in zero field (obviously, only the odd derivatives are non-zero for \(T > T_c\)), for the square lattice 2d Ising model. Various combinations of these are universal.

For the renormalized coupling constant amplitude combination, we have the series estimates (Baker, 1977) \(g\chi(0) = 14.66 \pm 0.06\) and \(g\chi(0) \approx 24\), for the 2d and 3d Ising \((n = 1)\) models, respectively. Series analysis of the correction-to-scaling amplitude ratio \(a_g/a\chi\), by Chang and Rehr (1983), yields values in the range \(a_g/a\chi = -2.2 \pm 0.5\) \((n = 1, 3d)\). The only other 3d Ising correction ratio estimated by series analysis is \(a_\xi/a\chi\). Recent estimates by several authors (Nickel and Rehr, 1981; Zinn-Justin, 1981; Nickel and Dixon, 1982; George and Rehr, 1984) cover the range \(a_\xi/a\chi = 0.8 \pm 0.1\) \((n = 1, 3d)\). This ratio was also estimated for certain \(n = 2\) models (in 3d) by Rogiers et al. (1979). Their results fall in the range \(a_\xi/a\chi = 0.61 \pm 0.08\) \((n = 2, 3d)\).

Theoretical (and experimental) estimates of the 3d surface tension amplitude combinations \(R_{\xi\xi}\) and \(R_{\eta\eta}\) (defined in (2.58) and (2.59)) have been reviewed by Moldover (1985). MC and series results for the relevant amplitude \(\sigma_0\), by Binder (1982) and Mon and Jasnow (1985a), as well as the \(\epsilon\)-expansion result (6.62) by Brézin and Feng (1984), have been combined with \(\xi_0\)
and $\xi_0/\xi'_0$ estimates by Tarko and Fisher (1975). The resulting range $R_{\sigma\xi} = 0.22 \pm 0.08$ is inconsistent with experiments, and with the mean-field type estimate in 3d, $R_{\sigma\xi} \approx 0.45$ (Fisk and Widom, 1969). However, more recently, new MC (Mon, 1988) and series (Shaw and Fisher, 1989) studies have been reported, favouring values $R_{\sigma\xi} = 0.36 \pm 0.01$ and $R_{\sigma A} = 0.75 \pm 0.025$.

Finally, we turn to polymer solutions and mixtures where, as described in Section 5, the dependence of amplitudes on the polymerization index $N$ is of interest. Sariban and Binder (1987) found by MC simulations up to $N = 32$, of a solution of two distinct but equal-$N$ polymer species, that the universal combinations $R_x$ and $\Gamma'/\Gamma''$ are indeed independent of $N$. However, their numerical $R_x$ estimates, covering the range $R_x = 3.1 \pm 0.3$, are inconsistent with other 3d Ising values, $R_x \approx 1.75$, quoted above. (No $\Gamma'/\Gamma''$ value is given, but a similar discrepancy was indicated.)

### 6.3 n-Vector models: summary of the results for $d = 3$

In this section we collect the results for the 3d universality classes of $n = 1$ (Ising), $n = 2$ ($XY$) and $n = 3$ (Heisenberg). The reason for this choice is that these models are the most studied, both theoretically and experimentally. More detailed results as well as literature sources have been surveyed in Section 6.2 above, classified by techniques and amplitude combinations. Here we summarize the estimates by models. Thus, in Table 6.1 we list several leading 3d amplitude combinations for $n = 1$. We present either ranges of values with error bars, when available, or just numerical values, as reviewed

**Table 6.1 Leading amplitude combinations for the 3d Ising universality class.**

<table>
<thead>
<tr>
<th>Combination</th>
<th>$\varepsilon$-Expansion</th>
<th>FT$^a$ in 3d</th>
<th>Series/MC$^b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A/A'$</td>
<td>$0.524 \pm 0.010$</td>
<td>$0.541 \pm 0.014$</td>
<td>$0.523 \pm 0.009$</td>
</tr>
<tr>
<td>$\Gamma/\Gamma'$</td>
<td>4.9</td>
<td>4.77 $\pm$ 0.30</td>
<td>4.95 $\pm$ 0.15</td>
</tr>
<tr>
<td>$R_x$</td>
<td>1.6 $\pm$ 0.1</td>
<td></td>
<td>1.75</td>
</tr>
<tr>
<td>$R_C$</td>
<td>0.066</td>
<td>0.0594 $\pm$ 0.0011</td>
<td>0.0581 $\pm$ 0.0010</td>
</tr>
<tr>
<td>$\xi_0/\xi'_0$</td>
<td>1.91</td>
<td></td>
<td>1.96 $\pm$ 0.01</td>
</tr>
<tr>
<td>$R_{\sigma\xi}$</td>
<td>0.27</td>
<td>0.2700 $\pm$ 0.0007</td>
<td>0.2659 $\pm$ 0.0007</td>
</tr>
<tr>
<td>$Q_2$</td>
<td>1.13</td>
<td></td>
<td>1.21 $\pm$ 0.04</td>
</tr>
<tr>
<td>$Q_3$</td>
<td>0.922</td>
<td></td>
<td>0.90 $\pm$ 0.01</td>
</tr>
<tr>
<td>$R_{\sigma A}$</td>
<td>0.2</td>
<td>0.39 $\pm$ 0.03</td>
<td>0.36 $\pm$ 0.01</td>
</tr>
<tr>
<td>$g_*(0)$</td>
<td>-</td>
<td>23.85 $\pm$ 0.02</td>
<td>24</td>
</tr>
</tbody>
</table>

$^a$ Field theory.

$^b$ Monte Carlo.
Table 6.2 Correction amplitude ratios for the 3d Ising universality class.

<table>
<thead>
<tr>
<th></th>
<th>(\varepsilon)-Expansion</th>
<th>FT in 3d</th>
<th>Series</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a_z/a_x)</td>
<td>0.65</td>
<td>0.64</td>
<td>0.8 ± 0.1</td>
</tr>
<tr>
<td>(a_M/a_x)</td>
<td>0.85</td>
<td>0.90 ± 0.21</td>
<td>—</td>
</tr>
<tr>
<td>(a_C/a_x)</td>
<td>—</td>
<td>8.6 ± 0.2</td>
<td>—</td>
</tr>
<tr>
<td>(a_C/a_C')</td>
<td>—</td>
<td>0.96 ± 0.25</td>
<td>—</td>
</tr>
<tr>
<td>(a_z/a_x')</td>
<td>—</td>
<td>0.315 ± 0.013</td>
<td>—</td>
</tr>
<tr>
<td>(a_y/a_x)</td>
<td>-2.22</td>
<td>-2.85 ± 0.06</td>
<td>-2.2 ± 0.5</td>
</tr>
<tr>
<td>(a_y/a_z)</td>
<td>-3.78</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

in Section 6.2. The error bars quoted are those given in the original works and usually represent some assessment of statistical spread of estimates, but do not include possible systematic errors and trends. The columns in Table 6.1 (and other tables below) are self-explanatory. Some entries in the tables are empty. This means that either that particular combination has not been

Table 6.3 Leading and correction amplitude combinations for the \(XY\) \((n = 2)\) and Heisenberg \((n = 3)\) universality classes, in 3d.

<table>
<thead>
<tr>
<th>(n)</th>
<th>(\varepsilon)-Expansion</th>
<th>FT in 3d</th>
<th>Series</th>
<th>Large-(n)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A/A')</td>
<td>2 1.029 ± 0.013</td>
<td>1.05</td>
<td>1.08</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 1.521 ± 0.022</td>
<td>1.58</td>
<td>1.52</td>
<td>—</td>
</tr>
<tr>
<td>(R_x)</td>
<td>2 —</td>
<td>—</td>
<td>—</td>
<td>1.37</td>
</tr>
<tr>
<td></td>
<td>3 1.33</td>
<td>—</td>
<td>—</td>
<td>1.20 ± 0.15</td>
</tr>
<tr>
<td>(R_C)</td>
<td>2 —</td>
<td>—</td>
<td>0.35 ± 0.01</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 0.17</td>
<td>—</td>
<td>0.165</td>
<td>—</td>
</tr>
<tr>
<td>(\xi_0/\xi_T)</td>
<td>2 0.33</td>
<td>0.50</td>
<td>—</td>
<td>0.140</td>
</tr>
<tr>
<td></td>
<td>3 0.38</td>
<td>0.56</td>
<td>—</td>
<td>0.208</td>
</tr>
<tr>
<td>(R_+^\xi)</td>
<td>2 0.36</td>
<td>0.3606 ± 0.0020</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 0.42</td>
<td>0.4347 ± 0.0020</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(R_T^\xi)</td>
<td>2 1.0 ± 0.2</td>
<td>0.78</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 0.9 ± 0.2</td>
<td>0.73</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(Q_3)</td>
<td>2 0.919</td>
<td>—</td>
<td>0.91 ± 0.02</td>
<td>—</td>
</tr>
<tr>
<td>(a_z/a_x)</td>
<td>2 0.63</td>
<td>0.615 ± 0.005</td>
<td>0.61 ± 0.08</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 —</td>
<td>0.60 ± 0.01</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(a_C/a_x)</td>
<td>2 —</td>
<td>5.95 ± 0.15</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 —</td>
<td>4.6</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(a_C/a_C')</td>
<td>2 1.17</td>
<td>1.6</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 —</td>
<td>1.4</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(a_C/a_p_)</td>
<td>2 —</td>
<td>-0.045</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 —</td>
<td>-0.69</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td>(a_y/a_x)</td>
<td>2 —</td>
<td>-2.08 ± 0.05</td>
<td>—</td>
<td>—</td>
</tr>
<tr>
<td></td>
<td>3 —</td>
<td>-1.65 ± 0.04</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
April 10, 1997

Professor V. Privman
Department of Physics
Clarkson University
Potsdam, NY 13699

Dear Professor Privman,

I have found what appears to be a small error in your useful review "Universal Critical-Point Amplitude Relations," published in 1991 with P. C. Hohenberg and A. Aharony. In Eq. (2.60) you state that "a general thermodynamic function will be written

\[ \phi_i = A_i t^{-\phi_i} \left( 1 + a_i t^{\theta} + \cdots \right), \]

where \( \theta > 0 \) is the leading correction exponent...". In Table 6.3 you then report a value of the corrections to scaling amplitude ratio \( a_c/a_\chi \) in the neighborhood of 8.6 for the 3d Ising universality class. Pure fluid papers, however, report values for this ratio in the neighborhood of 0.9 [see, e.g., Eq. (10) of Sengers and Levelt Sengers, Ann. Rev. Phys. Chem. 37, 189 (1986)], a sizable discrepancy. The resolution appears to be given by Bagnuls and Bervillier [Phys. Rev. B 24, 1226 (1981)], who also report a value of around 8.6, but give in their Eq. (1.4)

\[ C_S = \frac{A^+}{\alpha} t^{-\alpha(1 + \alpha a_\xi t^{0v} + \cdots)}, \]

The extra factor of \( \alpha \) inside the parentheses appears to resolve the discrepancy.
estimated at all, or that the available value is too rough and qualitative to be useful. (We generally list such a rough estimate if it is the only one available.) The consistency of the various estimates in Table 6.1 is quite satisfactory for most amplitude combinations listed.

In Table 6.2, we list various correction-to-scaling amplitude ratios for the 3d Ising universality class. Whenever estimates by more than one method are available, there are sufficient discrepancies to suggest that further studies would be useful.

We now turn to \( n > 1 \). Table 6.3 presents the leading and correction amplitude results for \( n = 2 \) and \( n = 3 \). While a consistent pattern of amplitude combination values seems to emerge, there are still discrepancies in the \( n = 2, 3 \) results. For \( n = 2 \) the amplitude combination \( R^T_r \) is of interest (see (3.54)). The second-order \( \varepsilon \)-expansion (Bervillier, 1976; Hohenberg et al., 1976a) suggests \( R^T_r \sim 0.95 \) and 0.88, for \( n = 2, 3 \), respectively. The uncertainty in both values is at least 20\%

Thus the 3d results \( R^T_r = 1.0 \pm 0.2 \) \((n = 2)\), \( R^T_r = 0.9 \pm 0.2 \) \((n = 3)\). All the FT values in Table 6.3 with no error limits are shown from the recent high-order perturbative work by Krause et al. 1990, Schloms and Dohm 1990.

### 6.4 Dipolar, random and dilute magnetic systems

Aharony and Hohenberg (1976) considered the dipolar Ising model (Larkin and Khmelnitzkii, 1969; Aharony, 1973) at the upper-marginal dimensionality \( d = 3 \). The amplitude definitions must then be modified to

\[
M = \hat{B}(-t)^{1/2}|\ln(-t)|^{1/3}, \quad t < 0, \quad H = 0, \tag{6.91}
\]

\[
\chi = \hat{f}t^{-1}|\ln t|^{1/3}, \quad t > 0, \quad H = 0, \tag{6.92}
\]

\[
\chi = \hat{f}'(-t)^{-1}|\ln(-t)|^{1/3}, \quad t < 0, \quad H = 0, \tag{6.93}
\]

\[
C = \hat{A}|\ln t|^{1/3}, \quad t > 0, \quad H = 0, \tag{6.94}
\]

\[
C = \hat{A}'|\ln(-t)|^{1/3}, \quad t < 0, \quad H = 0, \tag{6.95}
\]

\[
H = \tilde{D}_c M^3|\ln|M||^{-1}, \quad t = 0, \tag{6.96}
\]

\[
\chi = \hat{f}_c |H|^{-2/3}|\ln|H||^{1/3}, \quad t = 0. \tag{6.97}
\]

For the universal combinations, in a self-explanatory notation, we have the following explicit results (Aharony and Halperin, 1975; Aharony and Hohenberg, 1976; Brézin, 1975):

\[
\hat{A}/\hat{A}' = \frac{1}{4}, \quad \hat{f}/\hat{f}' = 2, \quad \hat{R}_\chi = \frac{1}{2}, \quad \hat{R}_c = \frac{1}{6}. \tag{6.98}
\]

Note also the generalized two-scale-factor universality relation (4.59).
For random-field models various authors have proposed a dimensional reduction scheme whereby the properties of the random system can be related to those of the pure system in dimension $d - 2$ (e.g. see Parisi and Sourlas, 1979). Although it has become clear that the method is not generally valid (in particular it predicts an incorrect lower critical dimension (Imbrie, 1984)), the possibility remains that the perturbative $(6 - \varepsilon)$-expansion is correct to all orders (Fisher, 1986). The only actual calculation of an amplitude ratio, that for $R_x$, is known for the spherical model ($n = \infty$) in the Gaussian random field: Tanaka (1977) reported a replica-calculation result $R_x = 1$ ($n = \infty$, $6 > d > 4$).

For the randomly dilute Ising model (e.g. Grinstein and Luther, 1976), the $\sqrt{\varepsilon}$-expansion values ($\varepsilon = 4 - d$) are known for three amplitude ratios (Newlove, 1983):

$$A/A' = -\frac{1}{2} + O(\sqrt{\varepsilon}), \quad (6.99)$$

$$\frac{\Gamma}{\Gamma'} = 2 + \sqrt{\frac{6 \varepsilon}{53}} \left( \ln 2 - \frac{3}{2} \right) + O(\varepsilon) \simeq 1.7, \quad (6.100)$$

$$\frac{\xi_0}{\xi'_0} = \sqrt{2} \left[ 1 - \frac{6 \varepsilon}{53} \left( \frac{9}{16} - \frac{1}{4} \ln 2 \right) + O(\varepsilon) \right] \simeq 1.2. \quad (6.101)$$

The numerical values in (6.100) and (6.101) were obtained by simply setting $\varepsilon = 1$. Further discussion of certain amplitude properties was given by Pelcovits and Aharony (1985). Aharony (1976b) and Schuster (1977, 1978) studied the effect of quenched impurities on the dipolar-Ising critical behaviour. While the leading power laws in (6.91)–(6.97) are not changed, the logarithmic factors are replaced by different terms (logarithmic or essentially singular). Some amplitude combinations are changed, for example, $\hat{R}_c = \sqrt{2}/6$ (compare (6.98)). However, the ratios $\hat{A}/\hat{A}'$ and $\hat{f}/\hat{f}'$ are not affected. The theoretical situation in 2d is controversial; see, e.g., Section 7.4.7.2 below.

6.5 Percolation, Potts and related models

6.5.1 Percolation

In percolation models (e.g. reviewed by Essam, 1980) the concentration of occupied bonds (or sites), $p$ (where $0 \leq p \leq 1$) plays the role of the temperature variable; it is appropriate then to define $t \equiv (p_c - p)/p_c$. The equivalent of the magnetic field is the “ghost” field (Reynolds et al., 1980). Various properties in percolation, e.g. the mean number of clusters, the percolation probability, the mean-squared cluster size and the pair-
connectedness correlation length can be regarded as analogous to, respectively, the free-energy, magnetization, susceptibility and correlation length quantities. With these identifications, Aharony (1980) considered the standard universal amplitude combinations in the $\varepsilon$-expansion, where $\varepsilon = 6 - d$ for percolation. (Note that Aharony (1980) uses the free-energy amplitudes $F_\pm$, which, however, are related straightforwardly to the specific-heat amplitudes via $A = -\alpha(1 - \alpha)(2 - \alpha)F_+,$ etc.) Thus, we have

$$A/A' = -\frac{1}{3}(1 + \frac{26}{35}\varepsilon) + O(\varepsilon^2),$$

$$\Gamma/\Gamma' = \gamma/\beta + O(\varepsilon^3) = \delta - 1 + O(\varepsilon^3),$$

$$R_\chi = 2\delta^{-2} + O(\varepsilon^3),$$

$$R_C = \frac{1}{4}(1 + \frac{3}{2}\varepsilon) + O(\varepsilon^2),$$

$$\zeta_0/\zeta_0' = 1 + \frac{5}{42}\varepsilon + O(\varepsilon^2),$$

$$\left(R_{\xi}^+\right)^d = \frac{7K_d}{2\varepsilon} \left(1 - \frac{397}{1764}\varepsilon\right) + O(\varepsilon).$$

In these expressions, $K_d$ (see (6.46)) and the exponents can be further $\varepsilon$-expanded. For example,

$$\delta = 2 + \frac{7}{2}\varepsilon + \frac{365}{882}\varepsilon^2 + O(\varepsilon^3).$$

Note that the mean-field exponents for percolation are $\alpha = -1, \beta = 1, \gamma = 1, v = \frac{1}{2}.$ At $d = 6$, Aharony (1980) also obtained the logarithmically modified critical behaviour with these leading exponents, and derived the equivalent of (4.58),

$$\xi_6 f_\varepsilon \approx \frac{7}{1536\pi^3} |\ln t| \text{ for } t > 0.$$  

For the correction-to-scaling ratios, Aharony (1980) shows that $a_M/a_C, a_M/a_\chi, a_C/a'_C, a_\chi/a'_\chi$ and $a_\xi/a'_\xi$ are all given by $1 + O(\varepsilon)$, and also that $a_M/a_\xi = \frac{12}{5} + O(\varepsilon),$ etc. For some further discussion of correction amplitudes see Adler et al. (1983).

Aharony's (1980) numerical estimates for $d = 2, 3, 4, 5$ are summarized in Table 6.4, where series and MC values are also listed (see below). Obviously, extrapolation of the $\varepsilon$-expansion from $d = 6$ down to $d = 4, 3, 2$ provides, at best, rough estimates.

The $d = 2, 4, 5$ values of $\Gamma/\Gamma'$ in the series/MC category come from the MC studies by Nakanishi and Stanley (1980) and Hoshen et al. (1979), including $\Gamma/\Gamma' \sim 200$ in 2d. Jan and Stauffer (1982) confirm the value $\sim 200$ for correlated percolation in 2d. Aharony (1980) also reviews certain earlier series and MC estimates, which vary widely. A recent series estimate by Meir (1987) yields $\Gamma/\Gamma' = 220 \pm 10.$ However, Kim et al. (1987) got, by the MC
Table 6.4 Universal amplitude combinations for percolation.

<table>
<thead>
<tr>
<th></th>
<th>$d = 2$</th>
<th>$d = 3$</th>
<th>$d = 4$</th>
<th>$d = 5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma/\Gamma'$</td>
<td>(expansion) 3.6–17.0</td>
<td>2.7–4.3</td>
<td>1.9–3.0</td>
<td>1.4–1.7</td>
</tr>
<tr>
<td>$\Gamma'/\Gamma''$</td>
<td>(series/MC) $\sim$14–200</td>
<td>10.0 ± 1.6</td>
<td>$\sim$5</td>
<td>$\sim$4</td>
</tr>
<tr>
<td>$Q^{-1}_{1/d}$</td>
<td>(expansion) 1.7–2.3</td>
<td>1.4–1.8</td>
<td>1.3–1.4</td>
<td>1.1–1.2</td>
</tr>
<tr>
<td>$R_{A/A'}^{1/d}$</td>
<td>(series/MC) 1.1–1.3$^b$</td>
<td>1.1–1.6$^b$</td>
<td>$\sim$1.2</td>
<td>$\sim$1.5</td>
</tr>
<tr>
<td>$A'/A$</td>
<td>(expansion) $-0.79$</td>
<td>$-0.65$</td>
<td>$-0.50$</td>
<td>$-0.35$</td>
</tr>
<tr>
<td>$A/A'$</td>
<td>(series/MC) $-1.0$</td>
<td>$-1.3$</td>
<td>$-1.5$</td>
<td>$-1.6$</td>
</tr>
<tr>
<td>$R_C$</td>
<td>(expansion) 0.96</td>
<td>0.79</td>
<td>0.61</td>
<td>0.43</td>
</tr>
<tr>
<td>$R_C$</td>
<td>(series/MC) 4.1–4.2</td>
<td>4.0 ± 0.5</td>
<td>2.0 ± 0.5</td>
<td>1.12–1.15</td>
</tr>
<tr>
<td>$\xi_0/\xi_0'$</td>
<td>(expansion) 1.5–2.0</td>
<td>1.36–1.64</td>
<td>1.23–1.40</td>
<td>1.12–1.15</td>
</tr>
<tr>
<td>$\xi_0'/\xi_0$</td>
<td>(MC) 4.0</td>
<td>2.0</td>
<td>1.36</td>
<td>1.23</td>
</tr>
<tr>
<td>$R_\zeta$</td>
<td>(expansion) 0.21</td>
<td>0.30</td>
<td>0.34</td>
<td>0.38</td>
</tr>
</tbody>
</table>

$^a$Results for correction-to-scaling amplitude ratios are described in the text following (6.109).

$^b$The series values are also reviewed in Sect. 4.5 of Essam’s (1980) review.

method, $\Gamma/\Gamma' \sim 14$, for a related continuum model, randomly diluted. For percolation in 2d, $\alpha \equiv -\frac{3}{2} < 0$, so that this model must be in the percolation universality class by the Harris (1974) criterion. For a continuum (off-lattice) percolation model in 2d, Gawlinski and Stanley (1981) find $\Gamma/\Gamma' = 50 \pm 26$. Nakanishi (1987) finds $\Gamma/\Gamma' = 139 \pm 24$ for a certain mixed-species model. We believe that the spread of values reflects numerical difficulties, and that future work will yield more “universal” estimates; see also Stauffer (1986).

Finally, the 3d value $\Gamma/\Gamma' = 10.0 \pm 1.6$ was taken from the MC work by Herrmann et al. (1982), who also estimated $\Gamma/\Gamma' = 2.7 \pm 0.8$ for kinetic gelation in 3d (not equivalent to percolation). The MC values for $\xi_0/\xi_0'$ in $d = 2, 3$ were obtained by Corsten et al. (1989). These authors also estimate $49 < \Gamma/\Gamma' < 115$ in 2d, and $6 < \Gamma/\Gamma' < 11$ in 3d, and discuss a result by Chayes et al. (1989), who obtain the exact value, 2, for a correlation length amplitude ratio in 2d, albeit for a definition different from the second-moment $\zeta$. The remaining series/MC values in Table 4, those for $R_\zeta$, $A/A'$ and $R_C$, were collected by Aharony (1980), primarily from data by Domb and Pearce (1976), Marro (1976), Stauffer (1976) and Nakanishi and Stanley (1980).

