Finite size scaling renormalization technique is studied by examining a test model, which is exactly
solvable, and the problems of lattice animals and self-avoiding walks. Bulk estimates of the leading
irrelevant RG eigenvalue are found and compared with independent estimates from finite-size data.
We introduce an extrapolation technique which is applicable to non-monotonic sequences of exponent
approximants.

1. Introduction

One of the applications of the finite-size scaling theory\(^1,2\) (see ref. 3 for a review)
is the “phenomenological renormalization” method introduced by Nightingale\(^4,5\)
(see also refs. 6 and 7 and the recent review: ref. 8). Accurate numerical estimates
of critical exponents of various two-dimensional models are obtained by this
method. The asymptotic rate of convergence of approximants to a critical
exponent, say \(\nu\), is determined by the form of corrections to finite size scaling.
Various sources of these corrections have been discussed\(^9\), but our present
theoretical understanding of some types of corrections is still rather limited.

The convergence of approximants is normally governed\(^9\) by the bulk irrelevant
scaling field which has the leading negative renormalization group (RG) eigen-
value \(\gamma = -\gamma (\gamma > 0)\). One thus anticipates

\[
\nu_L = \nu [1 + CL^{-\gamma} + E(L)],
\]

where \(\nu_L\) is the approximant calculated\(^4,8\) from data for strips of width \(L\) and
\((L - 1)\) according to

\[
1 + \frac{1}{\nu_L} = \ln \left[ \frac{\partial \xi (T, H, L)}{\partial T} \right]_{T=\tau_c}^{H=0} \left\{ \frac{\partial \xi (T, H, L - 1)}{\partial T} \right\}
\]

and where \(E(L)\) denotes more rapidly vanishing terms which are normally higher
negative powers of \(L\) (some of which were classified in ref. 9). Several authors\(^10-16\)
have observed that the apparent convergence exponent \(x\) found by fitting the
approximants for $L \leq 10$ to

$$v_L = v_0(1 + cL^{-x}),$$  \hspace{1cm} (1.3)

where $v_0$, $c$ and $x$ are adjustable parameters, is usually larger than the exponent $y$ (which can be found from bulk calculations, e.g. by using series analysis). In fact the ratio $x/y$ ranges from above 1 to 4 (one frequently finds $x \approx 2$ and $y < 1$). This apparent discrepancy is due to the fact\textsuperscript{8,10} that the $L^{-x}$ term in (1.3) approximates the decay of both the $L^{-y}$ and higher order terms $E(L)$ in (1.1); more elaborate techniques\textsuperscript{9} for studying the convergence do yield estimates for $y$ consistent with bulk values. In ref. 9 the $d = 2$ percolation and $q = 3$ Potts models were considered: here we report some further studies on the convergence properties of the “phenomenological renormalization” method.

In section 2, we derive the leading correction to finite size scaling in the infinite range (Husimi–Temperley) model. This model provides an example of $x < y$ and shows that there is no general mechanism underlying the rule of thumb $x \sim 2y$. It also provides a convenient test case to evaluate various techniques\textsuperscript{9–17} of estimating $v$ or $y$. We extend and test a method for estimating $v$ and $y$ which is based on the series analysis technique of refs. 18 and 19 and which was introduced in ref. 9. Non-monotonic $v_L$ sequences can be analyzed by this method [unlike a fit of (1.3)].

In sections 3 and 4, we study the problems of lattice animals (LA) and of self-avoiding random walks (SAW), respectively. In both cases we first obtain bulk series estimates of $y$ and compare them with previously known results. These problems have rich patterns of convergence of $v_L$; some sequences are non-monotonic. For the monotonic sequences one finds $x/y \approx 3$ (for LA, ref. 15) and $x/y \approx 2.5$ and 4 (for SAW sequences of refs. 16 and 11, respectively). Furthermore, these monotonic SAW sequences appear to approach values of $v$ which are larger than the presumably exact value\textsuperscript{20}$v = 3/4$; thus these sequences will exhibit non-monotonic behavior for higher $L$ values\textsuperscript{16}.

In the balance of this section we introduce some further notation and comment on some properties of data fits to the form (1.3). The specific procedures used to fit the values of $v_0$, $c$ and $x$ in (1.3) are somewhat subjective\textsuperscript{10–16}. In order to be more concrete, we could use, e.g., a 3-point fit definition of the apparent convergence exponent $x \simeq x^{(3)}$: this is defined by requiring that (1.3) holds exactly for $L$, $(L - 1)$ and $(L - 2)$. Thus $x^{(3)}$ is a solution $x$ of

$$\frac{v_L - v_{L-1}}{v_{L-1} - v_{L-2}} = \frac{L^{-x} - (L - 1)^{-x}}{(L - 1)^{-x} - (L - 2)^{-x}},$$  \hspace{1cm} (1.4)

When $v$ is known or an accurate $v$ estimate is available from other sources, one can use a 2-point fit, $x \simeq x^{(2)}$, which is defined by requiring that (1.3) holds exactly
for $L$ and $(L - 1)$ with $v_0 = v$. This yields

$$x_L^{(0)} = -\ln\left(\frac{v_L - v}{v_{L-1} - v}\right) / \ln\left(\frac{L}{L - 1}\right).$$