Recently, Adler et al. (1986, 1990a, b) studied percolation equivalents of the universal ratios $\Xi_p\Xi_q/(\Xi_r\Xi_s)$, with $p + q = r + s$ (see relation (6.89) and the discussion following it). Ratios corresponding to the $(p, q, r, s)$ choices $(2, 4, 3, 3)$, $(3, 5, 4, 4)$, $(2, 5, 3, 3)$ and $(2, 5, 3, 4)$ were obtained by $\epsilon$-expansion near $d = 6^-$, by series analysis for $d = 2, 3, 4, 5$ and by exact calculations for $d = 1$ and for the Bethe lattice. The agreement of the $\epsilon$-expansion and series estimates for $d = 2, \ldots, 5$, as well as among series estimates for different lattices (2d and 3d only), is excellent. For a study of the amplitude ratio
(2, 4, 3, 3), in the case of directed percolation in 4d, see Adler et al. (1988). Finally, Harris et al. (1987, 1990) reported series and $\varepsilon$-expansion (see also Harris and Lubensky, 1987) studies of certain ratios of moments for the diffusion and resistance distributions on percolation clusters at $p_c$. These ratios are also relevant to the crossover from zero-temperature percolation to finite temperature, for dilute spin systems with continuous symmetry. Adler et al. (1990b) review many more amplitude ratios concerning physical properties near the percolation threshold.

### 6.5.2 Cluster shape ratios for percolation and lattice animals

Results on the cluster shape ratios (see (5.11)-(5.13)) for percolation (at $p_c$) and lattice animals are relatively recent. They are summarized in Table 6.5. The MC values of $A_2$ are taken from Family et al. (1985), while those for $D_2$, $D_3$ from Quandt and Young (1987). All the series estimates are from Straley and Stephen (1987). The upper critical dimensions are $d > 6$ and 8, for percolation and lattice animals, respectively. The ratios $D_d$ and $J_d$ are then equal (for $d \geq 8$) and known exactly (for $d \geq d_\infty$).

#### Table 6.5 Universal shape ratios for percolation and lattice animals.$^a$

<table>
<thead>
<tr>
<th></th>
<th>Percolation</th>
<th>Lattice animals</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_2$ (MC)</td>
<td>0.4</td>
<td>0.29</td>
</tr>
<tr>
<td>$A_2$ (series)</td>
<td>0.346 ± 0.006</td>
<td>0.280 ± 0.003</td>
</tr>
<tr>
<td>$D_2$ ($\varepsilon$-expansion)</td>
<td>0.37</td>
<td>0.39</td>
</tr>
<tr>
<td>$D_2$ (series)</td>
<td>0.38 ± 0.01</td>
<td>0.466 ± 0.002</td>
</tr>
<tr>
<td>$D_2$ (MC)</td>
<td>0.325 ± 0.006</td>
<td>—</td>
</tr>
<tr>
<td>$D_3$ ($\varepsilon$-expansion)</td>
<td>0.31</td>
<td>0.33</td>
</tr>
<tr>
<td>$D_3$ (series)</td>
<td>0.30 ± 0.03</td>
<td>0.390 ± 0.003</td>
</tr>
<tr>
<td>$D_d \geq 6$ (exact)</td>
<td>$(2 + d)/(2 + 6d)$</td>
<td>—</td>
</tr>
<tr>
<td>$D_d \geq 8$ (exact)</td>
<td>0</td>
<td>$(2 + d)/(2 + 6d)$</td>
</tr>
<tr>
<td>$P_2$, $P_2$ (exact)</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$P_3$ (series)</td>
<td>0.23 ± 0.02</td>
<td>0.27 ± 0.01</td>
</tr>
<tr>
<td>$P_3$ ($d = 3$ in large-$d$ formula below)</td>
<td>0.16</td>
<td>0.16</td>
</tr>
<tr>
<td>$J_d \geq 6$ (exact)</td>
<td>$47(d^2 + 6d + 8)/(797d^2 + 822d + 376)$</td>
<td>—</td>
</tr>
<tr>
<td>$J_d \geq 8$ (exact)</td>
<td>$47(d^2 + 6d + 8)/(797d^2 + 822d + 376)$</td>
<td>—</td>
</tr>
<tr>
<td>$D_2$ (series)</td>
<td>0.282 ± 0.002</td>
<td>0.384 ± 0.004</td>
</tr>
<tr>
<td>$D_2$ (MC)</td>
<td>0.258 ± 0.006</td>
<td>—</td>
</tr>
<tr>
<td>$D_3$ (series)</td>
<td>0.2865 ± 0.0075</td>
<td>0.36 ± 0.015</td>
</tr>
<tr>
<td>$P_3$ (series)</td>
<td>0.54 ± 0.045</td>
<td>0.594 ± 0.0045</td>
</tr>
</tbody>
</table>

$^a$For definitions of cluster shape ratios see Section 5.1.5.
V. Privman et al. (1987) calculated the $\varepsilon$-expansions for percolation ($\varepsilon \equiv 6 - d$),

$$\mathcal{D}_d = \frac{2 + d}{2(1 + 3d)} + \frac{607}{4410} \frac{d(d + 2)}{(1 + 3d)^2} \varepsilon + O(\varepsilon^2),$$  \hspace{1cm} (6.110)

and lattice animals ($\varepsilon \equiv 8 - d$),

$$\mathcal{D}_d = \frac{2 + d}{2(1 + 3d)} + \frac{29}{288} \frac{d(d + 2)}{(1 + 3d)^2} \varepsilon + O(\varepsilon^2).$$  \hspace{1cm} (6.111)

The $\varepsilon$-expansion values in Table 6.5 were obtained from (6.110) and (6.111) as they are, without any resummation (Aronovitz and Stephen, 1987). Straley and Stephen (1987) also calculate the quantities $\mathcal{D}_d$ and $\mathcal{F}_d$, defined as in (5.12) and (5.13), but with averaging after taking the ratio, not before. Their estimates are also listed in Table 6.5.

6.5.3 Potts models

Among the $q$-state Potts models (e.g. Wu, 1982) with $q > 2$, only the 2d, $q = 3$ and 4 models have continuous transitions. (Recall that $q = 2$ is the Ising model, while the limit $q \to 1$, when properly defined, reproduces the percolation model.) The bulk critical-point amplitude ratios have not been studied extensively. As already mentioned, Kaufman and Andelman (1984) argue that $A/A' \equiv 1$ in 2d. (In fact, they conjecture the form of the $q$ dependence of $A$.) Stella et al. (1987) obtain $A/A' \simeq 0.925$ by a numerical transfer matrix study of the $q = 3$ model. Bartelt et al. (1987a) estimated the $q = 3$ amplitude ratios $\xi_0/\xi'_0 = 4.1 \pm 0.2$ and $\Gamma/\Gamma' = 43 \pm 3$ by a MC study. Bartelt et al. (1987b) and Stella et al. (1987) also discuss amplitude ratios for the three-state chiral clock model. These results, however, must be regarded as preliminary, and will not be detailed here.

For the $q = 3$ Potts model, Jasnow et al. (1986) estimated $R_{\sigma\xi} = 0.94 \pm 0.1$ and $1.01 \pm 0.1$, for the square and triangular lattices (in 2d) by combining MC results for $\sigma_0$ with series results for the second-moment $\xi_0$. True correlation lengths differ from the second-moment ones by about 2% in this case (Jasnow et al., 1986). An exact value, $R_{\sigma\xi} = 1$, is known (with "true" $\xi_0$), for the hard-hexagon model (Baxter and Pearce, 1982), which is in the $q = 3$ Potts universality class.

6.6 Multicritical points

The crossover scaling form for bicritical points was described in Section 3.4. Universal parameters of interest, both experimentally and theoretically, are the crossover exponent $\phi$ and the universal ratio $Q_0$, associated with the
shape of the critical lines meeting at the bicritical point. For two Ising \((n = 1)\) critical lines meeting at a bicritical point of \(XY\) \((n = 2)\) symmetry, RG arguments (Fisher and Nelson, 1974; Fisher, 1975b) and series studies (Pfeuty et al., 1974) suggest (in 3d)

\[
\phi = 1.175 \pm 0.015, \quad Q_b \equiv 1 \quad (2 \leftrightarrow 1 + 1). \quad (6.112)
\]

Another well-studied case is that of the Heisenberg symmetry \((n = 3)\) bicritical point at which Ising \((n = 1)\) and \(XY\) \((n = 2)\) lines meet. In this case the \(\epsilon\)-expansion yields (Bruce, 1975; Fisher, 1975b)

\[
Q_b = 2\phi + O(\epsilon^3) \quad (3 \leftrightarrow 2 + 1), \quad (6.113)
\]

where we can use the theoretical range \(\phi = 1.25 \pm 0.02\) to obtain estimates \(Q_b = 2.3\) to 2.4 in 3d. Series analysis by Fisher et al. (1980) gives a consistent value:

\[
Q_b = 2.34 \pm 0.08. \quad (6.114)
\]

For studies of \(Q_b\)-type universal ratios but for higher-order multicritical loci meeting, see Domany et al. (1977) and Blankschtein and Mukamel (1982). Some results are also available for two critical lines meeting at a Lifshitz point (see Mukamel and Luban (1978) and references therein).

### 6.7 Dynamics

#### 6.7.1 Model C

The simplest model with interesting dynamical coupling (Halperin et al., 1974) is purely relaxational, and involves a non-conserved order parameter \(\psi\) interacting with a conserved density \(m\), which represents either the energy or else the concentration of a set of mobile impurities (e.g. see Chapter IV of Hohenberg and Halperin (1977)). The dynamical scaling law in this case is \(z = z_m = 2 + 2\bar{\alpha}/2\), where \(\bar{\alpha} = \max(\alpha, 0)\). The associated dimensionless amplitude ratio is

\[
\mu \equiv \lim_{q \to 0} \left[ \frac{\omega_m(q)}{\omega_\psi(q)(q_\zeta)^2} \right], \quad (6.115)
\]

which has the \(\epsilon\)-expansion (Halperin et al., 1976a,b)

\[
\mu = 1 + \epsilon(9 \ln 2 - \frac{14}{3} \ln 3 - \frac{11}{18}) \quad (6.116)
\]

for \(n = 1\). In addition, a number of universal amplitudes characterizing nonlinear relaxation have been determined to lowest order in \(\epsilon\) by Bausch et al. (1979) and Eisenriegler and Schaub (1980).
6.7.2 Liquid–gas and binary fluid transitions

A model which describes the critical dynamics of the order parameter $\psi$ near liquid–gas and binary fluid critical points consists of the equation of motion for $\psi$ coupled to a mass current $j$ (e.g. see Chapter V of Hohenberg and Halperin (1977)). The corresponding characteristic frequencies $\omega_\psi(q)$ and $\omega_j(q)$ have transport coefficients $\lambda$ and $\lambda_j$, respectively (see (2.67)). For pure fluids, $\lambda$ is the thermal conductivity, while it is the concentration conductivity in binary mixtures. The coefficient $\lambda_j$ is the shear viscosity, which we denote $\eta$ (to distinguish it from the static exponent $\eta$).

A mode-coupling calculation in $d = 3$ (Kawasaki, 1970) yields a Lorentzian shape for the spectrum $\tilde{W}$ (see (3.70)), on the critical isochore,

$$\tilde{W}(\infty, \tilde{y}, \tilde{w}) = \pi^{-1}(\tilde{w}^2 + 1)^{-2}, \quad (6.117)$$

and evaluates the characteristic frequency for $\psi$ (see (2.65)) as

$$\omega_\psi(q) = (q^3/16\eta)\tilde{\Omega}(q\xi), \quad (6.118)$$

with exponent $z = 3$ and

$$\tilde{\Omega}(\tilde{y}) = (2/\pi\tilde{y}^3)[1 + \tilde{y}^2 + (\tilde{y}^3 - \tilde{y}^{-1})\tan^{-1}\tilde{y}]]. \quad (6.119)$$

Note that equation (5.15) of Hohenberg and Halperin (1977) is missing a factor of $k^2$ on the right-hand side. Equation (6.119) implies the universal ratio (see (3.75) with $p = 2$)

$$\tilde{R}_+ = 3\pi/8. \quad (6.120)$$

The dynamical coupling of the $\psi$ and $j$ modes leads to a scaling relation between the exponents for the transport coefficients $\lambda$ and $\eta$ ($\lambda \sim \xi^{x_\lambda}, \eta \sim \xi^{x_\eta}$), which reads, in $d$ dimensions,

$$x_\lambda + x_\eta = 4 - d + \eta. \quad (6.121)$$

Associated with this scaling law is the universal amplitude $R \sim \lambda\eta \xi^{d-2}/\chi_\psi$, which interestingly enough involves a product of frequencies, expressed in dimensional units as

$$R = \lim_{q \to 0} \left[ \frac{\omega_\psi(q)\omega_j(q)}{\omega_\xi(q)} \right] = \frac{\lambda\eta \xi^{d-2}}{k_B T \chi_\psi}. \quad (6.122)$$

In the above equation we have $\omega_\psi = (\lambda/\chi_\psi)q^2 = Dq^2$, $\omega_j = (\eta/\rho)q^2$, and $\omega_\xi = D_\xi q^2$, where $\rho$ is the mass density, $D_\xi = v_\xi \xi$, and $v_\xi$ is a characteristic velocity given by $v_\xi^2 = k_B T/\rho \xi^{d}$. Note that some authors define $R$ with the factor $(6\pi)$ on the right-hand side of (6.122) (e.g. see Sengers, 1982). Equation (6.122) is often rewritten as the Stokes–Einstein relation (Kawasaki,
1 Universal critical-point amplitude relations

in terms of the diffusion constant \( D = \lambda / \chi_\psi \),
\[
D = R(\k_B T_c / \bar{\eta}^{d-2}),
\]
(6.123)

and (6.118) and (6.119) imply the mode-coupling value
\[
R = 1/6\pi.
\]
(6.124)

The RG can be formulated for the above model, and the \( \varepsilon \)-expansion yields (Siggia et al., 1976) for 3d,
\[
R = K_d \frac{19}{24} \varepsilon^{-1} [1 + 0.06 \varepsilon + O(\varepsilon^2)] \simeq \frac{0.8}{6\pi},
\]
(6.125)
\[
\bar{R} = 1 + \frac{9}{38} \varepsilon + O(\varepsilon^2) \simeq 1.24.
\]
(6.126)

An attempt to combine mode coupling in 3d with the \( \varepsilon \)-expansion was devised by Siggia et al. (1976) and yields the value
\[
R = 1.2/6\pi,
\]
(6.127)
as a correction to (6.124). There is no a priori way to assess the reliability of this result. An alternative calculation directly in 3d was carried out by Paladin and Peliti (1982), who found (see Erratum, 1984)
\[
R = 1.0375/6\pi,
\]
(6.128)
and rather small corrections to the Kawasaki form of the scaling function (6.119). The variation among the different values of the ratio \( R \) provides a reasonable estimate of the uncertainty of the theory, but we agree with Paladin and Peliti (1982) that the inherent simplicity of their 3d calculation makes it rather more believable a priori.

6.7.3 The superfluid transition in \( ^4\text{He} \)

In this case the \( n = 2 \) order parameter is not conserved by the dynamics, and it is coupled to a conserved scalar field, denoted \( m \), which plays the role of the entropy density (see Hohenberg and Halperin, 1977). Above \( T_c \), we have
\[
\omega_m(q) = (\lambda / C_p)q^2,
\]
(6.129)
in terms of the transport coefficient \( \lambda_m \equiv \lambda \), which is the thermal conductivity, and the static susceptibility \( \chi_m \) which is the specific heat \( C_p \). Below \( T_c \), the spectrum of \( m \) has a propagating mode, known as second sound:
\[
\omega_2 = \pm c_2 q + \frac{i}{2} D_2 q^2,
\]
(6.130)
where

$$c_2^2 = g_0^2 \rho_s / C_p, \quad (6.131)$$

$\rho_s$ is the stiffness constant, $g_0$ is a measurable coupling constant which sets the frequency scale and $D_2$ is the damping constant of the second sound.

The dynamical scaling assumption for the superfluid leads to the scaling law (Halperin and Hohenberg, 1967, 1969; Ferrell et al. 1967, 1968)

$$z_m = z = \frac{d}{2} + \frac{\tilde{\alpha}}{2v}, \quad (6.132)$$

where $\tilde{\alpha} \equiv \max(\alpha, 0)$. To find the corresponding universal amplitude ratios, let us define a characteristic frequency scale

$$\omega_C = g_0 / (C_p \varepsilon)^{1/2}, \quad (6.133)$$

in terms of which we write

$$R_\lambda \equiv \lim_{q \to 0} \left[ \frac{\omega_m(q)}{(q \xi)^2 \omega_C} \right] = \frac{\lambda \varepsilon^{1/2}}{g_0 C_p^{1/2}}, \quad (6.134)$$

We also define

$$R_2 \equiv \lim_{q \to 0} \left[ \frac{\text{Im}[\omega_2(q)]}{\text{Re}[\omega_2(q)](q \xi_T)} \right] = \frac{D_2}{2c_2 \xi_T}, \quad (6.135)$$

where $\xi_T$ is given by (2.17).

In addition to the amplitude ratios (6.134) and (6.135), the full dynamical correlation function $\tilde{G}_m(q, \omega)$ can be measured by light-scattering techniques, and the shape of the frequency spectrum $\tilde{W}_m(\tilde{x} = \pm \infty, \tilde{y}, \tilde{w})$ calculated as a function of $q, t$ and $\omega$, both above and below $T_c$ (see Hohenberg and Halperin (1977) and references therein). From the characteristic frequency $\tilde{\xi}_m^\pm(y)$, one can extract, besides the universal ratios $R_\lambda$ and $R_2$ defined above, also the ratio $\tilde{R}_m$ introduced in (3.75). Specifically, we write, for $t < 0$,

$$\tilde{R}_m = \lim_{q \to 0, t \to 0} \left[ \frac{\omega_m(q, t = 0)}{\omega_m(q \to 0, -t)} \right](q \xi_T)^{1-z}, \quad (6.136)$$

since $p = 1$, i.e. $\omega_m = \omega_2 \propto q$ for $t < 0$.

The amplitude ratios were calculated in the $\varepsilon$-expansion by Halperin et al. (1976a,b) (an error in the second order was corrected by Dohm (1978); see also De Dominicis and Peliti (1978) and Erratum (1980) of Halperin et al. (1976b)). The values are

$$R_\lambda = (K_d / \varepsilon)^{1/2} \left[ 1 + 0.47\varepsilon + O(\varepsilon^2) \right] \simeq 0.33, \quad (6.137)$$

$$R_2 - \varepsilon = (K_d / \varepsilon) \left[ 1 + 0.311\varepsilon + O(\varepsilon^2) \right] \simeq 0.07, \quad (6.138)$$

$$\tilde{R}_m = (K_d / \varepsilon)^{1/2} \left[ 1 + 1.4\varepsilon + O(\varepsilon^2) \right] \simeq 0.54. \quad (6.139)$$
(Equation (6.137) replaces (6.30) and the corresponding entry in Table 2 of Hohenberg and Halperin (1977).) An alternative evaluation using a mode-coupling formalism in 3d (Hohenberg et al., 1976b) yields

\[
R_\lambda = 0.19, \quad (6.140)
\]
\[
R_2 = 0.09, \quad (6.141)
\]
\[
\tilde{R}_m^- = 0.42. \quad (6.142)
\]

It was pointed out by De Dominicis and Peliti (1978) that the dynamical scaling fixed point of the superfluid is nearly unstable and close to a crossover to a different fixed point. This means that the \(\varepsilon\)-expansion will yield poor estimates of the asymptotic critical behaviour. Subsequently, Dohm and Folk (1982a) studied the values of the ratios (6.137)–(6.139), taking the near instability of the scaling fixed point into account approximately. It was pointed out by a number of authors, however, that the asymptotic values would be unobservable, since near the crossover there are large corrections to scaling, and effective exponents must be considered (De Dominicis and Peliti, 1978; Ferrell and Bhattacharjee, 1979a; Hohenberg et al., 1980; Dohm and Folk, 1981). These corrections lead to a complicated temperature dependence of the dynamical amplitude ratios in the experimental range. Another important effect, which was found by Ahlers et al. (1981), is the anomalously small value of the bare dynamical coupling constant in \(^4\)He. This non-universal effect leads to a severe reduction of the dynamical critical region and a different crossover, to a regime without dynamical fluctuations for \(t \gtrsim 10^{-3}\) (see also Ferrell and Bhattacharjee, 1979b). An important consequence is that light-scattering measurements of \(\hat{G}_m(q, \omega)\), which are carried out for \(q \xi_0 \gtrsim 2.5 \times 10^{-3}\), can never probe the critical behaviour (Hohenberg and Sarkar, 1981). Despite these difficulties, it has proved possible to include the various universal and non-universal corrections into the dynamical formalism, and to evaluate the scaling function \(\hat{W}_m\) and the ratios \(R_2, R_\lambda, \tilde{R}_m^-\) with all appropriate corrections, in terms of one or two experimentally determined parameters (Ferrell and Bhattacharjee, 1979c; Ahlers et al., 1981; Dohm and Folk, 1981; Dohm, 1987). We will comment on the comparison with experiment in Section 8. Finally, we mention that results for the superfluid \(^3\)He–\(^4\)He mixtures were reported by Dohm and Folk (1983), building on the RG formulation of Siggia and Nelson (1977).

### 6.7.4 Magnetic transitions

The formalism and models for ferromagnets and antiferromagnets are similar to those describing superfluid helium (see Hohenberg and Halperin, 1977). In the magnetic case the characteristic frequency \(\omega_\phi(q)\) for the order
parameter is also of experimental interest, so one considers the scaling functions $\tilde{W}$ and $\tilde{\mathcal{O}}^\pm$ for the order parameter $\psi$.

In the antiferromagnet the auxiliary field $m$ is the total magnetization, and $\psi$ is the staggered magnetization. The dynamical scaling law is (Halperin and Hohenberg, 1967)

$$ z = z_m = d/2, \quad (6.143) $$

which leads to the amplitude ratios $R_{\tilde{\lambda}}$, as in (6.134), and

$$ R_{\tilde{\chi}} = \lim_{q \to 0} \left[ \frac{\omega_{\chi}(q)(q^2)}{\omega_C(q)} \right] = \frac{\Gamma_{\chi_m}^{1/2} \xi^{d/2}}{g_0 \chi_{\psi}}, \quad (6.144) $$

where, in analogy with (6.133), we have defined $\omega_C(q) = g_0/(\chi_m \xi^d)^{1/2}$. The $\varepsilon$-expansion yields the expressions

$$ R_{\tilde{\lambda}} = K_{d}^{1/2}(3\varepsilon)^{-1/2} \left[ 1 + 0.229 \varepsilon + O(\varepsilon^2) \right] \simeq 0.16 - 0.17, \quad (6.145) $$

$$ R_{\tilde{\chi}} = K_{d}^{1/2}(3\varepsilon)^{1/2} \left[ 1 - 0.562 \varepsilon + O(\varepsilon^2) \right] \simeq 0.17 - 0.25, \quad (6.146) $$

(see Erratum (1980) by Halperin et al. (1976b)). The present expressions supersede those in Table 2 of Hohenberg and Halperin (1977). Freedman and Mazenko (1975) have also evaluated the scaling function $\tilde{W}$ for $\psi$ to lowest order in $\varepsilon$. Calculations of amplitude ratios and scaling functions have been carried out using mode coupling in 3d as well (Joukoff-Piette and Résibois, 1973; Huber and Krueger, 1970; Wegner, 1969). The result

$$ R_{\tilde{\chi}} / R_{\tilde{\lambda}} \approx 3 \quad (6.147) $$

found from those calculations agrees with the lowest-order $\varepsilon$-expansion, but not with an extrapolation including the next term. Note that for the antiferromagnet there is no evidence for any instability of the dynamical-scaling fixed point.

In the ferromagnet there is no auxiliary field coupled to the order parameter, but dynamical scaling can be applied to $\psi$ itself, to give the relation (Halperin and Hohenberg, 1967)

$$ z = \frac{1}{2}(d + 2 - \eta). \quad (6.148) $$

The associated amplitude ratio is

$$ R_{\tilde{\chi}} = \lim_{q \to 0} \left[ \frac{\omega_{\chi}(q, t)}{\omega_C(q, t)} \right] = \frac{\lambda_{\varepsilon}^{\varepsilon(d-4)/4}}{g_0 \chi_{\psi^{1/2}}^{1/2}}, \quad (6.149) $$

where $\omega_C(q) = (g_0 \varepsilon^{(4-d)/2} / \chi_{\psi^{1/2}})q^2$ is a characteristic frequency which is universally related to the spin-wave frequency $\omega_{\tilde{\psi}}(q) = (g_0 \rho_s / M)q^2$ (see Hohenberg and Halperin, 1977).

The $\varepsilon$-expansion in this case is in terms of $\varepsilon \equiv 6 - d$ (Ma and Mazenko, 1975), and the universal scaling function $\tilde{W}(\tilde{g} = \infty, \tilde{w})$ for the order parameter
1 Universal critical-point amplitude relations

at \( T_c \) has been obtained by a number of authors. The results of Dohm (1976) and Bhattacharjee and Ferrell (1981) are in substantial agreement with each other, and disagree with those of Nolan and Mazenko (1977). The same scaling function was calculated earlier in mode coupling by Wegner (1968), who agrees with the first two papers. It is thus likely that Nolan and Mazenko (1977) made a computational error. More accurate evaluations of the scaling functions were recently carried out by Frey and Schwabl (1988) using mode coupling.

Bhattacharjee and Ferrell (1981) also calculated a universal ratio analogous to \( R_\lambda \) in (6.149), but for \( q\xi = \infty \), and they find

\[
\tilde{R}_\lambda \equiv R_\lambda \tilde{R} = \left( \frac{\omega_p(q, t = 0)}{\omega_c(q)} \right)(q\xi)^{1-d/2} = \left( \frac{2}{3\varepsilon} \right)^{1/2} \left[ 1 + 0.07\varepsilon \right] \simeq 0.87.
\] (6.150)

Finally, we cite the analysis by Frey et al. (1988) of the dynamical crossover from short-range to dipolar ferromagnet (see also Frey and Schwabl, 1988).

6.7.5 Other results

Dengler and Schwabl (1987) considered the universal amplitude ratio of the \( t > 0 \) and \( t < 0 \) coefficients of the critical sound wave attenuation in the zero-frequency limit, which they denote \( \alpha_+ / \alpha_- \). Their \( \varepsilon \)-expansion results are

\[
\frac{\alpha_+}{\alpha_-} \simeq \left( \frac{A}{A'} \right)^{-1/2} 2^{\alpha + vz}(1 + 0.118\varepsilon) \simeq 5 \quad \text{(liquid–gas)}, \quad (6.151)
\]

\[
\frac{\alpha_+}{\alpha_-} \simeq \left( \frac{A}{A'} \right)^{-2} 2^{\alpha + vz}(1 + 0.118\varepsilon) \simeq 15 \quad \text{(binary mixtures)}, \quad (6.152)
\]

\[
\frac{\alpha_+}{\alpha_-} \simeq \frac{\varepsilon}{72} 2^{\alpha + vz} \left( 1 + \frac{29}{27} \varepsilon \right) \simeq 0.072 \quad \text{(uniaxial magnets)}, \quad (6.153)
\]

where the numerical values are for 3d. These authors also have results for several other universality classes, as well as for the scaling functions

\[
\hat{g}_\pm (\omega / \omega_p) = \alpha_\pm (\omega, t) / \text{const} \omega^2 |t|^p.
\] (6.154)

The critical properties of the (first) sound mode at the superfluid transition of \(^4\text{He}\) were calculated by Ferrell and Bhattacharjee (1980) and more systematically by Pankert and Dohm (1986, 1989a,b), who evaluated the universal scaling functions analogous to (6.154). Dynamical properties of multicritical points were reviewed by Dohm (1984).
7 Experimental results: statics

7.1 General comments

Since the scaling and RG theories make predictions concerning universal amplitude ratios, it is natural to attempt to test these predictions against experiments. Generally speaking there are at least three different ways in which measurements of universal amplitudes contribute to our understanding of critical phenomena.

First, one can attempt to estimate the amplitudes for a wide range of substances and examine trends and regularities. This was done in the mid 1970s (Stauffer et al., 1972; Aharony and Hohenberg, 1976; Hohenberg et al., 1976a) with generally good agreement between experiment and theory, within the rather large uncertainties of both (±50–100%). Note that since amplitude ratios, unlike critical exponents, can easily differ from their mean-field values by factors of more than 2 or 3, agreement between experiment and theory, even within 50%, may be significant. Moreover, the approximate constancy of amplitude ratios between different members of a universality class is also significant, since the amplitudes themselves can differ by a large amount. For example the dimensionless specific heat amplitude $A'$ at the liquid–gas critical point is smaller in $^4$He than in CO$_2$ by a factor 0.22, in violation of the law of corresponding states, whereas the ratios $A/A'$ agree in the two substances to within 10% (Moldover, 1982).

A second level of study of universal amplitudes attempts to test the theory in a quantitative way, with an accuracy of a few per cent. Such tests are quite rare, however, because of the difficulties of determining amplitudes reliably, in both theory and experiment. For example, there is such a high correlation between exponents and amplitudes that the comparison is only meaningful if both experiment and theory are analysed with the same exponent. Of course, if one imposes a theoretical exponent in the fit to experiment, it is important to check that such a constraint is statistically allowed by the data. Similarly, correction-to-scaling amplitudes are very sensitive to the precise form of the correction terms assumed, i.e. to the inclusion of regular corrections and higher-order terms proportional to $t^{2\nu}$, $t^{12}$, etc. In spite of these limitations a number of significant quantitative comparisons have been carried out, and they will be described below. It must always be remembered, however, that from a quantitative point of view experiment is most useful for checking consistency or inconsistency with a particular theory, not for measuring universal exponents and amplitudes in an absolute sense.

Finally, information concerning amplitude ratios can be useful in a qualitative sense, when dealing with a transition belonging to an unknown
universality class. Since universal amplitude combinations vary more widely than the corresponding exponents, it is sometimes possible to identify or to eliminate one or another universality class, on the basis of the value of an amplitude ratio.

7.2 Liquid–gas and binary fluid critical points

The most comprehensive data available on the equation of state and static correlations are for this class of systems, since properties of many different fluids have been measured, and the full equation of state is accessible. Measurements carried out between 1965 and 1975 were generally analysed in terms of a single exponent along each path, in a rather wide range, typically $5 \times 10^{-5} < |t| < 10^{-2}$ (Sengers and Levelt-Sengers, 1978). Typical values of the thermodynamic exponents and amplitude combinations in pure fluids are given in the first row in Table 7.1. These numbers represent averages over many substances and experiments of differing accuracy, and the quoted errors are a rough estimate of the spread in values among different substances. The overall picture was encouraging, at the time, but neither the consistency between different substances nor the agreement between experiment and theory were better than semiquantitative.

In the more modern analyses, the exponents are often fixed at their RG values, and the corresponding amplitudes then extracted from the data. Results for a number of fluids are displayed in Table 7.1, and show definite improvement in consistency and agreement with theory. Note that the specific heat of Xe, analysed by Güttinger and Cannell (1981), did not allow for the constant $C_B$ (see (2.21), (2.24)), and leads to values of $A/A'$ and $R_C$ which suggest that the data do not correspond to the asymptotic critical behaviour. In general, the amplitude combinations are almost always obtained by combining data from different experiments, analysed over different domains of temperature and pressure. This may account for the residual spread in values (see Table 7.1).

Turning to binary fluid critical points, a comprehensive survey of early results, for 10 mixtures, was published by Beysens (1982) who concluded that the exponents and amplitude ratios were in rather good agreement with theory. This agreement is probably due to the smaller value of the non-universal correction-to-scaling amplitudes, as compared to pure fluids. Specifically, the ratios $R^+_{\xi}$ and $R^-_{\xi} R_C^{-1/3} = \xi_0 (B^{-2} \Gamma)^{-1/3}$ were determined, and the overall situation seemed promising, though the accuracy of the amplitude determination was sometimes rather low.

In Tables 7.2 and 7.3, we summarize results for several binary fluids, including recent experiments and data reanalyses and also some earlier
Table 7.1 Experimental results on universal amplitude combinations for simple fluids.

<table>
<thead>
<tr>
<th>Substance</th>
<th>$A/A'$</th>
<th>$\Gamma / \Gamma'$</th>
<th>$R_x$</th>
<th>$R_C$</th>
<th>$R_{\xi}R_C^{-1/3}$</th>
<th>$R_{\xi}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Equation of state</td>
<td>0.54 ± 0.10</td>
<td>4.4 ± 0.3</td>
<td>1.6 ± 0.2</td>
<td>0.049 ± 0.012</td>
<td>0.25 ± 0.02</td>
<td></td>
</tr>
<tr>
<td>Xe</td>
<td>0.63$^b$</td>
<td>4.5$^c$</td>
<td>1.57$^b$</td>
<td>0.062$^b$</td>
<td>0.68$^b$, 0.69 ± 0.04$^d$</td>
<td>0.27$^b$</td>
</tr>
<tr>
<td>CO$_2$</td>
<td>0.50$^e$, 0.53$^f$</td>
<td>4.96$^e$</td>
<td>1.72$^e$</td>
<td>0.052$^c$</td>
<td>0.69 ± 0.06$^g$</td>
<td></td>
</tr>
<tr>
<td>H$_2$O</td>
<td>0.532$^h$</td>
<td>4.886$^h$</td>
<td>1.693$^h$</td>
<td>0.0568$^h$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C$_2$H$_4$</td>
<td>0.50$^i$</td>
<td>5.31$^i$</td>
<td>1.76$^i$</td>
<td>0.051$^i$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^4$He</td>
<td>0.48–0.50$^f$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N$_2$</td>
<td></td>
<td>4.8 ± 0.6$^j$</td>
<td>1.71 ± 0.5$^j$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ne</td>
<td></td>
<td>4.8 ± 0.8$^j$</td>
<td>2.05 ± 0.8$^j$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF$_6$</td>
<td></td>
<td>4.9$^e$</td>
<td>1.66$^c$</td>
<td>0.68 ± 0.04$^g$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF$_6$ + 0.1 % N$_2$</td>
<td></td>
<td>4.97$^c$</td>
<td>1.73$^c$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>NH$_3$</td>
<td></td>
<td></td>
<td></td>
<td>0.71 ± 0.05$^k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>C$_2$H$_6$</td>
<td></td>
<td></td>
<td></td>
<td>0.72 ± 0.05$^k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>CF$_3$Cl</td>
<td></td>
<td></td>
<td></td>
<td>0.75 ± 0.05$^k$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^3$He</td>
<td></td>
<td></td>
<td></td>
<td>0.047 ± 0.010$^l$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>H$_2$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.27 ± 0.01</td>
<td></td>
</tr>
<tr>
<td>CHF$_3$</td>
<td>0.53 ± 0.02</td>
<td>4.9 ± 0.2</td>
<td>1.7 ± 0.1</td>
<td>0.057–0.066</td>
<td>0.67–0.70</td>
<td></td>
</tr>
</tbody>
</table>

$^a$ These empirical equation-of-state estimates were obtained with effective exponent values $\alpha = 0.10$, $\beta = 0.35$, $\gamma = 1.20$, $\delta = 4.43$, $\nu = 0.63$, with no allowance for singular corrections (see Aharony and Hohenberg, 1976; Hohenberg et al., 1976a). All other experimental estimates given for specific substances assume RG exponent values, typically, $\alpha = 0.110$, $\beta = 0.325$, $\gamma = 1.241$, $\delta = 4.818$, $\theta \approx 0.5$, etc. $^b$Güttinger and Cannel (1981). $^c$Hocken and Moldover (1976). $^d$Sengers (1982). $^e$Albright et al. (1987). $^f$Moldover (1982). $^g$Sengers and Moldover (1978). $^h$Levelt-Sengers et al. (1983). $^i$Levelt-Sengers and Sengers (1981). $^j$Pestak and Chan (1984). $^k$Tufeu et al. (1982). $^l$Pittman et al. (1979). $^m$De Bruyn and Balzarini (1989). $^n$Närger and Balzarini (1989). $^o$These are rather wide representative ranges (see Table 6.1).
Table 7.2 Amplitude combinations $R_\xi^+$ and $R_\xi^+ R_C^{-1/3}$ for binary fluids: experimental results.

<table>
<thead>
<tr>
<th>System</th>
<th>$R_\xi^+$</th>
<th>$R_\xi^+ R_C^{-1/3}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T–W: triethylamine–H$_2$O</td>
<td>0.267 ± 0.036$^a$</td>
<td>b</td>
</tr>
<tr>
<td>T–D: triethylamine–D$_2$O</td>
<td>0.27 ± 0.02$^c$</td>
<td></td>
</tr>
<tr>
<td>I–W: isobutyric acid–H$_2$O</td>
<td>0.258 ± 0.009$^a$</td>
<td>0.66 ± 0.03$^d$</td>
</tr>
<tr>
<td>M–N: 3-methylpentane–nitroethane</td>
<td>0.274 ± 0.004$^a$</td>
<td>b</td>
</tr>
<tr>
<td>N–H: nitrobenzene–n-hexane</td>
<td>0.27 ± 0.03$^e$</td>
<td>0.73 ± 0.05$^e$</td>
</tr>
<tr>
<td>N–I: nitroethene–isoctane</td>
<td>0.255$^a$</td>
<td>b</td>
</tr>
<tr>
<td>P–C: perfluoromethylcyclohexane–carbon tetrachloride</td>
<td>0.267 ± 0.027$^a$</td>
<td>b</td>
</tr>
<tr>
<td>A–C: aniline–cyclohexane</td>
<td>0.26 ± 0.03$^b$</td>
<td></td>
</tr>
<tr>
<td>M–C: methanol–cyclohexane</td>
<td>0.269$^a$</td>
<td>0.71 ± 0.02$^f$</td>
</tr>
<tr>
<td>Methanol–deuterated cyclohexane</td>
<td>0.68 ± 0.06$^g$</td>
<td></td>
</tr>
<tr>
<td>M–C with 11.6% deuterated cyclohexane</td>
<td>0.75 ± 0.09$^g$</td>
<td></td>
</tr>
<tr>
<td>Methanol–n-heptane</td>
<td>0.245 ± 0.034$^a$</td>
<td></td>
</tr>
<tr>
<td>2,6-Dimethyl pyridine–H$_2$O</td>
<td>0.305 ± 0.024$^a$</td>
<td>0.67 ± 0.10$^h$</td>
</tr>
<tr>
<td>Isobutyric acid–D$_2$O</td>
<td>0.284 ± 0.022$^i$</td>
<td></td>
</tr>
<tr>
<td>Theory (representative ranges)</td>
<td>0.27 ± 0.01</td>
<td>0.67–0.70</td>
</tr>
</tbody>
</table>

$^a$ Fast and Yun (1986); for earlier estimates of $R_\xi^+$ see also Klein and Woermann (1978) and Beyens and Bourgou (1979). $^b$ Beyens et al. (1982); they also estimated $R_\xi^+ R_C^{-1/3}$ for several substances (marked $^b$) but with large uncertainties; the values cover the range 0.4–0.9. $^c$ Bloemen et al. (1980). $^d$ Andrew et al. (1986). $^e$ Zalczer et al. (1983). $^f$ Jacobs (1986). $^g$ Houessou et al. (1985). $^h$ Jungk et al. (1987). $^i$ Gansen and Woermann (1984).

estimates. Agreement with the 3d Ising theoretical values is quite satisfactory. However, the spread of experimental values among the different substances suggests the need for further work. The difficulty can be traced to the spread in individual amplitude values when measured in several experiments for a given system and may be largely attributable to uncertainties in allowing for various correction-to-scaling terms. These differences in amplitude values apparently cancel out in part, when universal combinations are formed.

The above-mentioned difficulties with allowing for corrections to scaling are illustrated by the correction ratio data presented in Table 7.4 (for both simple and binary fluids). The available experimental values (and, in fact, some theoretical estimates) can be at best considered semiquantitative. For binary mixtures, the smaller absolute value of the correction terms means that the universal ratios are more difficult to measure. Aharony and Ahlers (1980) have argued that if experimental results from the temperature range...
Table 7.3 Experimental values of amplitude ratios $A/A'$, $R_C$, $\Gamma/\Gamma''$, $\xi_0/\xi'_0$, $R_\chi$, $Q_2$ and $Q_3$ for binary fluids.

<table>
<thead>
<tr>
<th></th>
<th>$A/A'$</th>
<th>$\Gamma/\Gamma''$</th>
<th>$\xi_0/\xi'_0$</th>
<th>$R_C$</th>
<th>$R_\chi$</th>
<th>$Q_2$ or $Q_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>T–W</td>
<td>0.63 ± 0.25$^a$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T–D</td>
<td>0.57 ± 0.01$^{a,b}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>I–W</td>
<td></td>
<td>4.9 ± 2.0$^c$</td>
<td>2.0 ± 0.4$^c$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M–N</td>
<td>0.56 ± 0.02$^d$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>N–H</td>
<td>4.3 ± 0.3$^f$</td>
<td>1.9 ± 0.2$^f$</td>
<td>0.050 ± 0.015$^f$</td>
<td>1.75 ± 0.30$^f$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>N–I</td>
<td>0.56 ± 0.09$^a$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>P–C</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A–C</td>
<td>Theory$^g$</td>
<td>4.9 ± 0.2</td>
<td>1.91–1.98</td>
<td>0.057–0.066</td>
<td>1.7 ± 0.1</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Q_2 \simeq 1.1–1.25$</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>$Q_3 \simeq 0.9–0.92$</td>
</tr>
</tbody>
</table>

Note that chemical specifications of substances (T–W, etc.) are given in Table 7.2.

$^a$Beysens et al. (1982); they also estimated $R_C$ for several substances (marked $^a$), but with large uncertainties. The values cover the range 0.04–0.10.

$^b$Bloemen et al. (1980).

$^c$Hamano et al. (1986).

$^d$Sanchez et al. (1983).

$^e$Chang et al. (1979).

$^f$Zalczer et al. (1983).

$^g$Representative ranges.
Table 7.4 Correction-to-scaling amplitude ratios for fluids.

<table>
<thead>
<tr>
<th>Substance</th>
<th>$a_M/a_X$</th>
<th>$(a_M/a_X)_{AA}$</th>
<th>$a_C/a_X$</th>
<th>$a_C/a_M$</th>
<th>$a_C/a_C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO₂</td>
<td>0.58</td>
<td>0.59</td>
<td></td>
<td>0.99–1.40</td>
<td></td>
</tr>
<tr>
<td>$^4$He</td>
<td>0.95 ± 0.13</td>
<td>0.59</td>
<td></td>
<td>0.69–0.78</td>
<td></td>
</tr>
<tr>
<td>N₂</td>
<td>0.9 ± 0.2</td>
<td>0.56</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Ne</td>
<td>∼1.4, ∼1.5</td>
<td>0.86</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Xe</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>H₂O</td>
<td>1.60</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>O₂</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$^3$He</td>
<td>0.41 ± 0.20</td>
<td>0.46, 0.50</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>SF₆</td>
<td>∼0.7</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>T–W</td>
<td>0.47 ± 0.07</td>
<td>7.9 ± 1.2</td>
<td>16.8 ± 0.09</td>
<td></td>
<td></td>
</tr>
<tr>
<td>M–N</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theory</td>
<td>0.85–0.92</td>
<td>0.85–0.92</td>
<td>8.6 ± 0.2</td>
<td>9–10.5</td>
<td>∼1</td>
</tr>
<tr>
<td></td>
<td></td>
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<td></td>
</tr>
</tbody>
</table>

The values $(a_M/a_X)_{AA}$ are effective estimates by Aharony and Ahlers (1980) (see text). Note that chemical specifications of substances T–W and M–N are given in Table 7.2.


$|t| \leq t_i$, for a quantity $\Phi_i$ (we use the notation of (2.60)) are fitted to a pure power law without allowing for corrections to scaling, then the effective exponent value will be shifted according to

$$\phi_i^{(eff)} - \phi_i = -a_i \theta t_i^\theta.$$  \hspace{10cm} (7.1)

By using some “old” exponent values with comparable $t_i$ one can therefore estimate

$$a_i/a_j \simeq (\phi_i^{(eff)} - \phi_i)/(\phi_j^{(eff)} - \phi_j).$$  \hspace{10cm} (7.2)

The numerical values for fluids are shown in the column $(a_M/a_X)_{AA}$ in Table 7.4.

An interesting determination of the scaling function $\tilde{h}(\tilde{x})$ in the equation of state of binary mixtures (see (3.6)) has been proposed by Calmettes (1979, 1982). The method relies on measurements of the viscosity, the variation of which as a function of concentration and temperature is described by a scaling function obtainable from $\tilde{h}(\tilde{x})$ (the dynamical critical exponent $\tilde{x}$ enters, but otherwise the quantities are purely static). Calmettes (1979, 1982) analysed data for several fluids, and found good agreement with the scaling function for $n = 1$ obtained from the second-order $\varepsilon$-expansion (Brézin et al., 1974).

A careful measurement of the static correlation function $\hat{G}(q, t)$ on the critical isochore in the binary fluid M–N (see Table 7.2) has been carried...
out by Chang et al. (1979). The amplitude ratio $Q_3$ was thus determined and, in fact, the full universal scaling function $\tilde{Z}(\tilde{x} = 0, \tilde{y})$ (see (3.40)) was shown to agree well with the corresponding Ising results.

Experimental values of the amplitude combinations $R_{\sigma \xi}$ and $R_{\sigma A}$ involving the surface tension amplitude $\sigma_0$ (see (2.56)--(2.59)) have been surveyed by Gielen et al. (1984), Moldover (1985) and Chaar et al. (1986). A comprehensive list of experimental values of $R_{\sigma \xi}$ and $\chi^{-2/3}R_{\sigma A}^{-1}$ is given by Chaar et al. (1986) (their Table 1 (columns $U_1^+$ and $Y^+$, respectively)). Representative ranges suggested by these data are

$$R_{\sigma \xi} = 0.38 \pm 0.03 \quad \text{and} \quad R_{\sigma A} = 0.76 \pm 0.025. \quad (7.3)$$

As mentioned in Section 6, only the most recent numerical MC estimates (Mon, 1988), listed in Table 6.1, agree with these experimental ranges.

Experimental results for several polymer solutions have been surveyed by Nose et al. (1984) and Shinozaki et al. (1982). Although the power law estimates (of the $N$ dependence) are somewhat scattered at the present time, and more systems must be studied, the following are typical ranges:

$$\xi_0 \propto N^{0.28 \pm 0.03}, \quad (7.4)$$

$$\Gamma \propto N^{0.48 \pm 0.03}, \quad (7.5)$$

$$B \propto N^{-(0.34 \pm 0.11)}, \quad (7.6)$$

$$\sigma_0 \propto N^{-(0.44 \pm 0.03)}. \quad (7.7)$$

The inconsistency with the theoretical predictions (5.6), (5.7) and (5.9), namely $\xi_0 \sim N^{0.185}, \sigma_0 \sim N^{-0.370}, \Gamma/B^2 \sim N^{0.555}$, should not be taken too seriously since the experimental data as quoted above also give $R_{\sigma \xi} \propto N^{-(0.12 \pm 0.06)}, R_{\sigma A}^{1/4} \propto N^{-(0.11 \pm 0.11)}$, which are inconsistent or barely consistent with $N^0$. Thus, better experimental data are needed. Muthukumar (1986) proposed a phenomenological mean-field relation $B \propto N^{-2/9}$, which is not inconsistent with (7.6). (For some further discussion of the $N$ dependence in connection with amplitude universality see Kholodenko and Qian (1989).)

Finally, we cite some intriguing and puzzling results of Pitchford et al. (1985) on the smectic $A$–hexatic $B$ transition in a series of liquid crystal compounds, which showed large specific heat exponents ($\chi \approx 0.5–0.67$) accompanied by $A’/A$ values in the range 0.7–1.0, in complete contradiction to the usual situation for continuous transitions.

### 7.3 Superfluid transition of $^4$He

The superfluid transition in $^4$He provides an opportunity to test the modern theory of critical phenomena at a quantitative level. This is because accurate
experiments are possible at various pressures along the $\lambda$-line, thus permitting a direct test of universality (see Ahlers, 1978, 1980, 1982). The ratios which are obtained are $R^T_\xi$ (defined in (3.54)) and $A/A'$, from measurements of the superfluid density $\rho_s$ and the specific heat or thermal expansion, as well as a number of ratios of correction amplitudes. There are two types of tests of the theory: first one can see if the ratios are universal as a function of pressure along the $\lambda$-line, where non-universal quantities vary by 15–30\%, typically. This test, which involves experiment alone, should be possible with an accuracy of a few per cent. Alternatively, one can examine how closely the measured values agree with theory.

The constancy of $A/A'$ was first tested by analysis of $C_p$ values obtained from measurements of $C_v$ and other thermodynamic functions to correct for $C_p - C_v$ (Ahlers, 1973). A variation of 10\% was found in $A/A'$ above a pressure of 15 bar. Independent measurements of the thermal expansion coefficient $\beta_p$ were carried out by Mueller et al. (1976), and yielded a universal $A/A'$. Later specific heat experiments (Takada and Watanabe, 1982) resolved the difficulty by re-evaluating the $C_p - C_v$ corrections at high pressure and confirmed the universality of $A/A'$. At the superfluid transition in $^4$He–$^3$He mixtures for concentrations of up to about 50\% of $^3$He (mole fraction) the universality of $A/A'$ was verified by Gasparini and Gaeta (1978) and by Takada and Watanabe (1980).

In early analyses of experimental data a more serious problem was found with the ratio $R^T_\xi$, which involves measurements of $\rho_s$ and $C_p$ below $T_c$ (Ahlers, 1978). Although the data were consistent with the RG exponent values, the $R^T_\xi$ values varied with pressure by about 15\% while the estimated errors were only 3\% in this case. However, a later careful reanalysis by Singsaas

<table>
<thead>
<tr>
<th></th>
<th>$A/A'$</th>
<th>$R^T_\xi$</th>
<th>$a'<em>c/a</em>{\rho_s}$</th>
<th>$a_c/a'_c$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$^4$He</td>
<td>1.067  $\pm$ 0.003\textsuperscript{a}</td>
<td>0.85 $\pm$ 0.02\textsuperscript{a}</td>
<td>$-0.068 \pm 0.03$\textsuperscript{a}</td>
<td>1.03 $\pm$ 0.3\textsuperscript{a}</td>
</tr>
<tr>
<td></td>
<td>1.088  $\pm$ 0.007\textsuperscript{b}</td>
<td></td>
<td></td>
<td>0.85 $\pm$ 0.2\textsuperscript{b}</td>
</tr>
<tr>
<td>$^4$He + $^3$He</td>
<td>1.09 $\pm$ 0.02\textsuperscript{c}</td>
<td></td>
<td></td>
<td>$\sim 1.088$\textsuperscript{d}</td>
</tr>
<tr>
<td></td>
<td>$\sim 1.088$\textsuperscript{d}</td>
<td></td>
<td></td>
<td>$\sim 1.1$\textsuperscript{d}</td>
</tr>
<tr>
<td>Theory\textsuperscript{e}</td>
<td>1.05</td>
<td>0.78</td>
<td>$-0.045$</td>
<td>1.6</td>
</tr>
</tbody>
</table>

\textsuperscript{a}Singsaas and Ahlers (1984). These values supersede earlier estimates based on results of Mueller et al. (1976) and Greywall and Ahlers (1973). The estimates shown here are stable for all pressures ($p < 30$ bar). \textsuperscript{b}Takada and Watanabe (1982). \textsuperscript{c}Takada and Watanabe (1980). \textsuperscript{d}Gasparini and Gaeta (1978). \textsuperscript{e}Theoretical values (Tables 6.3 and 6.4) are from the 3d field theory.
and Ahlers (1984), allowing for correction terms and incorporating newer experimental data, resolved this apparent breakdown of hyperuniversality.

In Table 7.5, we summarize the latest results by various groups for the best studied amplitude combinations (see Singsaas and Ahlers (1984) for some other ratios). The consistency with the theoretical values is encouraging.

7.4 Magnetic and some other transitions

7.4.1 Isotropic three-dimensional systems \((n = 3)\)

In Table 7.6 we list experimental results for three extensively studied isotropic \((n = 3, 3d)\) magnetic materials, Ni, EuO and RbMnF₃, and we also list some \(A/A'\) estimates for the most familiar ferromagnet, iron. A comprehensive listing of several exponent and universal amplitude combinations (experimental values) at magnetic transitions has been compiled by Stierstadt et al. (1984, 1989). At a glance, only the most studied ratio, \(A/A'\) (see Table 7.6), shows consistency with the theoretical predictions, provided correction terms are taken into account in the analysis (Ahlers and Kornblit, 1975). For other universal combinations, experiment and theory agree only qualitatively.

**Table 7.6 Experimental results on universal amplitude combinations for isotropic \((n = 3)\) magnetic systems in 3d.**

<table>
<thead>
<tr>
<th></th>
<th>Ni</th>
<th>EuO</th>
<th>RbMnF₃</th>
<th>Fe</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A/A')</td>
<td>(F)</td>
<td>(F) (b)</td>
<td>(A)</td>
<td>(F) (b)</td>
<td>1.58</td>
</tr>
<tr>
<td></td>
<td>(1.33 \pm 0.08^{b})</td>
<td>(1.5 \pm 0.2^{f})</td>
<td>(1.36 \pm 0.02^{e})</td>
<td>(1.036 \pm 0.01^{g})</td>
<td>0.42–0.44</td>
</tr>
<tr>
<td></td>
<td>(1.65^{c})</td>
<td>(1.22 \pm 0.06^{f})</td>
<td>(1.48 \pm 0.09^{g})</td>
<td>(1.46 \pm 0.01^{e})</td>
<td>0.73</td>
</tr>
<tr>
<td></td>
<td>(1.15 \pm 0.02^{d})</td>
<td>(1.52 \pm 0.02^{e})</td>
<td>(1.28 \pm 0.02^{e})</td>
<td>(1.41 \pm 0.05^{e})</td>
<td>(~0.17)</td>
</tr>
<tr>
<td></td>
<td>(1.49 \pm 0.02^{c})</td>
<td>(1.22 \pm 0.03^{e})</td>
<td>(1.15 \pm 0.02^{g})</td>
<td>(~1.1–1.4)</td>
<td>(~1.1–1.4)</td>
</tr>
<tr>
<td></td>
<td>(1.40 \pm 0.01^{e})</td>
<td>(1.28 \pm 0.02^{e})</td>
<td>(1.14 \pm 0.01^{e})</td>
<td>(1.41 \pm 0.05^{e})</td>
<td>(1.4)</td>
</tr>
<tr>
<td>(R_x^+)</td>
<td>(0.45^{f})</td>
<td>(0.45^{f})</td>
<td>(0.45^{f})</td>
<td>(0.45^{f})</td>
<td>(0.45^{f})</td>
</tr>
<tr>
<td>(R^+_y)</td>
<td>(~0.17)</td>
<td>(~0.17)</td>
<td>(~0.17)</td>
<td>(~0.17)</td>
<td>(~0.17)</td>
</tr>
<tr>
<td>(R_C)</td>
<td>(0.06^{j})</td>
<td>(0.11^{j})</td>
<td>(0.06^{j})</td>
<td>(0.11^{j})</td>
<td>(0.06^{j})</td>
</tr>
<tr>
<td>(R_x)</td>
<td>(1.4^{j})</td>
<td>(1.5^{j})</td>
<td>(1.4^{j})</td>
<td>(1.5^{j})</td>
<td>(1.4^{j})</td>
</tr>
<tr>
<td>(a_C/a_C)</td>
<td>(1.2 \pm 0.9^{b})</td>
<td>(1.2 \pm 0.9^{b})</td>
<td>(1.2 \pm 0.9^{b})</td>
<td>(1.2 \pm 0.9^{b})</td>
<td>(1.2 \pm 0.9^{b})</td>
</tr>
<tr>
<td>((a_M/a_x)_{AA})</td>
<td>(0.29^{k})</td>
<td>(0.33^{k})</td>
<td>(0.29^{k})</td>
<td>(0.33^{k})</td>
<td>(0.6 + O(x)^{k})</td>
</tr>
</tbody>
</table>

(F), denotes ferromagnetic substances.
(A), denotes an antiferromagnetic substance.
\(^a\) Possible dipolar-induced deviations from pure \(n = 3\) behaviour. \(^b\) Geldart and Malmström (1982). \(^c\) Maszkiewicz (1978). \(^d\) Kollie (1977). \(^e\) Ahlers and Kornblit (1975). The second line of estimates is with constant background term only, without allowance for singular corrections to scaling. \(^f\) Kornblit and Ahlers (1975). The second estimate is without allowance for corrections to scaling. \(^g\) Lederman et al. (1974). \(^h\) Balberg et al. (1978). \(^i\) Hohenberg et al. (1976a). \(^j\) Barmatz et al. (1975). \(^k\) Aharony and Ahlers (1980).
7.4.1.1 Ferromagnets

The equation of state has been studied by macroscopic MHT measurements in a number of ferromagnets in the critical region, and the scaling form (3.1) has been verified explicitly (e.g. see Ho and Litster, 1969). From the scaling function \( h(x) \) obtained in the fit it is straightforward to calculate the universal amplitude ratios using the formulas of Section 3. As was noted by Barmatz et al. (1975), however, the specific heat singularity predicted by these equations is often quite far from the experimentally observed one. Analyses of data which treat magnetization and thermal properties consistently were carried out by Barmatz et al. (1975) for EuO and Ni, and the implications for universal amplitude ratios were discussed by Aharony and Hohenberg (1976). The general conclusion was that the numbers agreed roughly with theoretical predictions for the Heisenberg model (see also Table 7.6). On the other hand, uncertainties in the analysis, as well as in the precise treatment of weak dipolar interaction effects (especially for EuO) and of effects that smear the transition, precluded more quantitative tests.

7.4.1.2 Antiferromagnets

For antiferromagnets, the equation of state is difficult to measure since a staggered field cannot be applied simply, though some quantities are obtainable by measuring relaxation properties (see also Section 7.4.2 below, for the Ising case). Values of \( A/A' \) have been obtained from specific heat results for various materials (Stierstadt et al., 1984, 1989). Hohenberg et al. (1976a) considered in detail the isotropic antiferromagnet RbMnF\(_3\) (see Table 7.6). Crude estimates of the correlation length have been made in RbMnF\(_3\) from neutron-scattering data, both above and below \( T_c \) (see Hohenberg et al., 1976b). Since the measured exponents do not satisfy scaling with high precision, the amplitude ratios are not finite as \( t \to 0 \), but estimates for \( t \) in the experimental range can still be made. The \( R_\xi^+ \) and \( R_\xi^- \) values thus obtained are shown in Table 7.6.

7.4.2 Uniaxial three-dimensional systems (\( n = 1 \))

In Table 7.7, we list experimental results for three well-studied uniaxial (\( n = 1, 3d \)) antiferromagnets and also for the order–disorder transition in \( \beta \)-brass (\( n = 1 \); Als-Nielsen, 1976a,b). Uniaxial ferromagnetic and antiferromagnetic systems are less common than isotropic ones. A compilation of data for many substances has been given by Stierstadt et al. (1984, 1989). The specific heat amplitudes are measured by standard methods. However, evaluation of the staggered susceptibility and correlation amplitudes is more
Table 7.7 Experimental results on universal amplitude combinations for uniaxial \((n = 1)\) 3d magnetic systems and for \(\beta\)-brass.

<table>
<thead>
<tr>
<th></th>
<th>FeF(_2)</th>
<th>MnF(_2)</th>
<th>CoF(_2)</th>
<th>(\beta)-Brass</th>
<th>Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td>(A/A')</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(A)</td>
<td>(A)</td>
<td>(A)</td>
<td>(OD)</td>
<td></td>
</tr>
<tr>
<td>0.543 ± 0.020(^a)</td>
<td>0.491 ± 0.014(^a)</td>
<td>0.538(^f)</td>
<td>0.53 ± 0.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.528 ± 0.010(^a)</td>
<td>0.596 ± 0.015(^a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5441 ± 0.0011(^a)</td>
<td>0.5295 ± 0.0009(^a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.5437 ± 0.0014(^a)</td>
<td>0.5343 ± 0.0028(^a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.53 ± 0.05(^b)</td>
<td>0.74 ± 0.25(^d)</td>
<td>0.60 ± 0.10(^d)</td>
<td>0.27(^e)</td>
<td>0.27</td>
<td></td>
</tr>
<tr>
<td>0.49(^e)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(R_\uparrow^+)</td>
<td>0.36(^a)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(\Gamma/\Gamma')</td>
<td>6.1 ± 1.0(^f)</td>
<td>4.8 ± 0.5(^a)</td>
<td>4.9 ± 0.5(^h)</td>
<td>5.46(^i)</td>
<td>4.9 ± 0.2</td>
</tr>
<tr>
<td>(\xi_0/\xi_0)</td>
<td>2.06 ± 0.20(^f)</td>
<td>1.7 ± 0.3(^a)</td>
<td>1.93 ± 0.10(^h)</td>
<td>1.96</td>
<td></td>
</tr>
<tr>
<td>(a_c/a_C)</td>
<td>0.5 ± 0.8(^a)</td>
<td>1.2–2.5(^a)</td>
<td></td>
<td></td>
<td>0.96 ± 0.25</td>
</tr>
</tbody>
</table>

\(^a\)Belanger et al. (1985). First and third lines, specific heat data fit; second and fourth lines, birefringence data fit. The narrower ranges (lines three and four) are with imposed \(\varepsilon = 0.110\).


\(^i\)Als-Nielsen (1976a,b). \(^j\)Ramos et al. (1989).

Consideration of values in Table 7.7 suggests that consistency with theoretical predictions can be generally claimed, although in some cases the ranges are still wide enough to warrant further studies.

7.4.3 Systems with \(n > 3\)

In a number of cases, amplitude ratios can be used to make qualitative statements about the universality class of a particular material. Examples are the rare-earth metals Ho and Dy, which are predicted to correspond to \(n = 4\) (Bak and Mukamel, 1976), for which the \(\varepsilon\)-expansion yields \(\varepsilon \simeq -0.17\) and \(A/A' \simeq 1.7\). Measurements of the electrical resistance anomaly lead to \(\varepsilon \simeq -0.27\) and \(A/A' \simeq 1.7 \pm 0.14\) in Ho (Singh and Woods, 1981), and \(\varepsilon \simeq -0.20\), \(A/A' \simeq 1.6\) in Dy (Malmström and Geldart, 1980). The value \(\varepsilon \simeq -0.27\) in Ho is closer to the \(n = 6\) prediction, but the analysis leading to this value neglected correction terms, so it is not clear that the data exclude the lower-\(\varepsilon\) value. It should be noted that Balberg and Maman (1979) had earlier found \(\varepsilon \simeq -0.04\) and \(A/A' \simeq 1.44\) in Dy.
A case where neither the exponents nor the amplitude ratios are understood is Fe(S₂CN(C₂H₅)₂)₂Cl, measured by De Fotis and Pugh (1981), which was expected to be Ising like. An analysis of equation of state and thermal measurements led to \( \gamma \approx 1.165, \beta \approx 0.245, \alpha > 0.3 \) and \( R_C \approx 0.2 \), which do not correspond to any known universality class.

An interesting controversy has been initiated by Inderhees et al. (1988). These authors measured the specific heat of the high-\( T_c \) superconductor YBa₂Cu₃O₇₋δ. As can be seen from (1.32), the crossover from mean-field to critical behaviour is a function of the scaling variable \((Bu/\varepsilon)t^{-\varepsilon/2}\). This is in fact a general result, true beyond the \( \varepsilon \)-expansion that led to (1.32). If one expands the specific heat to leading order in \((Bu/\varepsilon)t^{-\varepsilon/2}\), the leading correction diverges as \( C_{\pm}|t|^{-\varepsilon/2} \), with \( C_+/C_- = n/2^{d/2} \). Inderhees et al. (1988) fitted this lowest-order form, with \( \varepsilon = 1 \) (\( d = 3 \)), and found \( C_+/C_- = 2.8 \pm 0.8 \), which they interpreted as \( n = 7.9 \pm 2.3 \), apparently ruling out the standard \( n = 2 \) Ginzburg–Landau theory. Since their data show clear systematic deviations from \( |t|^{-\varepsilon/2} \) for small \( |t| \), we feel that one should attempt fitting the full crossover function (1.32) before drawing any conclusions.

### 7.4.4 Two-dimensional transitions

A number of antiferromagnets (see Table 7.8) exhibit strongly enhanced in-plane coupling with, at the same time, an easy-axis anisotropy. They therefore have the 2d Ising \(( n = 1 \) critical behaviour. The most studied universal ratio is \( A/A' \), which has been determined both from power law fits (which also give \( \alpha \approx 0 \); see Hatta and Ikeda, 1980) and from fits to a logarithmic specific heat singularity (which also yield a symmetric background

<table>
<thead>
<tr>
<th></th>
<th>( A/A' )</th>
<th>( R_C(R_x^\pm)^{-2} )</th>
<th>( \Gamma/\Gamma' )</th>
<th>( \xi_0/\xi' )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( K_2CoF_4 )</td>
<td>0.92–1.03(^a)</td>
<td>0.0565 ± 0.0075(^c)</td>
<td>32.6 ± 3.7(^c)</td>
<td>1.85 ± 0.22(^c)</td>
</tr>
<tr>
<td>( Rb_2CoF_4 )</td>
<td>0.84 ± 0.19(^b)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Ba_2NiF_6 )</td>
<td>0.97 ± 0.07(^b)</td>
<td>0.043 ± 0.002(^d)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( Rb_2NiF_4 )</td>
<td>0.94(^a)</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>( K_2MnF_4 )</td>
<td>1.10(^a)</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>( K_2NiF_4 )</td>
<td>1.04–1.08(^a)</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Theory(^x)</td>
<td>0.96(^a)</td>
<td></td>
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<tr>
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<tr>
<td>( K_2MnF_4 )</td>
<td>1.10(^a)</td>
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<tr>
<td>Theory(^x)</td>
<td>0.96(^a)</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

\(^a\)Hatta and Ikeda (1980). \(^b\)Ikeda et al. (1976). \(^c\)Cowley et al. (1984). \(^d\)Hagen and Paul (1984). \(^x\)Exact results. Note that \( \xi_0/\xi' \) and \( R_x^\pm \) are defined here with the “true” correlation lengths (see Bruce, 1981a).
term as expected). For substances with different anisotropies the values of $A$ vary by an order of magnitude (Hatta and Ikeda, 1980) so that constancy of $A/A' \approx 1$ provides a gratifying verification of the theory.

7.4.5 Dipolar Ising systems in three dimensions

Some ferromagnets have dipolar forces dominating their ordering. The most studied material is LiTbF$_4$, although some results for amplitudes are also available for other substances (Brinkman et al., 1978; Stierstadt et al., 1984, 1989). The critical behaviour of the dipolar Ising model has been discussed in Section 6.4 (see (6.91)–(6.97)). For LiTbF$_4$, the experimental values are $\tilde{A}/\tilde{A}' = 0.244 \pm 0.009$ (Ahlers et al., 1975), $\tilde{\tilde{A}}/\tilde{\tilde{A}}' \approx 2$ (Als-Nielsen, 1976a), $\tilde{R}_x \approx 0.5$ (Frowein et al., 1979), while the estimates for $\tilde{R}_c$ range from $\tilde{R}_c^{-1} \approx 6$ (Ahlers et al., 1975; Frowein et al., 1979) to $\tilde{R}_c^{-1} = 7.8 \pm 0.7$ (Ahlers et al., 1975; Beauvillian et al., 1980). These “leading” results are consistent with the exact RG values (6.98) at the upper-marginal dimensionality. There is, however, some uncertainty in the treatment of the double-logarithmic correction terms (Brézin, 1975).

The full equation of state was investigated by Frowein et al. (1979), who found good agreement with the theory, while Als-Nielsen (1976a) verified the logarithmic violation of hyperscaling (see (4.59)).

7.4.6 Bicritical points

Among the multicritical points, bicritical behaviour (of 3d antiferromagnets) is the only extensively studied case as far as universal amplitude ratios are concerned. For the $(2 \leftrightarrow 1 + 1)$ bicritical points (see Section 6), the experimental results on GdAlO$_3$ and NiCl$_2$·6(H$_2$O) are, respectively, $Q_b = 0.9 \pm 0.2$ (Rohrer and Gerber, 1977) and $Q_b = 1.06 \pm 0.22$ (Oliveira et al., 1978), consistent with the theoretical prediction $Q_b = 1$. Note that the experimental crossover exponent values are around $\phi \sim 1.2$, which compare well with the theoretical prediction (6.112).

The $(3 \leftrightarrow 2 + 1)$ bicritical behaviour has been studied in MnF$_2$ (King and Rohrer, 1979). In this case the $\phi$ value is consistent with theory, though the experimentally obtained $Q_b = 1.56 \pm 0.35$ is clearly inconsistent with the theoretical value (6.114).

7.4.7 Random-exchange and random-field systems

7.4.7.1 Dilute Heisenberg systems

The experimental situation for disordered magnetic materials of the isotropic ($n = 3$) 3d random-exchange type has been surveyed by Chang and
Hohenemser (1988). Typically, the critical exponents are unchanged by disorder, as predicted by Harris (1974), since \( \alpha < 0 \) for \( n = 3 \). However, we are aware of only one experimental determination of amplitude ratios: Papp (1983) measured \( A/A' = 1.4 \pm 0.05 \) and \( a_c/a'_c = 0.6 \pm 0.4 \) for the ferromagnet Ni with up to 10% by weight of non-magnetic Cu impurities. These values are consistent with pure Ni and with the theoretical predictions for pure \( n = 3 \) systems.

### 7.4.7.2 Dilute Ising systems

Since \( \alpha > 0 \) for the pure 3d Ising model, new critical behaviour is expected for random-exchange systems (see (6.99)–(6.101)). Experimental results for four dilute Ising antiferromagnets were surveyed by Mitchell et al. (1986). Average experimental values, \((\xi_0/\xi'_0)^{-1} = 0.70 \pm 0.02 \) and \( \Gamma/\Gamma' = 2.4 \pm 0.2 \), agree with the theoretical predictions and are well away from the pure Ising values.

For \( \text{Rb}_2\text{Co}_{0.7}\text{Mg}_{0.3}\text{F}_4 \), a substance with 2d random Ising antiferromagnetism, Hagen et al. (1987) measured \( \Gamma/\Gamma' = 19.1 \pm 5.0 \), \( \xi_0/\xi'_0 = 1.02 \pm 0.20 \), \( R_C(R_\xi^*)^{-2} = 0.062 \pm 0.010 \), which should be compared with the pure 2d Ising results for \( \text{K}_2\text{CoF}_4 \) (Table 7.8): the values are clearly different. However, the theoretical situation for this borderline case (\( \alpha_{\text{pure}} = 0 \)) is controversial (e.g. see Shankar (1987) and references therein).

### 7.4.7.3 Random-field systems

The antiferromagnetic substance \( \text{Fe}_{0.6}\text{Zn}_{0.4}\text{F}_2 \) in applied fields of about 20 kOe exhibits random-field 3d Ising critical behaviour. Experimentally, one finds \( A/A' \approx 1.0 \) and a very small value of \( \alpha \) (= 0.001 \pm 0.027), which is consistent with scaling (see Belanger et al., 1984). This has been interpreted as evidence for dimensional reduction from \( d = 3 \) to \( d = 2 \) but, as mentioned in Section 6, there is no theoretical justification for applying dimensional reduction at \( d = 3 \).

### 8 Experimental results: dynamics

#### 8.1 Liquid–gas and binary fluid critical points

The most important universal quantities in the dynamics of fluids are the scaling function (3.73) for the characteristic frequency, \( \mathcal{Q}(\tilde{y}) \), and the ratio \( R \) (see (6.122)). Although it is difficult to obtain \( \mathcal{Q}(\tilde{y}) \) with very high accuracy, many measurements, starting with the pioneering work of Bergé et al. (1970), have verified the Kawasaki-type calculations to within roughly 50% (for a review of experimental work, see Swinney and Henry (1973)).
Experimental determinations of the ratio \( R \) have been undertaken by a number of groups, following publication of the theoretical predictions (6.124) and (6.127). The larger value \( R = 1.2/6n \) was found by Chen et al. (1978), Sorensen et al. (1978) and Beysens (1982) in most of the fluid mixtures analysed (the average over 10 mixtures was \( 6\pi R = 1.16 \pm 0.005 \)). However, a later study by Beysens et al. (1984) yielded \( 6\pi R = 1.07 \pm 0.07 \), averaged over five mixtures. A similar range, \( 6\pi R = 1.06 \pm 0.04 \), was found by Hamano et al. (1986, and references therein). Agosta et al. (1987) report \( 6\pi R = 1.05 \pm 0.01 \). This work is noteworthy since the quantities in (6.124) were measured in separate experiments at zero frequency (except for \( \zeta \)). Measurements by Güttinger and Cannell (1980) and by Burstyn and Sengers (1982) yielded \( 6\pi R = 1.01 \pm 0.06 \). As emphasized by these authors, and by Siggia et al. (1976) in the original theoretical work, the weakness of the viscosity divergence does not permit an unambiguous test of the asymptotic critical behaviour. An approximate treatment of correction terms suggests that the theory should remain valid in the experimental range if the full viscosity \( \bar{\eta} \) is used in the definition (6.122) of \( R \), rather than its singular part (Calmettes, 1977; Hohenberg and Halperin, 1977; Ohta, 1977; Beysens et al., 1984).

The uncertainties in the proper treatment of correction terms, and in the theoretical evaluation of the 3D asymptotic value of \( R \), are sufficient to accommodate any of the experimental values mentioned above. In fact, the present agreement between all experiments and all theories to within 20\% makes \( R \) one of the best controlled ratios. Nevertheless, it appears that there exists sufficient experimental and theoretical information to warrant a more complete analysis of correction terms, using nonlinear recursion relations in three dimensions analogous to those for the superfluid (see Section 6). In this way one might hope to predict the full temperature dependence of the transport coefficients without the necessity for arbitrary background subtractions (see Ahlers et al., 1981).

### 8.2 Superfluid \(^4\)He

An early determination of the amplitude ratio \( R_\lambda \) (see (6.134)) was provided by Ahlers (1968), who measured the divergence of thermal conductivity \( \lambda \). The value obtained was \( R_\lambda = 0.3 \), in good agreement with the theoretical estimates (6.137) and (6.140). Subsequent accurate experiments revealed, however, that this value was pressure and temperature dependent in the experimental range \( 10^{-6} < |t| < 10^{-2} \) (see Ahlers et al., 1982). As mentioned in Section 6.7, these variations have been successfully explained in terms of different slow transients which make the asymptotic dynamical critical region unattainable in superfluid \(^4\)He (for reviews see Dohm and Folk (1982b) and Hohenberg (1982)). The theory makes predictions on the variation of \( R_\lambda \) for
$10^{-8} < |t| < 10^{-3}$, and pressures all along the $\lambda$-line, which comprises an accessible experimental range. The full two-loop calculation of $R_\lambda$ for the asymmetric-spin model has been carried out by Dohm (1981, 1985), and tested in a set of careful measurements along the $\lambda$-line by Tam and Ahlers (1986), which were confirmed at saturated vapour pressure by Dingus et al. (1986). The agreement between experiment and theory is excellent, and involves both the temperature and the pressure dependence of $R_\lambda$, with only the non-universal amplitudes at $t = 10^{-2}$ as adjustable parameters. In the experimental range $10^{-6} < t < 10^{-4}$ the value of $R_\lambda$ is of the order of $0.25-0.35$, whereas the extrapolated asymptotic value (which is supposedly finite and universal) is estimated to be $R_\lambda \approx 1$, and would only be reached for $t \ll 10^{-15}$. We thus see that the RG is capable of making reliable predictions for semiuniversal amplitudes which are evolving slowly towards their fixed-point values.

The situation with regard to $R_2$ (see (6.135)) was even less clear since the early measurements of second-sound damping by Tyson (1968) yielded a value $R_2 \approx 0.5$, which was five times larger than theoretical estimates. This discrepancy, which did not seem explicable by the addition of correction terms, led to the conclusion that a serious problem existed (Hohenberg et al., 1976b; Hohenberg and Halperin, 1977). The situation was partially resolved by later measurements of second-sound damping (Ahlers, 1979; Mehrotra and Ahlers, 1984), which brought $R_2$ down to the order of $0.1$, a value much closer to theoretical estimates. The remaining temperature dependence of $R_2$ has been successfully explained on a semiquantitative level by the nonlinear RG theory (Dohm and Folk, 1980, 1981; Ahlers et al., 1982; Mehrotra and Ahlers, 1984). A quantitative theory of second-sound attenuation must await analysis of the asymmetric model F of Hohenberg and Halperin (1977), as well as a model which includes the transients associated with the first-sound mode (Dohm, 1987).

The dynamical density correlation function $\hat{G}_m(q, \omega)$ was measured by light-scattering techniques (Tarvin et al., 1977; Vinen and Hurd, 1978), and it failed to show the critical behaviour expected on the basis of dynamical scaling (Hohenberg et al., 1976b). The actual shape of the spectrum can be explained in terms of the nonlinear theory with transients (Ferrell and Bhattacharjee, 1979b; Hohenberg and Sarkar, 1981), but the difficulty of obtaining accurate data precludes a truly quantitative test.

### 8.3 Magnetic transitions

The main experimental tests of critical dynamics in magnetic systems have involved measurements of critical exponents, primarily using neutron scattering. By those methods it is difficult to obtain accurate information on absolute intensities, so little information is available on the amplitude combinations.
We may note the experimental values $R_\chi = 0.17$ and $R_\Gamma = 0.23$ obtained by Tucciarone et al. (1971) for RbMnF$_3$, in good accord with the $\varepsilon$-expansion values (see (6.145) and (6.146)). In addition, Bhattacharjee and Ferrell (1981) have compared their value of $\tilde{R}_3$ (see (6.150)) to the neutron-scattering data, on EuO, of Dietrich et al. (1976), which yield a value 30% higher. The shape function $\tilde{W}$ for the spectrum has also been measured in a number of ferromagnets and antiferromagnets, but we are not aware of any attempts to make detailed quantitative comparisons with theory. For the universal function $\tilde{\Omega}$ of the characteristic frequency, on the other hand, early experimental data have been shown to agree with mode-coupling calculations in the ferromagnet Fe (Parette and Kahn, 1971), and in the antiferromagnets FeF$_2$ and MnF$_2$, where anisotropy leads to a crossover from Heisenberg- to Ising-like behaviour (Bagnuls and Joukoff-Piette, 1975; Kawasaki, 1976).

More recently, the crossover in low-temperature ferromagnets (e.g. EuO) between short-range isotropic and long-range dipolar behaviour has been studied by a combination of spin-echo neutron-scattering techniques, electron-spin resonance, and hyperfine interaction experiments. The results have been reviewed by Frey and Schwabl (1988), who also show that their mode-coupling calculation removes previously found inconsistencies. These authors do not, however, show detailed comparisons involving universal amplitude combinations.

9 Statistics of polymer conformations

9.1 Fixed number of steps ensemble

In this section we consider configurational properties of self-avoiding walks (SAWs), and also some results for non-self-avoiding Gaussian walks, on regular lattices as well as in the continuum description. For lattice walks of $N$ steps, visiting $N + 1$ sites (which need not be distinct for Gaussian walks), we denote by $c_N$ the number of all the different $N$-step walks beginning at the origin. For large $N$, we expect

$$c_N \approx C \mu^N N^{7-1}, \quad (9.1)$$

and also

$$p_N \approx P \mu^N N^{3-3}, \quad (9.2)$$

where $p_N$ is the number of distinct $N$-step, $N$-site unrooted polygons, self-avoiding or Gaussian. (Unrooted polygons are counted without regard
to which site is the origin of the ring. The number of rooted, unoriented rings
beginning at the origin is thus $Np_N$.

Let $r_0, r_1, \ldots, r_N$ denote the coordinates of the sites visited by an $N$-step
walk (with, typically, $r_0 \equiv 0$). The mean-squared end-to-end distance is
defined by

$$R_N^2 = \langle (r_N - r_0)^2 \rangle,$$

(9.3)

where $\langle \rangle$ denotes an average over all $N$-step walks. The mean-squared
radius of gyration is defined by considering the distance from the centre of
mass:

$$G_N^2 = \langle \frac{1}{N+1} \sum_{j=0}^{N} \left( r_j - \frac{1}{N+1} \sum_{i=0}^{N} r_i \right)^2 \rangle$$

$$= \langle \frac{1}{2(N+1)^2} \sum_{j=0}^{N} \sum_{i=0}^{N} (r_j - r_i)^2 \rangle. \quad (9.4)$$

For large $N$, we expect

$$R_N^2 \approx \rho_R N^{2\nu}, \quad (9.5)$$

$$G_N^2 \approx \rho_G N^{2\nu}. \quad (9.6)$$

In the fixed fugacity, $z$, ensemble the large-$N$ behaviours (9.1), (9.2) and
(9.5), (9.6) translate to critical point-like divergences of generating functions.
For example,

$$\xi^2(z) \equiv \frac{1}{2d} \sum_{j=0}^{N} c_N R_N^2 z^N \sim (z_c - z)^{-2\nu} \quad (9.7)$$

behaves as the square of the second-moment correlation length. (Here $z_c \equiv 1/\mu.$) The correspondence can be made between SAW critical behaviour
and the $n \to 0$ limit of the $n$-vector model (De Gennes, 1972; Des Cloizeaux,
1975). However, the $n \to 0$ limit of the $n$-vector universal amplitude combinations
has not been studied extensively (Aharony and Hohenberg, 1976; Privman
and Redner, 1985; Cardy, 1988b). The reason is, in part, in that fixed-$z$
SAW properties correspond only to the high-temperature side of the
$n$-vector criticality. In the fixed-$N$ ensemble, however, several universal amplitudes
and amplitude combinations have been investigated. These results will be
reviewed in this section.

The ratio $\rho_R / \rho_G$ is the simplest universal $N$-ensemble quantity (Domb
and Hioe, 1969), defined in the limit $N \to \infty$. Selected results are listed in Table 9.1:
umerical values for several 2d and 3d lattices are consistent, and the
3d values are close to the second-order $\epsilon$-expansion result. Note that the
Gaussian walk results are generally reproduced by putting $\epsilon = 0$, for various
Table 9.1 Universal ratios of radial measures for self-avoiding walks and rings.

<table>
<thead>
<tr>
<th>$d$</th>
<th>Value</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 (exact)</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>2 (MC)</td>
<td>$7.131^a$, $7.126^b$</td>
<td>Rapaport (1985a)</td>
</tr>
<tr>
<td>(series)</td>
<td>$7.14 \pm 0.05^{a,b}$</td>
<td>Domb and Hioe (1969)</td>
</tr>
<tr>
<td>3 (MC)</td>
<td>$6.41 \pm 0.06^c$, $6.31 \pm 0.05^d$</td>
<td>Rapaport (1985b)</td>
</tr>
<tr>
<td>(series)</td>
<td>$6.383 \pm 0.014^e$</td>
<td>Privman (1986)</td>
</tr>
<tr>
<td>(series)</td>
<td>$6.45 \pm 0.04^f$</td>
<td>Wall and Hioe (1970)</td>
</tr>
<tr>
<td>(series)</td>
<td>$6.45 \pm 0.04^{c,d,e}$</td>
<td>Domb and Hioe (1969)</td>
</tr>
<tr>
<td>(MC)</td>
<td>$\sim 6.25^g$</td>
<td>Schäfer and Baumgärtner (1986)</td>
</tr>
<tr>
<td>(ε-expansion)</td>
<td>6.2–6.3</td>
<td></td>
</tr>
<tr>
<td>$4 - \varepsilon$</td>
<td>$6(1 + \varepsilon/96 + 0.0307\varepsilon^2 + \ldots)$</td>
<td>Benhamou and Mahoux (1985)</td>
</tr>
<tr>
<td>$\rho_G^{\text{(polygons)}}/\rho_G$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3 (series)</td>
<td>$0.53 \pm 0.01^e$</td>
<td>Rapaport (1975)</td>
</tr>
<tr>
<td>(series)</td>
<td>$0.535 \pm 0.005^f$</td>
<td>Wall and Hioe (1970)</td>
</tr>
<tr>
<td>(ε-expansion)</td>
<td>$\sim 0.57$</td>
<td></td>
</tr>
<tr>
<td>$4 - \varepsilon$</td>
<td>$\frac{1}{2}(1 + \frac{13}{96}\varepsilon + \ldots)$</td>
<td>Prentis (1982)</td>
</tr>
</tbody>
</table>

$a$ Triangular lattice. $b$ Square lattice. $c$ SC lattice. $d$ BCC lattice. $e$ FCC lattice. $f$ Diamond lattice. $g$ Off-lattice simulation.
quantities considered here and below. (The upper critical dimension for SAWs is 4.)

The radius of gyration can be also defined for N-step, N-site polygons, by using relations similar to (9.4) and (9.6). The amplitude ratio \( \rho_{g}^{(\text{polygons})}/\rho_{g} \) must be universal (see Table 9.1 for some numerical values). It should be noted that numerous data on walk and ring size measures are available in the literature, mostly for 2d and 3d lattice models. Most of these data, however, have not been analysed to estimate amplitude ratios. Results for ratios other than \( \rho_{R}/\rho_{G} \) and \( \rho_{G}^{(\text{polygons})}/\rho_{G} \) include higher-moment definitions of the radii (Domb et al., 1965; Rapaport, 1975; Duplantier, 1986a,b), as well as certain other radial measures (e.g. Domb and Hioe, 1969; Wall and Hioe, 1970). In 2d, some conformal invariance results have been derived recently by Cardy and Saleur (1989).

Experimentally, one can measure \( G_{N}^{2} \), and also the “hydrodynamic radius” defined by using the \((-1)\)st moment instead of the 2nd moment in (9.4). However, experimentally (e.g. Akcasu and Han, 1979), the “hydrodynamic” exponent \( v \) value is not consistent with the “radius-of-gyration” and theoretical values (\( v \approx 0.588 \) in 3d). Schäfer and Baumgärtners (1986) attribute this discrepancy to short-chain effects. They also calculate a universal amplitude ratio involving the two radii, to \( O(\varepsilon) \), by MC estimation (in 3d).

Several different measures of the asymmetry of polymer chain shape have been introduced (e.g. Domb et al., 1965; Mazur et al., 1973; Rubin and Mazur, 1975, 1977; Bishop and Michels, 1985a,b; and references therein). Here we consider the measures described in Section 5. Let \( r_{i}^{(1)}, r_{i}^{(2)}, \ldots, r_{i}^{(d)} \) denote the components of \( r_{i} \). The radius-of-gyration tensor \( \mathcal{G} \) is defined in analogy with (9.4):

\[
\mathcal{G}_{ij} = \frac{1}{2(N+1)^{2}} \sum_{k=0}^{N} \sum_{m=0}^{N} (r_{k}^{(i)} - r_{m}^{(i)})(r_{k}^{(j)} - r_{m}^{(j)}). \tag{9.8}
\]

The asymmetry measures \( \mathcal{A}_{d} \) and \( \mathcal{S}_{d} \), relations (5.12) and (5.13), have been calculated in the \( \varepsilon \)-expansion by Aronovitz and Nelson (1987):

\[
\mathcal{A}_{d} = \frac{2(2 + d)}{4 + 5d} \left[ 1 + \frac{745\varepsilon}{3584(4 + 5d)} + \ldots \right], \tag{9.9}
\]

\[
\mathcal{S}_{d} = \frac{8(d^{2} + 6d + 8)}{35d^{2} + 84d + 64} + O(\varepsilon), \tag{9.10}
\]

where near \( d = 4 \) one can further expand by putting \( d = 4 - \varepsilon \). However, (9.9) and (9.10) as shown, with \( \varepsilon = 0 \), also give the correct answers for Gaussian walks for any \( d \), and for SAWs above four dimensions (Rudnick and Gaspari, 1986; Aronovitz and Nelson, 1987). The results for polygons
are limited (Gaspari et al., 1987) to the Gaussian expression

\[ D^{\text{polygons}} = \frac{2 + d}{2 + 5d}. \]  

(9.11)

As mentioned in Section 6, the quantities \( \mathcal{G}_d \) and \( \mathcal{D}_d \), calculated with averaging after forming the ratios in (5.12) and (5.13), are also of interest. The latest analytical results for these ratios are still limited to the Gaussian expressions for \( \mathcal{G}_d \) (Diehl and Eisenriegler, 1989),

\[ \mathcal{D}^{\text{polygons}} = \frac{d + 2}{2d + 2} \mathcal{G}_d, \]  

(9.12)

and

\[
\mathcal{D}_d = \frac{d(d + 2)}{4(d + 1)} \left[ 3 + d - d \int_0^\infty x^{d+1}(\sinh x)^{-d} \, dx \right] \\
= \frac{2}{5} - \frac{12}{175d} + O(d^{-2}).
\]

(9.13)

Bishop and Michels (1986), and Bishop and Saltiel (1986, 1988) obtained MC estimates of various \( \mathcal{D} \)- and \( \mathcal{G} \)-type shape ratios. Their values generally agree with the \( \epsilon \)-expansion and large-\( d \) results, although their simulations were done for relatively short chains/rings, and the estimates show a wide spread.

### 9.2 Hyperuniversality and conformal invariance

As exemplified by (9.7), the radial measures of SAW sizes generate correlation length-like quantities in the fixed-fugacity ensemble. Similarly, the numbers of unrooted polygons, \( p_N \), generate a free energy-like quantity,

\[ f(z) = a^{-d} \sum_N p_N z^N \sim (z_c - z)^{2-\alpha}, \]  

(9.14)

where \( a^d \) denotes the unit cell volume. Note that \( a \) and the step length \( \sqrt{R_1^2} \) are not equal, except for hypercubic lattices. Privman and Redner (1985) introduced a hyperuniversal amplitude combination proportional to \( a^{-d} \mathcal{G}_R^{d/2} \) (see (9.2) and (9.5)). Certain caution is required, however, since (9.2) is applicable only for some close-packed lattices (e.g. triangular, FCC), for which \( p_N \) is a positive monotonic sequence (for \( N \geq 3 \)). For other lattices, the \( p_N \) are less regular (e.g. square, SC, BCC). For the square lattice, for
instance, only 4, 6, 8, 10, ... -step polygons can be formed. Thus, \( p_{2k+1} = 0 \),
and the amplitude \( P \) is defined for \( N = 2k \) only. The fixed-\( z \) ensemble reflection of this behaviour is the singularity at \(-z_c\), in addition to the “critical-point” singularity at \(+z_c\), shown in (9.14). This \(-z_c\) singularity is part of the “background”, formally, and it is in fact present in other generating functions as well. For most SAW quantities, the critical-point singularity dominates. However, for polygons the two are identical in strength and, in fact, \( f(z) \) is an even function of \( z \). Privman and Redner (1985) define, therefore, the amplitude combination

\[
W_R \equiv \rho^{d/2}_R P/(\tau a^d), \tag{9.15}
\]

where \( \tau \) compensates for the background singularity effect. Thus,

\[
\tau(\text{triangular, FCC, ...}) = 1, \tag{9.16}
\]

\[
\tau(\text{square, BCC, SC, ...}) = 2. \tag{9.17}
\]

For some looser-packed lattices a more complicated behaviour is found. For example, for the two-choice-square lattice (also known as the L-lattice), only \( p_4 \) and \( p_{12}, p_{16}, p_{20}, \ldots \) are non-zero. In this case there are four equal-strength singularities, at \( \pm z_c, \pm i z_c \), so that one takes

\[
\tau(\text{2-choice-square, ...}) = 4. \tag{9.18}
\]

Numerical series analysis results (Privman and Redner, 1985) are

\[
W_R(\text{square}) = 0.217 \pm 0.007, \tag{9.19}
\]

\[
W_R(\text{triangular}) = 0.216 \pm 0.007, \tag{9.20}
\]

\[
W_R(\text{2-choice-square}) = 0.24 \pm 0.025, \tag{9.21}
\]

\[
W_R(\text{FCC}) = 0.06 \pm 0.02, \tag{9.22}
\]

\[
W_R(\text{BCC}) = 0.08 \pm 0.02, \tag{9.23}
\]

\[
W_R(\text{SC}) = 0.08 \pm 0.015. \tag{9.24}
\]

Thus, hyperuniversality is checked quite accurately in 2d, however, further studies are needed for a conclusive verification in 3d.

Cardy (1988b) proposed to consider another hyperuniversal combination,

\[
W_G = [\rho^{(\text{polygons})}_d]^{d/2} P/(\tau a^d), \tag{9.25}
\]

in 2d. Indeed, he argued that the radius of gyration of polygons, when calculated with bond-centre coordinates instead of site coordinates, can be related to the energy-energy correlation length entering the conformal invariance prediction (1.17). The two radii of gyration are related (Cardy,
1988b) by
\[ G_N^2(\text{polygons, bonds}) = G_N^2(\text{polygons, sites}) - \frac{R_1^2}{4}, \] (9.26)
so that the leading amplitude \( p_0 \) is the same. The polymer equivalent of (1.17) is thus
\[ W_G \equiv \frac{5}{16\pi^2} \] (2d), (9.27)
which also checks against numerical data on the square lattice (Cardy, 1988b).

Fisher et al. (1984) considered the winding angle \( \theta \) distribution of 2d SAWs around the origin. Here \( \theta \) is the polar angle of the vector \( r_N \) measured with respect to \( r_1 \) (we assume \( r_0 \equiv 0 \)). The angle is varied in small increments as the walk proceeds, so that \(-\infty < \theta < \infty\), with no reduction of the angle to, say, \([0, 2\pi)\). Series and MC results of Fisher et al. (1984) on the square lattice suggest that the distribution of the \( \theta \) values of \( N \)-step walks approaches a scaling-limiting form, with \( \theta \) scaled by \( \sqrt{\ln N} \). Specifically, for the leading moments we have
\[ \theta_N^2 \equiv \langle \theta^2 \rangle \approx a_2 \ln N, \] (9.28)
\[ \theta_N^4 \equiv \langle \theta^4 \rangle \approx a_4 (\ln N)^2. \] (9.29)
What Fisher et al. (1984) actually estimated was that the exponents in (9.28) and (9.29) are very close to 1 and 2 numerically, and that the universal ratio,
\[ \frac{a_4}{a_2^2} = 2.9 - 3.2, \] (9.30)
is close to the Gaussian distribution value 3.

Duplantier and Saleur (1988) used conformal invariance to prove that the asymptotic probability distribution \( \mathcal{P}(x) \), in terms of \( x \equiv \theta/\sqrt{4 \ln N} \), is indeed Gaussian and universal,
\[ \mathcal{P}(x) \equiv \frac{e^{-x^2}}{\sqrt{\pi}}. \] (9.31)
It follows that \( a_4/a_2^2 \equiv 3 \) exactly and also that the coefficients in (9.28) and (9.29) are universal, specifically, \( a_2 \equiv 2 \), as is also supported by reanalysis of the data of Fisher et al. (1984).

Consideration of the winding angle in \( d > 2 \), is also possible (e.g. Prager and Frisch, 1967; Puri et al., 1986; and references therein). Rudnick and Hu (1988) established that for SAWs in \( 4 - \varepsilon \) dimensions (\( \varepsilon > 0 \)), the winding angle around a \((2 - \varepsilon)\)-dimensional “cylinder” is in \( O(\varepsilon) \) distributed as in
(9.31), but with \( x \equiv \sqrt{\varepsilon} / \sqrt{8 \ln N} \). However, the Gaussian distribution is probably not exact in \( O(\varepsilon^2) \) and higher.

The universal SAW amplitudes \( a_2, a_4 \), and the whole distribution function for \( \theta / \sqrt{\ln N} \), are hyperuniversal in that the appropriate quantities for Gaussian walks must be lattice dependent. This is suggested by numerical results of Rudnick and Hu (1987, 1988), and also by the following argument. For 2d Brownian motion, Spitzer (1958) established a result which can be loosely written as

\[
\mathcal{P}(x) = \frac{\pi^{-1}}{1 + x^2}, \quad x \approx \frac{2\theta}{\ln N}.
\]

For rigour, one must replace \( N \) by the Brownian motion time variable related to \( N \) by a lattice-dependent proportionality constant which vanishes as the lattice spacing goes to zero. Since the continuous Cauchy distribution (9.32) is not normalizable, it would yield infinite \( \theta_N^2, \theta_N^4 \), etc. However, the moments for lattice Gaussian walks are finite by definition. Therefore, their \( N \) dependence will be controlled by the ultraviolet lattice cutoffs and have a typical mean-field non-universal behaviour. Note also the change in the power of \( \ln N \), scaling \( \theta \): for lattice Gaussian walks, we thus expect \( \theta_N^2 \approx \tilde{a}_2 (\ln N)^2 \), with a non-universal coefficient \( \tilde{a}_2 \), etc.

## 10 Finite-size systems

### 10.1 Universal finite-size amplitude ratios

This section is devoted to universal amplitudes and amplitude combinations arising in the finite-size scaling description of rounded critical-point singularities. First, consider \( d < 4 \) systems, of roughly hypercubic shape, and volume \( L^d \), so that for the free-energy density we can use relation (1.5):

\[
f_s(t, H; L) \approx L^{-d} Y(K_t L^{1/\nu}, K_h H L^{\lambda/\nu}).
\]

By using relations (1.22), (1.23) and (5.2), which define \( M, \chi, \chi^{(nl)} \), with \( L < \infty \), one can check that the quantity, introduced by Binder (1981a,b),

\[
\tilde{g}_*(t; L) \equiv - \left[ \frac{\chi^{(nl)}}{L^d \chi^2} \right]_{H=0} = \frac{3 \langle \sigma^2 \rangle^2 - \langle \sigma^4 \rangle}{\langle \sigma^2 \rangle^2},
\]

scales according to

\[
\tilde{g}_*(t; L) \approx \tilde{G}(K_t L^{1/\nu}),
\]
where

\[ \bar{G}(x) = \left[ (\partial^4 Y/\partial y^4)/(\partial^2 Y/\partial y^2)^2 \right]_{y=0}, \quad Y = Y(x, y). \quad (10.4) \]

Relation (10.3) actually applies for any boundary conditions, not necessarily periodic ones. The definition (10.2) is reminiscent of the "renormalized coupling constant" (Section 5.1.3), whereas the second form shown, involving the dimensionless magnetization per spin of a given configuration,

\[ \sigma \equiv \sum_i \sigma_i / \sum_i 1, \quad (10.5) \]

suggests the term "cumulant ratio" for \( g_\ast \) (Binder, 1981a,b, 1990). (The dimensionless spin variables for the Ising case, \( \sigma_i = \pm 1 \), are those entering (6.1). For \( n > 1 \), the use of the longitudinal components of \( \sigma_i \), where \( |\sigma_i| = 1 \), is implied.)

As \( L \to \infty \), the quantity \( \bar{g}_\ast(0; L) \) approaches a universal constant, \( \bar{g}_\infty \). Numerical estimates of \( \bar{g}_\infty \) for hypercubic-shaped Ising \( (n = 1) \) models are listed in Table 10.1. All the 2d and 3d estimates (for both periodic and "sub-block" boundary conditions) are quite consistent.

| Table 10.1 Numerical estimates of the universal finite-size cumulant ratio \( \bar{g}_\infty \) for hypercubic-shaped Ising models. |
|---|---|---|---|
| \( d \) | \( \bar{g}_\infty \) | Method | Reference |
| 2 | 1.56 ± 0.03 \(^a\) | MC | Binder (1981a,b) |
| 1.74 \(^a\) | Approximate RG | Bruce (1981b) [Wilson (1971a,b)] |
| 1.835 | Transfer matrix MC | Burkhardt and Derrida (1985) |
| 1.832 ± 0.002 | MC | Bruce (1985) |
| 1.935 ± 0.03 | Microcanonical MC | Desai et al. (1988) |
| 0.63 ± 0.03 \(^a\) | MC | Binder (1981a,b) |
| 0.66 \(^a\) | Approximate RG | Bruce (1981b) [Wilson (1971a,b)] |
| \(~0.9\) | MC | Kaski et al. (1984) |
| 1.32 ± 0.06 | MC | Binder (1981a,b) |
| \(~1.2\) | \( \varepsilon \)-Expansion | Brézin and Zinn-Justin (1985) |
| 1.40 | MC | Barber et al. (1985) |
| 1.40 ± 0.01 | MC | Lai and Mon (1989a) |
| \(~1.35\) | MC, polymer mixture | Sariban and Binder (1987) |
| 4 | \(~0\) | MC | Binder (1981a,b) |
| 5 | \(~1.0\) | MC | Binder et al. (1985) |
| \( >4 \) | 0.81156... | Mean field | Brézin and Zinn-Justin (1985) |

\(^a\)These estimates where obtained for a geometry where a finite system \( L^d \) is a sub-block of a larger system. All other estimates are for periodic boundary conditions. Some results are also available for free boundary conditions (Binder, 1981a,b) and for Ising spin glasses (Bhatt and Young, 1988).
Recently, several studies have been reported of the $\varepsilon$-expansion for finite-size systems with periodic boundary conditions (Brézin and Zinn-Justin, 1985; Rudnick et al., 1985b; Nemirovsky and Freed, 1985, 1986; Guo and Jasnow, 1987; Jasnow, 1990). These developments have yielded several conceptual advances. For example, the conjecture (Brézin, 1982; Privman and Fisher, 1984) that for periodic systems one can take $f_{ns}(t; L) \approx f_{ns}(t; \infty)$ has been confirmed. Actual calculations, however, prove rather difficult in the $\varepsilon$-expansion for finite systems (which for cubic shapes is in powers of $\sqrt{\varepsilon}$), as compared to the bulk ($L = \infty$) case. For the cumulant ratio, Brézin and Zinn-Justin (1985) obtained the result

\[
3 - \tilde{g}_\infty \equiv \frac{\langle \sigma^4 \rangle}{\langle \sigma^2 \rangle^2} = \frac{n\Gamma^2(n/4)}{4\Gamma^2[(n + 2)/4]} \times \left\{1 - v_0 \left[ \frac{\Gamma[(n + 6)/4]}{\Gamma[(n + 4)/4]} + \frac{\Gamma[(n + 2)/4]}{\Gamma(n/4)} - 2 \frac{\Gamma[(n + 4)/4]}{\Gamma[(n + 2)/4]} \right] + v_0^2 \left[ 3 \frac{\Gamma^2[(n + 4)/4]}{\Gamma^2[(n + 2)/4]} - n - 1 + \frac{\Gamma[(n + 2)/4]\Gamma[(n + 6)/4]}{\Gamma(n/4)\Gamma[(n + 4)/4]} \right] + O(\varepsilon^{3/2}) \right\},
\]

where

\[
v_0 = \sqrt{\varepsilon} \left( -1.7650848 \ldots \right) \frac{n + 2}{\sqrt{2(n + 8)}} + O(\varepsilon).
\]

Higher-order $H = 0$ cumulant ratios, involving $\langle \sigma^6 \rangle$, etc., have been considered (Binder, 1981a,b; Privman, 1984; Brézin and Zinn-Justin, 1985), but no definite numerical results are available.

Finite-size scaling behaviour of higher-dimensional systems, $d > 4$, is strongly dependent on boundary conditions and system shape (Privman and Fisher, 1983; Binder et al., 1985; Rudnick et al., 1985a). For periodic, near-cubic systems, Binder et al. (1985) found that (10.1) is replaced by

\[
f_s(t, H; L) \approx L^{-d} Y(\tilde{K}_t, tL^{d/2}, \tilde{K}_h H L^{3d/4}).
\]

This form is confirmed by mean-field (zeroth-order $\varepsilon$-expansion) studies (Brézin and Zinn-Justin, 1985; Rudnick et al., 1985b). Relation (10.8) suggests that for this particular geometry (periodic, cubic), $\tilde{g}_\infty$ remains universal for $d > 4$. (Behaviour at $d = 4$, where finite-size power law $L$ dependences may be complicated by logarithmic correction factors (Brézin, 1982), has not been investigated in detail.) The consistency of the MC and mean-field values for $d = 5$, listed in Table 10.1, is poor; further studies are needed.
Privman (1984) considered the large-|x| behaviour of the n = 1 (Ising) universal scaling function \( \bar{G}(x) \) in (10.3). For \( x \to +\infty \), we have

\[
\bar{G}(x) \approx -\Gamma^{(nl)} \Gamma^{-2} K_{t}^{-2-\alpha} x^{-2},
\]

(10.9)

which, in fact, applies for all \( n \). The asymptotic form as \( x \to -\infty \) is more complicated, and its derivation relies on certain results on finite-size rounding of first-order transitions near the \( H = 0, T < T_{c} \) phase boundary (Privman and Fisher, 1983), which will not be reviewed here. Let us introduce the “nonlinear magnetization” of a bulk system:

\[
M^{(nl)} = \left( \frac{\partial^{2} M}{\partial H^{2}} \right)_{H \to 0^{+}} \approx B^{(nl)} (-t)^{\beta - 2\gamma}.
\]

(10.10)

Then the result for the \( x \to -\infty \) asymptotic form of \( \bar{G}(x) \) is (Privman, 1984), for \( n = 1 \) only,

\[
\bar{G}(x) \approx \frac{2B^{4} (-\bar{x})^{3(2 - \alpha)} - 4BB^{(nl)} (-\bar{x})^{2 - \alpha} - [\Gamma^{(nl)}]'}{(-\bar{x})^{2 - \alpha} [B^{2} (-\bar{x})^{2 - \alpha} + \Gamma']^{2}},
\]

(10.11)

where \( \bar{x} \equiv x / K_{t} \), while the primes on the susceptibility amplitudes denote, as usual, the \( t < 0 \) generalizations of (5.2), etc. The corrections in (10.11) and (10.9) are believed to be \( O(e^{-\text{const}|x|}) \) (see Privman, 1984). (This conclusion applies to (10.9) and (10.11) with \( n = 1 \) only (and with periodic boundary conditions); corrections to (10.9), and the leading expression equivalent to (10.11), have not been investigated for \( n > 1 \).)

For models other than Ising, we are only aware of the MC estimate \( g_{\infty} = 1.934 \pm 0.001 \) for 2d percolation, by Saleur and Derrida (1985). These authors also proposed to consider the equivalent of (10.2) in the long cylinder geometry, \( L^{d-1} \times \bar{L} \), with \( \bar{L} \to \infty \). The cross-section, \( L^{d-1} \), is hypercubic, with periodic boundary conditions. Thus, one defines

\[
\bar{g}_{\ast}(t; L) \equiv - \left[ \frac{\chi^{(nl)}}{L^{d} \chi^{2}} \right]_{H = 0, L = \infty} = \lim_{L \to \infty} \left\{ \frac{\bar{L}}{L} \frac{3 \langle \sigma^{2} \rangle^{2} - \langle \sigma^{4} \rangle}{\langle \sigma^{2} \rangle^{2}} \right\},
\]

(10.12)

\[
\bar{g}_{\infty} = \lim_{L \to \infty} \bar{g}_{\ast}(0; L).
\]

(10.13)

Results for \( \bar{g}_{\infty} \) in the periodic cylinder geometry are summarized in Table 10.2. While several “pure” models have been studied, only for the 2d Ising case do we have more than a single estimate, and all the values are quite consistent. For “disordered” 2d models, Derrida et al. (1987) observed a clear change in \( \bar{g}_{\infty} \) for the three-state Potts model (\( \alpha = \frac{1}{3} > 0 \)). However, for the borderline (\( \alpha = 0 \)) Ising case, no change in \( \bar{g}_{\infty} \) was found for different amounts of disorder.
Table 10.2 Estimates of the universal finite-size cumulant ratio $\bar{g}_\infty$ in the periodic cylinder geometry, $L^{d-1} \times \infty$.

<table>
<thead>
<tr>
<th>Model</th>
<th>$\bar{g}_\infty$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>2d Ising model</td>
<td>7.38132 ± 0.00006\textsuperscript{a}</td>
<td>Burkhardt and Derrida (1985)</td>
</tr>
<tr>
<td></td>
<td>7.38\textsuperscript{b}</td>
<td>Burkhardt and Derrida (1985)</td>
</tr>
<tr>
<td></td>
<td>$\sim 7.3$\textsuperscript{c}</td>
<td>Saleur and Derrida (1985)</td>
</tr>
<tr>
<td>Disordered 2d Ising</td>
<td>$\sim 7.38$\textsuperscript{c}</td>
<td>Derrida \textit{et al.} (1987)</td>
</tr>
<tr>
<td>2d Percolation</td>
<td>9.90 ± 0.06\textsuperscript{c}</td>
<td>Saleur and Derrida (1985)</td>
</tr>
<tr>
<td>2d, $q = 3$ Potts model</td>
<td>2.49 ± 0.09\textsuperscript{c}</td>
<td>Derrida \textit{et al.} (1987)</td>
</tr>
<tr>
<td>Disordered 2d, $q = 3$ Potts model</td>
<td>4.5 ± 0.9\textsuperscript{c}</td>
<td>Derrida \textit{et al.} (1987)</td>
</tr>
<tr>
<td>2d, Small-$\eta$ result\textsuperscript{d}</td>
<td>6/$\pi\eta$</td>
<td>Cardy (1987)</td>
</tr>
<tr>
<td>3d Ising model</td>
<td>$\sim 3$\textsuperscript{c}</td>
<td>Saleur and Derrida (1985)</td>
</tr>
<tr>
<td>3d Percolation</td>
<td>5.0 ± 0.02\textsuperscript{c}</td>
<td>Saleur and Derrida (1985)</td>
</tr>
</tbody>
</table>

\textsuperscript{a}Integration of the correlation function predicted by conformal invariance (Cardy, 1984a, b, 1987). Saleur and Itzykson (1987) get a wider range, 7.38144 ± 0.00015, by a different numerical method. \textsuperscript{b}Transfer matrix method. \textsuperscript{c}Transfer matrix combined with MC. \textsuperscript{d}This is the leading order small-$\eta$ result valid for any conformal-invariant 2d model with $\eta > 0$.

Burkhardt and Derrida (1985) considered 2d Ising squares with certain folded-periodic boundary conditions, for which numerical integration of the conformal invariance expressions for the critical-point correlation functions (Cardy, 1984a, b, 1987) yields $\bar{g}_\infty = 1.67 \pm 0.02$. Numerical transfer matrix MC estimates confirm this value to about 1%.

Privman and Schulman (1982a, b) introduced free energy-like quantities, $f_\pm$, calculable from the two largest transfer matrix eigenvalues below $T_c$, for periodic Ising cylinders $L^{d-1} \times \infty$ (in any dimension). These functions break the $H \leftrightarrow -H$ symmetry in a finite-size system (details of their definition are not given here). Recall that $f(T, H; L)$ is an even function of $H$ (for Ising models) as long as $L < \infty$. However, $f_+(T, H; L) = f_-(T, -H; L)$ have all their $H$ derivatives finite at $H = 0$, and these derivatives approximate the spontaneous magnetization, etc., below $T_c$. Let $Y_+(x, y)$ denote the scaling function as in (10.1), for $f_+$, and let $Y_+^{(k)} \equiv [\partial^k Y_+(x, y)/\partial y^k]_{x,y=0}$. Thus, up to numerical prefactors, $Y_+^{(k)}$ are the universal values at the origin of the scaling functions for the magnetization $M_+ (k = 1)$, susceptibility ($k = 2$), etc. Privman and Fisher (1983) considered universal combinations $w_k = Y_+^{(k+1)} Y_+^{(k-1)}/[Y_+^{(k)}]^2$ which turn out to be particularly convenient for numerical evaluation. (Several $Y_+^{(k)}$ have also been estimated.) Privman and Fisher (1983) calculated $w_k$ for $k = 2, 3, 4, 5, 6$, for the square- and triangular-lattice Ising models, by the transfer matrix method, with $L$ up to 10 lattice spacings. The values for the two lattices agree to within 4% for $k = 2$ and 6 but to better than 1% for $k = 3, 4, 5$ (see original work for numerical values).

Finally, we mention that Desai \textit{et al.} (1988) recently developed a finite-size scaling approach for the microcanonical ensemble. Their MC estimate of
\( \bar{g}_\infty = 1.935 \pm 0.03 \) (for 2d Ising squares) is somewhat higher than "canonical ensemble" estimates listed in Table 10.1.

10.2 Free-energy and correlation length amplitudes

As mentioned in Section 1, the finite-size scaling form (10.1) for the free energy, and the appropriate correlation length relation (1.6),

\[
\xi(t, H; L) \approx L^X(K_t t L^{1/\nu}, K_h H L^{A/\nu}),
\]

(Privman and Fisher, 1984) involve universal critical-point amplitudes:

\[
f_s(0, 0; L) \approx L^{-d} Y(0, 0),
\]

(10.15)

\[
\xi(0, 0; L) \approx L X(0, 0).
\]

(10.16)

Generally, (10.1) and (10.14) are hyperuniversal relations, which apply only for \( d < 4 \). While these relations are well established for systems with periodic boundary conditions, their use for other boundary conditions requires certain caution; their applicability depends on geometry (see below). In this section we describe various tests of relations (10.1), (10.14)–(10.16) for periodic and some non-periodic geometries for which they were found to apply. The appropriate modifications for cases where finite-size scaling relations are more complicated will be discussed in Section 10.3 below.

10.2.1 Spherical models

Singh and Pathria (1985a) (see also Shapiro and Rudnick, 1986) considered spherical models for \( 2 < d < 4 \), of finite extent in all dimensions, \( L_1 \times L_2 \times \ldots \times L_d \), as well as for some of the \( L_j \to \infty \), with periodic boundary conditions in all finite dimensions. The calculation can be carried out formally with both \( d \) and the number of finite dimensions, \( d^* \leq d \), varying continuously. Singh and Pathria (1985a) confirm relation (10.1) for the zero-field free energy. The scaling function \( Y(x, 0) \) is given by a complicated implicit equation; it is markedly dependent on the dimensionalities \( d \) and \( d^* \) and on system shape. In fact, for \( d - d^* > 2 \), the finite-size system has its own \((d - d^*)\)-dimensional shifted phase transition (see also Barber and Fisher, 1973) manifested as a singularity in \( Y(x, 0) \) at \( x = x_s \).

For spherical models with antiperiodic boundary conditions, Singh and Pathria (1985b) find generally similar (zero-field) results: relation (10.1) applies, with a different scaling function \( Y(x, 0) \). Singh and Pathria (1985c, 1987) also considered a class of Bose gas models with periodic boundary conditions, for \( 2 < d < 4 \) and \( d - 2 \leq d^* \leq d \), which are in the same universality class as the spherical model and, indeed, yield consistent results.
for $Y(x, 0)$. In all cases, the amplitudes $Y(0, 0)$ are given implicitly as solutions of rather complicated equations.

Brézin's (1982) calculations for the large-$n$ limit yield an implicit equation for the universal correlation length amplitude $X(0, 0)$ in the periodic cylinder geometry $L^{d-1} \times \infty$, with varying $2 < d < 4$. Numerically, he obtains $X(0, 0) \approx 0.6614$ in 3d, and

$$X(0, 0) \approx (4\pi^2 \varepsilon)^{-1/3}, \quad (10.17)$$

for small positive $\varepsilon = 4 - d$. Luck (1985) further quotes

$$X(0, 0) \approx (\pi \varepsilon)^{-1}, \quad (10.18)$$

for small positive $\varepsilon = d - 2$. For cylinders, the definition of the "true" correlation length by correlation function decay in the infinite direction is an obvious choice. For cubic samples, however, it is less clear which definition to use, and there are indications (Brézin, 1982; Privman and Fisher, 1983) that the applicability of the finite-size scaling relations, and the form of the scaling functions, are quite sensitive to the choice of the correlation length.

Since $\eta = 0$ in the large-$n$ limit, Brézin (1982) proposed to consider the length $\chi^{\eta / \gamma} = \sqrt{\lambda}$. The ratio $\sqrt{\chi} / L$ is indeed asymptotically constant at $T_c$, and he quotes the small-$\varepsilon$ result

$$\sqrt{\chi} / L \propto \varepsilon^{-1/4} \quad (\varepsilon = 4 - d). \quad (10.19)$$

Henkel (1988) considered the Hamiltonian version of spherical model cylinders, with antiperiodic boundary conditions in the $d - 1$ finite dimensions (with $2 < d < 4$). He derived an expression for $X(0, 0)$ for the spin–spin correlations and further established that the corresponding amplitude for the energy–energy correlations is then $2X(0, 0)$. A similar conclusion (factor of 2) applies for the periodic cross-section case.

10.2.2 $\varepsilon$-Expansion results

Substantiation of the free-energy scaling form (10.1) for periodic boundary conditions, in the framework of the $\varepsilon$-expansion, was discussed by Guo and Jasnow (1987) and Jasnow (1990). They also comment that a one-loop order expression for $Y(x, y)$, suitable for numerical calculations, can be derived from the results of Rudnick et al. (1985b). Some results for non-periodic boundary conditions will be mentioned in Section 10.2.3 below.

For the periodic cylinder geometry, Brézin and Zinn-Justin (1985) get

$$X(0, 0) = \left( \frac{48\pi^2 \varepsilon}{n + 8} \right)^{-1/3} h \left[ \left( -5.0289788 \ldots \right) \pi^{-s / \varepsilon} n + 2 \left( \frac{6\varepsilon}{n + 8} \right)^{1/3} \right]$$

$$\times \left[ 1 + O(\varepsilon) \right], \quad (10.20)$$
where \( h(x) \) is the inverse of the leading energy gap of the Hamiltonian 
\[
\frac{1}{2} [p^2 + xq^2 + \frac{1}{2} q^2]^{22}.
\]
For large \( n \), relation (10.20) reduces to (10.17).

Eisenriegler and Tomaszitz (1987) initiated an interesting study of
finite-size scaling in the fixed-\( M \) ensemble, i.e. with system properties
considered as functions of \( (t, M; L) \) instead of \( (t, H; L) \). Their results (not
reproduced here) include \( e \)-expansions for several universal scaling function
amplitudes of the Helmholtz free energy.

10.2.3 Numerical results in three dimensions

The universal free-energy amplitude for periodic 3d Ising cubes \((L^3)\) was
obtained by MC estimation (Mon 1985, 1990):

\[
Y(0, 0) = -0.657 \pm 0.03 \quad \text{and} \quad -0.643 \pm 0.04, \quad (10.21)
\]

for the SC and BCC lattices, respectively. Numerical results were also obtained
for 3d Ising cylinders \((L^2 \times \infty)\) in the Hamiltonian (extreme anisotropy)
limit: Henkel (1986, 1987a,b,c, 1990) established that, with proper rescalings
of lengths, the quantities \( Y(0, 0) \) and \( X(0, 0) \) (both spin–spin and energy–
energy) are universal for periodic boundary conditions in the cross-section.
His conclusions regarding \( X(0, 0) \) also include the case of antiperiodic
boundary conditions.

Next, we discuss results for 3d slabs, \( L \times (\infty)^2 \) with two surfaces. For
non-periodic systems, relation (10.1) does not always apply, as will be
discussed in Section 10.3 below. It turns out, however, that for 3d slabs, 2d
strips, etc., only the form of \( f_{ns}(t; L) \) is modified. Relation (10.1) for \( f_{s}(t, H; L) \)
can be used (with a boundary condition-dependent scaling function \( Y(x, y) \)).

Close to a surface (in a semi-infinite 3d geometry), the interactions may
be characterized by \( J_{\text{wall}} \), different from the bulk coupling \( J \). As long as
\( J_{\text{wall}}/J \) is not too large, the surface ordering, as \( T \) is decreased through \( T_c \)
(at zero field), will be driven by the bulk ordering. This is called the ordinary
\((O)\) surface transition. Two cases are of particular interest. First, for \( J_{\text{wall}} = J \)
we have free-boundary conditions. Secondly, it is believed that a continuous
field-theoretic description near \( T_c \) with the constraint that the order parameter
entering the Ginzburg–Landau Hamiltonian vanishes at the wall, corresponds
to the \( O \)-type surface ordering transition: we will include such a theory under
the term “free-boundary conditions”.

When \( J_{\text{wall}}/J \) increases, the surface may actually order above \( T_c \). However,
at the bulk \( T_c \) singularities still develop in surface properties. This is termed
the extraordinary \((E)\) transition. The multicritical borderline case is termed
the special \((Sp)\) surface transition. Finally, wall interactions may result in
positive (+) or negative (−) ordering fields acting at the surface. This type
of surface ordering is also believed to be universal with the case of a continuous field-theoretic order parameter fixed at positive or negative value at the surface.

From the point of view of finite-size scaling, the strength of the surface ordering field, and the precise value of $J_{\text{wall}}$ when there are no surface fields, are largely irrelevant. Each surface is characterized by having ordering type $O$, $E$, $Sp$, + or −. The reader should note that the surface types summarized above are, strictly speaking, valid only for the short-range Ising case. For other models, e.g. three-state Potts, etc., different classifications of boundary conditions are appropriate (e.g. see Park and Den Nijs, 1988). However, since most of the results we quote are Ising, we will not discuss other model boundary specifications here. Note also that in 2d only Ising types $O$, ± exist.

Indekeu et al. (1986) summarized results for the amplitudes $Y(0, 0)$ in slab geometry. Their 3d estimates are based on Migdal–Kadanoff real-space RG calculations:

\begin{align}
Y_{++}(0, 0) &\approx 0.279, \\
Y_{OO}(0, 0) &\approx -0.015, \\
Y_{Sp+}(0, 0) &\approx 0.017, \\
Y_{O+}(0, 0) &\approx 0.051, \\
Y_{SpSp}(0, 0) &\approx 0.019, \\
Y_{OSP}(0, 0) &\approx 0.017, \\
Y_{+ +}(0, 0) &\approx 0.279, \\
Y_{+ +}(0, 0) &\approx 0.
\end{align}

Note that the subscripts indicate boundary condition types at the two walls. The vanishing value of the $++$ amplitude (10.28) is an artefact of the Migdal–Kadanoff method. It is expected that $Y_{++}(0, 0) < 0$. Generally, real-space results are at best semiquantitative. However, for the amplitudes (10.22)–(10.27), Indekeu et al. (1986) expected better reliability, due to the success of the same numerical method in reproducing several known results in 2d and in the mean-field case. For the OO amplitude, first-order $\varepsilon$-expansion results are also available (Symanzik, 1981),

\begin{equation}
Y_{OO}(0, 0) \approx -0.012.
\end{equation}

Based on certain exact 2d results and mean-field calculations, Indekeu et al. (1986) conjectured the relation

\begin{equation}
Y_{AE}(0, 0) = Y_{A+}(0, 0)
\end{equation}
for \( A = O, E, Sp, +. \) (Note also the obvious symmetries like \( O+ \leftrightarrow O- \), etc.) They also derived a set of mean-field relations in 4d:

\[
\begin{align*}
Y_{O+}(0,0) &= Y_{-+}(0,0)/2^d, \\
Y_{Sp+}(0,0) &= Y_{++}(0,0)/2^d, \\
Y_{++}(0,0) &= -dY_{-+}(0,0)/16.
\end{align*}
\]

(10.31) (10.32) (10.33)

We emphasize that relations (10.30)–(10.33) have not been checked in 3d. Relations (10.31) and (10.33) are actually inconsistent with conformal invariance predictions in 2d (see below). Indekeu (1986) also speculated, based on certain indirect numerical evidence, that \( Y_{oo}(0,0) \sim -0.03 \) and \( Y_{oe}(0,0) \sim 0.10 \), for the 3d \( XY (n = 2) \) universality class. Mon and Nightingale (1987) estimated by MC that \( Y_{oo}(0,0) \) is in the range \(-0.07 \) through \(-0.03 \) in this case.

Mon and Nightingale (1985) considered a rather complicated model: they studied, by MC estimates, fully periodic boxes \( L \times \bar{L} \), and also boxes of the same size but with free boundary conditions in the \( L \) direction, i.e. at the two \( \bar{L} \) faces (while both \( L \) directions remain periodic). Their MC calculations yielded the difference of the singular parts of the free energies of these two systems, at the bulk critical point. Since each scales as in (10.1), the results are for

\[ \bar{\gamma} = Y_{\text{free-periodic}}(0,0) - Y_{\text{periodic-periodic}}(0,0), \]

(10.34)

where both amplitudes on the right, and their difference \( \bar{\gamma} \), depend on the aspect ratio \( L/\bar{L} \) of the boxes. Mon and Nightingale (1985) checked that \( \bar{\gamma} \) is indeed universal when calculated on the SC and BCC lattices, with \( l = L/\bar{L} \) ranging approximately from 0.05 to 1. The function \( \bar{\gamma}(l) \) is roughly linear in \( l^2 \), with \( \bar{\gamma}(0) \sim 0.1 \), \( \bar{\gamma}(1) = 0.41 \pm 0.03 \). Mon and Nightingale (1987) reported a similar study of the aspect ratio dependence for the \( XY \) model in 3d.

### 10.2.4 Amplitudes in 2d and conformal invariance

Conformal invariance (e.g. Cardy, 1987) predictions for the 2d critical exponents and finite-size amplitudes at \( T_c \), have far superseded earlier results of exact and numerical studies. The field has grown explosively in recent years, and new developments and ideas are being constantly published. Collections of reprints have been put together by Itzykson et al. (1988) and Cardy (1988c). Our review of conformal invariance-related results in this section will be limited in two ways. First, we quote specific results without attempting to survey the formalism underlying their derivation beyond providing references. Secondly, we focus on the most familiar models, selected from a wider class of 2d systems for which conformal invariance results are available. We emphasize that this section is not devoted solely to conformal
invariance; a substantial fraction of the results reviewed have been obtained by other analytical and numerical methods.

10.2.4.1 True correlation length for $L \times \infty$ strips

Cardy (1984a) established that the correlation length, defined by the exponential decay of the spin–spin correlation function along an $L \times \infty$ strip, satisfies relation (10.16) at $T_c$, with

$$X(0, 0) = \frac{1}{\pi \eta} \quad \text{(periodic boundary conditions)}, \quad (10.35)$$

$$X(0, 0) = \frac{2}{\pi \eta_\parallel} \quad \text{(free-boundary conditions)}. \quad (10.36)$$

Here $\eta$ is the exponent of the decay law, $r^{-\eta}$ in 2d, of the bulk spin–spin correlation function, while $\eta_\parallel$ is the exponent of the decay with distance of correlations between two points at the boundary in a half-plane geometry. Results for other correlations are similar. For example, if we denote the decay exponent of the bulk energy–energy correlation function $\eta_{ee}$ then, for strips, we have

$$X(0, 0) = \frac{1}{\pi \eta_{ee}} \quad \text{(periodic; energy–energy)}. \quad (10.37)$$

Following earlier “Coulomb gas” results reviewed by Nienhuis (1987), and the works by Belavin et al. (1984), Friedan et al. (1984) and Cardy (1984b), the exponents entering (10.35)–(10.37) and similar relations for other correlation lengths have been determined exactly for many 2d models, in particular $n$-vector models with $-2 \leq n \leq 2$ (including SAWs) and $q$-state Potts models with $0 \leq q \leq 4$ (including percolation). Specifically, for the Ising model we have $\eta = \frac{1}{4}, \eta_\parallel = 1, \eta_{ee} = 2$; for the three-state Potts model we have $\eta = \frac{4}{15}, \eta_\parallel = \frac{4}{3}, \eta_{ee} = \frac{8}{5}$; etc.

The periodic-strip relations (10.35) and (10.37) were established empirically, based on numerical transfer matrix and exact Ising calculations, before conformal invariance methods were developed (e.g. Derrida and De Seze, 1982; Luck, 1982; Nightingale and Blöte, 1983; Privman and Fisher, 1984). The free boundary condition result (10.36) has been tested mostly by exact 2d Ising calculations (e.g. Burkhardt and Guim, 1987; and references therein). Note that (10.36) applies to any type O identical walls. A similar relation also holds for + + or − − walls (Cardy 1986, 1987); however, the surface exponent is different. For example, we have $\eta_\parallel(OO) = 1$ but $\eta_\parallel(++) = \eta_\parallel(−−) = 4$, for the Ising model. Finally, for certain models (Ising, three- and four-state Potts), results have been derived for the cases of antiperiodic, mixed $(O\pm)$ and some other boundary conditions. (Note that we use the Ising surface-type nomenclature here, as we did in the 3d case, i.e. O, ±. For other
models, the reader should consult the original works.) These results and their exact (Ising) and numerical tests have been reviewed by Cardy (1986, 1987) and Burkhardt and Guim (1987), and will not be surveyed here. Another recent development which will not be reviewed here (see Chapter 6 of Itzykson et al., 1988) is connected with the Bethe ansatz solutions of several "integrable" models. A method of extracting finite-size behaviour of quantities related to the transfer matrix spectrum (correlation lengths, free energy, surface tension, etc.) has been developed by De Vega and Woynarovich (1985) and used to classify conformal invariance properties of various 2d systems at criticality.

Regarding more general tests of the scaling relation (10.14), these have been reported for the 2d Ising model, for which results, mostly for \( X(x, 0) \), are available for periodic-, antiperiodic-, free-, fixed- and some mixed-type boundary conditions (see Turban and Debierre, 1986; Burkhardt and Guim, 1987; Debierre and Turban, 1987; Henkel, 1987a,b,c, 1990; and references therein). The functions \( X(x, 0) \) are not simple, except in the limit \( x \to 0 \). Note that exact calculations for Ising models rely on the original solution by Onsager (1944), as well as on works by Fisher and Ferdinand (1967), Ferdinand and Fisher (1969), Au-Yang and Fisher (1975, 1980) and Fisher and Au-Yang (1980). While conformal invariance applies only at \( T_c \), several perturbation schemes expanding about the conformal invariance results have been proposed (Reinicke, 1987; Saleur and Itzykson, 1987). These calculations, however, are rather complicated, and specific results have been limited and restricted mostly to the Ising case. Turban and Debierre (1986) and Debierre and Turban (1987) also reported numerical tests of universality for several quantities derivable from \( X(x, y) \), for the three-state Potts model, for Ising models with spins \( \frac{1}{2} \) and 1, and for some other models, on the square and triangular lattices.

10.2.4.2 Second-moment correlation length for strips

For a strip of size \( L \times \infty \), let us introduce coordinates \(-\infty < x < \infty \) and \( 0 \leq y < L \). Let \( G(x, y; t; L) \) denote the two-point spin–spin correlation function between \( (x, y) \) and the origin \((0, 0)\), and put

\[
\bar{x}^2 = \frac{\iint x^2 G(x, y) \, dx \, dy}{\iint G(x, y) \, dx \, dy},
\]

(10.38)

\[
\bar{r}^2 = \frac{\iint (x^2 + y^2) G(x, y) \, dx \, dy}{\iint G(x, y) \, dx \, dy}.
\]

(10.39)

Since the strip is effectively one dimensional, the proper definition of the second-moment correlation length is

\[
\xi(t; L) = \sqrt{\bar{x}^2 / 2}.
\]

(10.40)
This quantity scales according to (10.14), as \( t \to 0, L \to \infty \). Specifically, at \( t = 0 \), Privman and Redner (1985) derived the conformal invariance prediction for the periodic case:

\[
X(0, 0) = \frac{1}{4\pi} \sqrt{\left[ 2\psi'\left(\frac{\eta}{4}\right) - \frac{\pi^2}{\sin^2(\pi\eta/4)} \right]} \quad \text{(periodic)},
\]

(10.41)

where \( \psi'(x) \equiv \sum_{k=0}^{\infty} (x + k)^{-2} \) is the derivative of the digamma-function \( \psi(x) \). Privman and Redner (1985) evaluated this expression for SAWs \((\eta = \frac{5}{24})\),

\[
X(0, 0) = 1.527 \ldots,
\]

(10.42)

and also obtained \( X(0, 0) \) by MC estimation on square lattice strips up to 15 lattice spacings wide. The MC estimates confirm the conformal invariance result (10.42) to about 2%.

For free-boundary conditions, Privman and Redner (1985) obtained MC data for \( x^2 \) and \( r^2 \) at \( t = 0 \) (see (10.38) and (10.39)) on strips of up to 25 lattice spacings, which extrapolate to

\[
\sqrt{x^2} / L \approx 0.71 \pm 0.01,
\]

(10.43)

\[
\sqrt{r^2} / L \approx 0.75 \pm 0.01.
\]

(10.44)

There are no analytical predictions for these quantities. However, it turns out (Cardy, 1987) that, generally for models with small \( \eta \), one can expect that “true” and second-moment spin–spin correlation lengths on strips are quite close. (Note that the leading contribution to (10.40) as \( \eta \to 0 \) is just (10.25).) It is therefore instructive to compare (10.43) and (10.44) with \( \sqrt{2X(0, 0)} \) of (10.36). Since \( \eta_\| = \frac{5}{24} \) for SAWs, relation (10.36) yields

\[
\sqrt{2X(0, 0)} = \frac{2\sqrt{2}}{\pi\eta_\|} = 0.72 \ldots,
\]

(10.45)

which is, indeed, close to the values in (10.43) and (10.44). Finally, Cardy (1987) reviewed work on conformal invariance predictions for the structure factors of 2d strips and other geometries, and some numerical tests. These studies entail several new correlation length definitions which will not be surveyed here.

10.2.4.3 Free-energy amplitudes

Blöte et al. (1986) and Affleck (1986) derived results for the universal finite-size amplitude \( Y(0, 0) \) in the strip geometry (see (10.15)). For periodic boundary
conditions, one finds

\[ Y(0, 0) = -\frac{\pi c}{6} \quad \text{(periodic),} \]

(10.46)

where the conformal anomaly number, \( c \), is a characteristic of a given 2d universality class. For example, \( c = \frac{1}{2} \) for the Ising model and \( c = \frac{1}{3} \) for the three-state Potts model. For free or fixed (same at both walls) boundary conditions, the appropriate result is

\[ Y_{\text{free}}(0, 0) = Y_{\text{fixed}}(0, 0) = -\frac{\pi c}{24}. \]

(10.47)

There are two points of caution to keep in mind when these results are considered as cases of general scaling relations like (10.1). First, for free or fixed boundary conditions, a more careful examination of the finite-size properties is needed to ascertain that (10.1) can be used for the strip geometry and to classify possible "non-singular background" contributions. This issue will be taken up in Section 10.3 below. Secondly, even for periodic boundary conditions there are certain complications when \( \alpha = 0 \) or a negative integer. A particularly important case is the 2d Ising model, which is a major source for testing conformal invariance and other theoretical predictions, and it has \( \alpha = 0 \). It turns out that in the \( \alpha \to 0 \) limit there develops a new \( L \)-dependent term in the free energy, proportional to \( t^2 \ln L \) and additive to a hyperuniversal part as in (10.1) (see Privman and Rudnick (1986) and Privman (1990) for a general discussion and references); explicit 2d Ising results are given by Ferdinand and Fisher (1969) and by Blöte and Nightingale (1985). This term has no significant implications for conformal invariance considerations, which are restricted to \( T = T_c \).

For boundary conditions other than those in (10.46) and (10.47), progress has been more limited in that results were obtained only for some models. Cardy (1986) studied the Ising and three-state Potts models. For illustration, we list here his results for the Ising case, \( Y(0, 0)/\pi = \frac{1}{6}, \frac{23}{48} \) and \( \frac{1}{24} \) for antiperiodic, mixed + - and mixed O+ (or O-) boundary conditions, respectively.

Numerical tests of conformal invariance predictions for \( Y(0, 0) \) have been largely limited to systems with periodic boundary conditions, e.g. the \( 0 \leq q \leq 4 \) Potts models, etc. (Blöte et al., 1986; Debierre and Turban, 1987; Henkel, 1987a,b,c). Analytical results on the universality of \( Y(x, 0) \) for \( x \neq 0 \), are still limited to the 2d Ising model; recent developments for Bethe ansatz-solvable models have thus far been focused on evaluation of \( c \) at \( T_c \) (e.g. De Vega and Karowski, 1987; De Vega, 1987; see also Chapter 6 of Itzykson et al., 1988).
For $q$-state Potts models ($0 \leq q \leq 4$), Park and Den Nijs (1988) derived the values of $Y(0, 0)$ for several types of boundary conditions by a "Coulomb gas"-type approach (reviewed by Nienhuis, 1987). Their results are fully consistent with conformal invariance predictions, and were obtained for a general $L_1 \times L_2$ rectangle geometry. Indeed, it has been recently recognized that partition functions at $T_c$, and thus $Y(0, 0)$, can be derived for a parallelogram geometry, both by conformal invariance and Coulomb gas methods (see Chapter 5 of Itzykson et al. (1988) and also work by Di Francesco et al. (1987) and reference therein). The results for $Y(0, 0)$ are rather complicated, involving $\theta$-functions, etc., as found earlier by Ferdinand and Fisher (1969) for the 2d Ising model. Mon (1985, 1990) obtained an MC estimate

$$Y(0, 0) = -0.6687 \pm 0.006,$$

for the square lattice Ising model in the periodic-square geometry. The exact value given by Ferdinand and Fisher (1969) is

$$Y(0, 0) = -\ln(2^{1/4} + 2^{-1/2}) = -0.6399\ldots \quad (L_1 = L_2).$$

Certain exact (Park and Widom, 1989) and numerical MC (Wang et al., 1990) results, not reviewed here, were also obtained recently for antiferromagnetic Potts models.

10.3 Surface and shape effects

10.3.1 Finite-size free energy for free boundary conditions

In this section we consider systems with flat surfaces, straight edges (in 3d) and corners, i.e. finite polygons in 2d, etc. We assume free boundary conditions and review recent scaling (Privman, 1988a, 1990), conformal invariance (Cardy and Peschel, 1988) and MC (Lai and Mon, 1989; Mon, 1990) results. Extensions to other fixed boundary conditions are quite straightforward, and one can also consider curved boundaries (Cardy and Peschel, 1988; Privman, 1988a). However, we restrict our discussion to the case of flat, "free" surfaces which illustrates the new finite-size effects involved. For simplicity, we put $H = 0$ in this section.

Consider first size effects away from $T_c$, when the bulk correlation length $\xi^{(b)}(t; L = \infty)$ is small compared to a characteristic system size, $L$. For systems with no soft modes (spin waves), which we consider here for simplicity, it has been well established (Fisher, 1971) that the free energy (and, in fact,
other thermodynamic quantities) can be expanded as

\[ f(t; L) - f^{(b)}(t) = \frac{1}{L} f^{(s)}(t) + \frac{1}{L^2} f^{(e)}(t) + \frac{1}{L^3} f^{(c)}(t) + O(e^{-L/\xi^{(b)}}) \quad (d = 3), \]

\[ (10.50) \]

\[ f(t; L) - f^{(b)}(t) = \frac{1}{L} f^{(s)}(t) + \frac{1}{L^2} f^{(e)}(t) + O(e^{-L/\xi^{(b)}}) \quad (d = 2), \]

\[ (10.51) \]

where \( f^{(b)} \equiv f(t; \infty) \) is the bulk free energy, while the contributions proportional to \( f^{(s)} \), \( f^{(e)} \) and \( f^{(c)} \) are attributed to surfaces, edges (in 3d) and corners. For 3d slabs \( (L \times (\infty)^2) \) only the \( f^{(s)} \) term is present. Similarly, \( f^{(c)} = 0 \) for 2d strips.

There is accumulating evidence that, near the critical point, the non-singular part \( f_{ns}(t; L) \) has a similar expansion. For example, in 2d,

\[ f_{ns}(t; L) = f^{(b)}_{ns}(t) + \frac{1}{L} f^{(s)}_{ns}(t) + \frac{1}{L^2} f^{(e)}_{ns}(t) + O(L^{-2}). \]

\[ (10.52) \]

For the leading critical scaling form of the singular part of the free energy in general \( d \), let us tentatively accept the form (10.1), although later we will find that modifications are needed in some instances. The \( o(L^{-2}) \) corrections in (10.52) and similar terms, \( o(L^{-3}) \), in 3d, are expected to be \( \sim L^{-(d+1)} \), etc. \( (d = 2, 3) \). However, the leading “geometry-associated” terms in the expansion of \( f_{ns}(t; L) \) closely parallel the series (10.50) and (10.51). For example, for slabs or strips, as mentioned above, we anticipate that the edge and corner terms are not present, i.e. only \( f^{(b)}_{ns} \) is non-zero. Thus, for instance, for the strip geometry, conformal invariance at \( T_c \) (Blöte et al., 1986) predicts non-universal \( O(1) \) and \( O(L^{-1}) \) terms, and a universal \( O(L^{-2}) \) term, in \( f(0; L) \). These terms are then identified as follows:

\[ f(0; L) = f^{(b)}_{ns}(0) + \frac{1}{L} f^{(s)}_{ns}(0) + \frac{1}{L^2} Y(0, 0) + \ldots, \]

\[ (10.53) \]

with \( Y(0, 0) \) given by (10.47). (Note that \( f^{(b)}_{ns}(0) = f^{(b)}(0), f^{(s)}_{ns}(0) = f^{(s)}(0) \).)

Thus, there is no non-universal “corner” term additive to the \( Y(0, 0) \) term for strips. Similarly, numerical results for 3d slabs, reviewed in Section 10.2.3, suggest that the scaling term (10.1) is the only \( O(L^{-3}) \) contribution near \( T_c \). In expansions analogous to (10.53), only the bulk and surface terms are present in the case of slabs.

Field-theoretic \( \varepsilon \)-expansion studies of systems with free boundaries may eventually substantiate various phenomenological expectations described above. The results are, however, limited (Symanzik, 1981; Eisenriegler, 1985;
Huhn and Dohm, 1988; Dohm, 1989), in part, due to difficulties in treating non-uniform order parameter profiles below $T_c$.

For fully finite systems, with edges (in 3d) and corners, expansions of the form (10.52) will have all the "geometry" terms present, which are proportional to inverse integral powers of $1/L$, in any $d$. However, the scaling contribution (10.1) is proportional to $L^{-d}$ with, generally, varying $d$. When $d$ passes through integer values (2 or 3), one should anticipate "resonant" divergences in $Y$ and $f_{ns}^{(c)}$, yielding logarithmic terms (Privman, 1988a), reminiscent of the mechanism for the logarithmic specific heat as $x \to 0$ (Section 1.3). Without going into technical details (Privman, 1988a), let us quote the final result for $d = 3$:

$$f(t; L) = f_{ns}^{(b)}(t) + \frac{1}{L} f_{ns}^{(s)}(t) + \frac{1}{L^2} f_{ns}^{(e)}(t) + \frac{u \ln(L/a)}{L^3} + \frac{1}{L^3} f_{ns}^{(c)}(t)$$

$$+ \frac{1}{L^3} \tilde{Y}(K, tL^{1/\nu}) + \ldots,$$

(10.54)

where the terms are ordered by their significance at $T_c$ (and $H \equiv 0$). The 2d result is similar; $a$ is some arbitrary microscopic length scale introduced to make $L/a$ dimensionless; $u$ is a universal amplitude. For integer $d = 2, 3$ at $T_c$, there are thus two universal contributions, $\tilde{Y}(0)L^{-d}$ and $uL^{-d}\ln L$.

Thus far, the scaling prediction (10.54) has been tested only at $T_c$, where the universal, and non-universal $L^{-d}$ contributions cannot be separated. However, one can study the $uL^{-d}\ln L$ term. In 3d, Lai and Mon (1989b) obtained, by MC estimation,

$$u = 0.009 \pm 0.005 \quad \text{and} \quad 0.012 \pm 0.003$$

(10.55)

for the SC and BCC lattice Ising models, respectively. In 2d, conformal invariance predictions for the universal term $uL^{-2}\ln L$ in the critical-point free energy are available (Cardy and Peschel, 1988). Their results are for geometries much more general than polygons considered here. In fact, relations of the type (10.54) apply generally to systems with curved boundaries, or in curved-space geometries. The amplitude $u$ is then a global property, and, in 2d, it is known exactly in many cases (Cardy and Peschel, 1988).

If the only source of the $uL^{-d}\ln L$ contribution are corners, with otherwise flat geometry and boundaries, then we can attribute $u_j$ to each corner (see further below), so that

$$u = \sum_j u_j.$$
In 2d, each corner is fully specified by its inner opening angle $\delta_j$. Cardy and Peschel (1988) derived the result

$$u_j = \frac{c \delta_j}{24\pi} \left[ 1 - \left( \frac{\pi}{\delta_j} \right)^2 \right] \quad (d = 2). \quad (10.57)$$

Universal terms proportional to $L^{-d} \ln L$ arise naturally in integral-$d$ Gaussian models (e.g. Cardy and Peschel, 1988; see also Section 10.3.4 below). However, the mechanism of emergence of such terms described above does not apply for Gaussian models (Privman, 1988a).

Regarding other instances of logarithmic size dependence, Privman (1985) considered the finite-size (constrained) behaviour of polymer chains (SAWs) of $N$ monomers, and emphasized a universal term $(\gamma - 1)N^{-1} \ln N$ in the free energy of both unconstrained and constrained chains, in the fixed-$N$ ensemble. Duplantier (1987) found that self-avoiding $D$-dimensional manifolds (membranes), of $N$ “tethered” monomers embedded in $d$ space dimensions, generally have a universal shape-dependent term $(\gamma - 1)N^{-1} \ln N$ in the free energy. (In both cases, the free energy is defined per monomer and per $k_B T$ as usual.)

### 10.3.2 Surface and corner free energies

We now turn again to systems with free boundary conditions and consider expansions away from $T_c$ (relations (10.50) and (10.51)). Although the coefficients $f^{(b)}(t)$, $f^{(s)}(t)$, etc., are defined with $L \to \infty$ and $t \neq 0$, one can also consider their limiting behaviour as $t \to 0$. For simplicity, we take $t \to 0^+$ (and $H = 0$ as before). The critical behaviour of the bulk, surface, etc., free energies must match that predicted by (10.54), in the limit $L \gg (K_t t)^{-\nu}$. In fact, the singular contributions to $f^{(b)}(t)$, etc., come only from the asymptotic form of the scaling term ($\tilde{Y}$) in (10.54). (Note that, for a fully finite system of volume $V$ and surface area $S$, the surface free energy per unit area is actually given by $(V/LS)f^{(s)}$. Note also that $f^{(c)}$ is dimensionless in our definition.)

Privman (1988a) used standard arguments (e.g. Fisher, 1971) to conclude that, for large positive $x$, one has

$$\tilde{Y}(x) = y_3 x^{3\nu} + y_2 x^{2\nu} + y_1 x^{\nu} - vu \ln x + y_0 + \ldots \quad (3d), \quad (10.58)$$

$$\tilde{Y}(x) = y_2 x^{2\nu} + y_1 x^{\nu} - vu \ln x + y_0 + \ldots \quad (2d), \quad (10.59)$$

where the universal coefficients $y_i$ (and $u$) depend on $d$ and geometry. While the results that follow for the bulk, surface and edge free energies are quite standard (e.g. $f^{(s)}_s \sim t^{(d-1)\nu}$, etc.; note also that $y_d = Q_+(0)$, see (1.18) and (1.19)), the leading singular behaviour of the corner free energy is predicted
(Privman, 1988a) to take the form

\[ f_s^{(\epsilon)}(t) \approx -v u \ln|t|, \]  

(10.60)

where we also included the appropriate \( t < 0 \) result. Relations (10.56) and (10.57) further suggest that each corner in a 2d or 3d system will have its own singular free-energy contribution \(-v u_j \ln|t|/L^d\), with universal amplitudes \(-v u_j\).

### 10.3.3 Universal surface amplitude combinations

The surface free energy (\( \propto f^{(\epsilon)} \), as mentioned above) can be studied as a function of \( t, H \) and also surface-coupling enhancement \( J_{\text{wall}}/J \) and surface field \( H_{\text{wall}} \). A variety of thermodynamic surface quantities can then be introduced by differentiating the surface free energy with respect to \( H, H_{\text{wall}} \), etc. Several authors classified universal amplitude combinations that can be formed from surface magnetizations, susceptibilities and other quantities; in most cases, these are quite similar to the bulk universal combinations defined in Section 2. A survey of these studies is outside the scope of the present review. However, we list below some literature on surface amplitudes. Universal surface amplitude combinations have been considered by Okabe and Ohno (1984), Eisenriegler (1984), Diehl and Deitrich (1985) and Diehl et al. (1985). The theoretical results are much more limited than for bulk amplitude combinations and are restricted to mean-field theory or, in some cases, first-order \( \epsilon \)-expansions. For an MC study, see Kikuchi and Okabe (1985b). Huang et al. (1989) considered certain universal amplitude combinations at wetting transitions.

### 10.3.4 Finite-size scaling for the surface tension

#### 10.3.4.1 Periodic interfaces near \( T_c \)

Consider \( L \times \bar{L} \) rectangles in 2d, or \( L^2 \times \bar{L} \) boxes in 3d, with periodic boundary conditions in the \( L^{d-1} \) dimensions. A “floating” periodic interface of size \( L^{d-1} \) can be introduced by imposing antiperiodic or \(+ -\) boundary conditions along \( \bar{L} \) (as before, we use the Ising nomenclature for boundary conditions). We also consider boxes with no interfaces, with, respectively, periodic and \( + + \) boundary conditions along \( \bar{L} \). The difference in the free energies can then be formed to get an excess interfacial free energy, \( \sigma(t; L, \bar{L}) \), defined per \( k_B T \) and per unit area (\( L^{d-1} \)). As \( L, \bar{L} \to \infty \), this quantity approaches the bulk surface tension \( \Sigma(t) \) for fixed \( t < 0 \), and vanishes for fixed \( t \geq 0 \).
For finite-size boxes, the interfacial free energy is described by the scaling ansatz

\[ \sigma(t; L, \bar{L}) \approx L^{1-d} \Xi(K, tL^{1/\nu}; \bar{L}/L), \]

entailing a universal amplitude \( \Xi(0; \bar{L}/L) \). This scaling form was introduced by Mon and Jasnow (1985b) in their study of 2d Ising models, in the \(+−\) square \((\bar{L} = L)\) geometry. They obtain an MC estimate \( \Xi(0; 1) \approx 1.3 \).

Abraham and Švrakić (1986) checked (10.61) analytically in the \(+−\) case for the general aspect ratio \( \bar{L}/L \). Their results include

\[ \Xi(0; 1) \approx 1.53 \quad (+− \text{ Ising}), \]

which is higher than the MC value. Jasnow et al. (1986) considered the three-state Potts model in 2d \((++)\) is replaced by fixing Potts variables at the two \( L \) walls at the same value, while \(+−\) corresponds to fixing them in different states). Their MC studies yield \( \Xi(0; 1) \approx 2.1 \). MC results have also been obtained in 3d (Ueno et al., 1989), testing (10.61) for \((+−) \times (\text{periodic})^2\) cubes. Their studies (Ueno, 1990) yielded the estimate \( \Xi(0; 1) = 1.4 \pm 0.15 \).

The antiperiodic Ising geometry was considered by Park and Den Nijs (1988). They calculate \( \Xi(0; \bar{L}/L) \) exactly, by Coulomb gas techniques. For example, they get

\[ \Xi(0; 1) = \ln(1 + 2^{3/4}) \approx 0.9865 \quad \text{(antiperiodic Ising)}. \]  

Note that in the limit of an infinite cylinder, \( \infty \times \bar{L} \), it would be more convenient to rescale (10.61) with \( \bar{L} \). If, however, we use the \( L \)-scaled form consistently, then the result by Onsager (1944) for antiperiodic Ising cylinders is

\[ \Xi(0; z) \approx \frac{\pi}{4z} \quad \text{as} \quad z \to 0, \]

which simply means that \( \sigma(0; \infty, \bar{L}) \approx \pi/(4\bar{L}) \).

Park and Den Nijs (1988) considered the equivalent of antiperiodic boundary conditions for general \( 0 \leq q \leq 4 \) Potts models (see the original work for details). Their results for amplitudes apply for general \( q \). For example, for \( q = 3 \), the amplitude in (10.63) is replaced by \( \approx 1.1874 \), while \( \pi/4 \) in (10.64) is replaced by \( 4\pi/15 \). The results for \( q = 3 \) and 2 (Ising), for three aspect ratios, including the case \( \bar{L}/L = 1 \), were also checked by MC simulations (Park and Den Nijs, 1989) with an accuracy of about 0.3%.

10.3.4.2 Pinned interfaces near \( T_c \)

For interfaces with pinned ends, finite-size scaling behaviour near \( T_c \) has been studied only in one 2d geometry. Consider an Ising strip, \( L \times \infty \), with
one side at the $x$ axis ($-\infty < x < \infty$), at $y = 0$, the other side at $y = L$, parallel to the $x$ axis. At $y = 0$, we fix the boundary conditions + for $x > 0$, and − for $x < 0$. At $y = L$, we fix + for $x > l > 0$, and − for $x < l$. These boundary conditions induce an interface across the strip, of length $L/\cos \phi$, tilted at an angle $\phi \equiv \tan^{-1}(l/L)$ with respect to the $y$ axis. The end points of the interface are thus pinned at $(x, y) = (0, 0)$ and $(l, L)$. The excess interfacial free energy is then given by

$$\sigma(t; L; \phi) = -\frac{\cos \phi}{L} \ln \left( \frac{Z_{+-}}{Z_{++}} \right),$$

(10.65)

where $Z_{+-}$ is the partition function of the original strip, while $Z_{++}$ is the partition function with all + boundary conditions. Strictly speaking, $Z_{+\pm}$ can be defined only for finite rectangles $L \times 2L$, i.e. $-L < x < L$. However, their ratio is finite as $L \to \infty$.

For the 2d Ising model, the partition function ratio entering (10.65) has been calculated exactly for $t < 0$ and arbitrary $L$ (Švrakić et al., 1988). Abraham (1989) has derived the scaling behaviour as $t \to 0^-$. His results suggest the following pattern of finite-size behaviour of the interfacial free energy (Privman, 1989): there is a scaling contribution similar to (10.61), as well as “background” terms of orders $1/L$ and higher; both are $\phi$ dependent. However, there is also an additional term $L^{-1} \ln L$ in $\sigma(t; L; \phi)$, additive to the scaling and background contributions, with a universal amplitude ($= 1$ here).

This behaviour of the interfacial free energy is reminiscent of the bulk free energy for non-periodic boundary conditions described in Section 10.3.1. Generally, for non-periodic interfaces near $T_c$, we thus expect (Privman, 1989) to have a universal term $\propto L^{1-d} \ln L$, in addition to the hyperuniversal scaling term (10.61) and background contributions. The universal amplitude will, however, depend on geometrical features such as boundary conditions, etc.

10.3.4.3 Interfacial fluctuations below $T_c$

For fixed $t < 0$, rough interfaces undergo Gaussian “capillary” fluctuations which are a critical-type soft-mode phenomenon. Note that 2d interfaces are rough for $0 < T < T_c$ whereas, for 3d lattice models, the interfaces are rough only for $T_R \leq T < T_c$, where $T_R > 0$ is the roughening temperature (which is actually interface orientation dependent). For finite-size systems, these fluctuations contribute a universal (geometry-dependent) term $vL^{1-d} \ln L$ to the difference $\sigma(t; L) - \sigma(t; \infty)$ (see Gelfand and Fisher, 1988; Privman, 1988b). Generally, for non-periodic interfaces, there will also be non-universal contributions of orders $L^{-1}, L^{-2}, \ldots, L^{1-d}, \ldots$ to the difference. Note,
however, that unlike for the critical-point behaviour, there is no “scaling”
term here. Also, a Gaussian contribution $S^{-1} \ln L$, with a universal coefficient
1, as well as a non-universal term $\sim S^{-1}$, are present even for periodic
interfaces, where $S \propto L^{1-d}$ is the interface area.

We will not survey all the available results (Gelfand and Fisher, 1988;
Privman, 1988b; Švrakić et al., 1988; Privman and Švrakić, 1989), but only
mention that in the 2d Ising strip geometry considered in the preceding
section in connection with critical-point behaviour, with the pinned-end
interface inclined at an angle $\phi$, we have, for small $\phi$ (Privman, 1988b, 1989),

$$v = \frac{1}{2} - \frac{1}{4} \phi^2 + O(\phi^4), \quad (10.66)$$

which is different from the critical-point amplitude of the $L^{1-d} \ln L$ term
(which is 1 for all $0 < \phi < \pi/2$, in this geometry).

10.4 Experimental results

Quantitative experimental results on (static) critical-point finite-size amplitudes
and, more generally, on scaling functions are quite rare (see Barber’s (1983)
review). A notable exception is the work of Gasparini and coworkers on the
specific heat $C_p$ and superfluid density $\rho_s$ in confined geometries near the
superfluid transition in $^4$He (Chen and Gasparini, 1978; Gasparini et al.,
1984; Rhee et al., 1989). Initially, no simple application of the finite-size
scaling relations of type (10.1) seem to explain the data, but a recent reanalysis
by Huhn and Dohm (1988), using standard RG recursion relations with no
adjustable parameters, has yielded an excellent fit to some of the specific heat
measurements. It turns out that the appropriate scaling “shift” variable
emerging naturally in these calculations is

$$\tilde{t}_L \propto T_c(L) - T_{\text{max}}(L), \quad (10.67)$$

rather than

$$t_L \propto T_c(\infty) - T_{\text{max}}(L), \quad (10.68)$$

where $T_{\text{max}}(L)$ locates the specific heat maximum, and $T_c(L)$ is an intrinsic
reference temperature obtained in the theory. It seems likely to us that the
same theory will also explain the recent $\rho_s$ data of Rhee et al. (1989).

Low-energy electron diffraction (LEED) experiments probing 2d surfaces
and monolayers (and also some specific heat measurements) can be analysed
to extract some finite-size amplitudes (e.g. Clark et al., 1986). Due to
experimental difficulties, comparison with theoretical results, including effective
critical-point shift and certain conformal invariance-related amplitudes, has
been possible only on a semiquantitative level (see a review by Einstein
(1987)).
11 Concluding remarks

Universal combinations of critical amplitudes emerge in all subfields of critical phenomena theory and experiment. The available research results, reviewed in Sections 1–10, reveal substantial differences in the level of understanding of the amplitude-related universal properties, and in numerical values, ranging from a rather complete and consistent picture for, for example, bulk n-vector models, to partially explored (e.g. polymers) or rapidly developing (e.g. finite-size) fields. Overall, the authors hope that this review provides both a status summary for the better explored topics, and an introduction to ongoing research in less developed subfields.

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**Note added in proof**

See page 364.
Note to Chapter 1, added in proof

The purpose of this note is to cite results which were published, or brought to the authors’ attention after the completion of the main text of the review. Some of the appropriate modifications and updates were incorporated in the main text. The materials collected here are those that would require more than minor changes in the main text. Our objective is to provide reference to, and citation of these recent results. Thus, no comprehensive or detailed coverage is attempted here.

Critical End Points. Our discussion of the multicritical points in Section 6.6 has been focused on the bicritical behaviour. Recently, however, a comprehensive study of the critical amplitudes and their universal combinations at critical endpoints has been initiated: Fisher and Upton (1990a, b), Fisher and Barbosa (1990), Barbosa and Fisher (1990), Barbosa (1991).

Quantum Critical Phenomena. Kim and Weichman (1991) classified hyperuniversal combinations of amplitudes in two-dimensional Bose systems undergoing phase transitions at $T = 0$ as a function of a field-like variable $\mu$ (see Fisher et al., 1989). This work also allowed extension of certain proposals (Fisher, 1987; Fisher et al., 1990) on the ac conductivity of granular superconductors, and of systems undergoing metal-insulator transitions.

Finite-Size and Surface Amplitudes. A combination of the Coulomb gas, conformal invariance, and transfer matrix methods to study finite size amplitudes in $2d$ models, similar to studies reviewed in Section 10.2.4, have been extended to chiral models and to incommensurate phases, see Li et al. (1990), Park (1990), Park and Widom (1990a, b), Li and Park (1991).

Several new results, both analytical and numerical, have appeared, involving conformal field theory studies of $2d$ models. Review articles include Christe and Mussardo (1990), and Henkel (1991). For new amplitude results in systems with corners see Kleban and Vassileva (1990).

Results in dimensions higher than two include new $\varepsilon$-expansion calculations in parallel plate (film) geometry (Krech and Dietrich, 1991), as well as general conformal invariance relations among certain finite-size and surface amplitudes, Cardy (1990). Privman (1990) predicted new, universal-amplitude logarithmic surface contributions for the superfluid density in helium films. These were apparently observed experimentally (Gasparini, 1990), and recently confirmed by numerical Monte Carlo simulations (Mon, 1991).

Further results on universal surface and defect associated amplitudes were presented by Berche and Turban (1990), and by Landau and Binder (1990). See also Binder and Landau (1990) for, respectively, $2d$ and $3d$ Ising models.
Conformal Charge. Singh and Baker (1991) developed a series-analysis approach to calculating the conformal charge based on Cardy’s ideas (e.g. Section 10.2.4) which lead, for instance, to relations (1.17); see the main text. Results were obtained for the 2d Ising and 3-state Potts models (Singh and Baker, 1991). An interesting consideration of the amplitudes associated with the various moment definitions of correlation lengths has been initiated by Singh (1991).

Percolation, Walks, Vesicles, Polymers. The controversial amplitude ratio $\Gamma/\Gamma'$ for percolation, Section 6.5.1, has been further studied by Lee (1990) and by Matthews-Morgan and Landau (1989). A study of universal properties of vesicles (i.e., closed surfaces with statistical weight assigned to the surface area and to the enclosed volume) was initiated, with most of the results obtained for polygons in 2d (Fisher, 1989; Maggs et al., 1990; Camacho and Fisher, 1990).

Lam (1990a,b) obtained new results on certain cluster radius ratios (Section 9.1) for self-avoiding walks in 2d and 3d. Amplitudes at the collapse transition of polymers in 2d were studied by Chang et al. (1990). Conformal invariance and associated universal distance ratios for self-avoiding walks were further investigated by Caracciolo et al. (1990). Finally, Zylka and Öttinger (1991) calculated several universal properties for dilute polymer solutions undergoing shear flow.

Field Theory in 3d. Some very recent results for $T < T_c$ were cited at the end of Section 6.2.3. Notably, Dohm and coworkers (work referred to in the main text, Section 6.2.3) extended previous approaches to $n > 1$. These authors use a $d = 3$ field theory combined with the minimal subtraction scheme, which is convenient for extensions to the ordered phase ($T < T_c$), as well as to nonzero $k$, $\omega$ and $L^{-1}$ (finite-size effects). An interesting general observation is that higher-order perturbative corrections have a rather small effect on the amplitude ratio values provided the geometric factors such as $K_d$, equation (6.46), are properly treated. The authors advocate keeping the quantity $K_d\Gamma(1 + \frac{k}{2})\Gamma(1 - \frac{k}{2})$ unexpanded.

Binary Fluid Experiments. Zalczer and Beysens (1990) measured several leading and correction amplitude combinations for the mixtures of triethylamine and water; see also Gastaud et al. (1990). An earlier work by Beysens et al. (1983) reported some additional amplitude estimates for binary fluids.

Experiments on Dynamics at the Superfluid Transition. Recently, Lipa (1991) presented new experimental values for the ratio $R_\lambda$ at the superfluid transition
in $^4$He, in the range $10^{-7} \leq t \leq 10^{-4}$, correcting earlier data by Lipa and Chui (1987). These results agree with the theory and with other experiments described in Sections 6.7.4 and 8.2.

Finally, we cite experimental work on the superfluid transition in $^3$He–$^4$He mixtures for which the amplitude ratios were measured, and compared with theory, by Meyer and coworkers (see Meyer, 1988).

**Random Media.** Recent numerical simulations in 2d (Andreichenko et al., 1990; Wang et al., 1990a, b) have helped clear up some of the earlier theoretical controversies. For a review of numerical studies for both random bond- and site-dilution see Selke (1991). The theoretical understanding is, however, far from being complete. Indeed, for weak bond-dilution, Ziegler (1990, 1991a, b) has argued for the presence of a new phase and for slow finite-size transients.

As far as amplitudes are concerned, Narajan and Fisher (1990) have derived a general relation between the ratios $R^T_\infty$ for the pure system and for a system with marginally irrelevant quenched disorder ($x = 0$).

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