(1.5)

Let us isolate the next-to-leading term in (1.1) by supposing

$$E(L) = E_i L^{-\gamma} + \cdots,$$

(1.6)

where normally $\gamma' > \gamma$. By (1.5), the asymptotic behavior of $x_L^{(2)}$ is

$$x_L^{(2)} = y + (E_i/C)(\gamma' - \gamma)L^{-(\gamma'-\gamma)} + \cdots$$

(1.7)

[and similarly for $x_L^{(3)}$ provided $(\gamma' - \gamma) < 1$ as is normally the case in the models of interest]. Since typical values of $(\gamma' - \gamma)$ for two-dimensional models are around 0.5 or less, the correction term is a slowly varying function of $L$. This could explain why apparently stable values of $x$, not equal to $y$, are usually found. However, (1.7) is valid only asymptotically and implies that $x > y$ when $E_i/C$ is positive. It was observed that when $x > y$ and $L \ll 10$ there is a cancellation between the $CL^{-\gamma}$ and $E(L)$ terms in (1.1). If $E(L)$ is accurately represented by (1.6), such cancellation demands a negative $E_i/C$, which in turn implies $x < y$. This contradiction forces us to conclude that higher order terms are needed in (1.6) and (1.7). Further arguments suggesting that the available finite size data for $L \ll 10$ is far from being in the $L \to \infty$ asymptotic regime and cannot be described by few power law terms in (1.1) were presented in ref. 9.

2. Infinite range model

We consider a "phenomenological renormalization" for an infinite range Ising model which yields mean-field theory in the thermodynamic limit. The Hamiltonian for an $N$ spin system is

$$\mathcal{H}/k_B T = -\frac{1}{2} \frac{K}{N} \left( \sum_{i=1}^{N} s_i \right)^2 - h \sum_{i=1}^{N} s_i, \quad s_i = \pm 1.$$

(2.1)

One can show that the partition function of this model is proportional to

$$Z_N(h, K) \propto \int_{-\infty}^{\infty} d\mu \ e^{N \xi(\mu, h, K)},$$

(2.2)

where

$$g(\mu) = -\frac{1}{2} K \mu^2 + \ln \cosh(K \mu + h).$$

For $T < T_c$ (given by $K_c = 1$), the magnetization is a positive (for $h \geq 0^+$) or a
negative (for $h \leq 0^-$) root of
\[ m = \tanh(Km + h). \tag{2.3} \]

For finite $N$, the thermodynamic quantities can easily be calculated on a computer from
\[ Z_N(h, K) = \sum_{i=0}^{N} \binom{N}{i} \exp \left\{ N \left[ \frac{K}{2} \left( 1 - \frac{2i}{N} \right)^2 + h \left( 1 - \frac{2i}{N} \right) \right] \right\}, \tag{2.4} \]

where the summation over configurations has been reduced to a summation over possible magnetization values $m_i = 1 - 2i/N$ ($i = 0, \ldots, N$) because $\chi^s(s_i)$ depends on spins only through $m(s_i) = (1/N) \Sigma_{i=1}^{N} s_i$.

We concentrate on the susceptibility $\chi_N(h, K)$. The finite-$N$ scaling form for $\chi_N(0, K)$ was conjectured\(^{22}\) to be
\[ \chi_N(0, K) \approx |t|^{-\gamma} F(N|t|^{d_x^*}), \tag{2.5} \]

where $t \propto K_c - K \sim T - T_c$, $\gamma = 1$, $v = \frac{1}{2}$ and $d_x = 4$, and this conjecture has been tested\(^{22,23}\) both analytically and numerically. It can also be obtained from universality by considering finite size scaling\(^{24}\) for Ising spin systems at $d > d_c$; use of the finite range model is equivalent to ignoring gradient terms in the Ginzburg–Landau expansion [see (2.7), below] and suppressing inhomogeneous configurations already at finite volume $V$, thus the only relevant length is $L_0 \equiv V^{1/d}$ and the arguments of ref. 24 suggest
\[ \chi_N(h, K) \approx |t|^{-\nu} R(L_0|t|^{2d_x^*}, h|t|^{-d}), \tag{2.6} \]

where $\Delta = 3/2$. This result obviously reproduces (2.5) when $h = 0$ (notice $L_0 \propto N^{1/d}$). In order to derive the leading correction to the scaling form (2.6) we must include the second irrelevant term ($\propto \sigma^6$) in the Ginzburg–Landau expansion which, for this infinite range model, is done in terms of the homogeneous order parameter $\sigma(\sim m)$
\[ f_{\nu}(h, t) \approx -\frac{1}{V} \ln \int_{-\infty}^{\infty} d\sigma \exp \left[ -\int_{V} d^d x \left( \frac{t \sigma^2}{2} - h \sigma + u \sigma^4 + u^2 \omega \sigma^6 + \cdots \right) \right], \tag{2.7} \]

where $u$ is the “dangerous irrelevant variable”\(^{25}\) [the bulk free energy density is $f_{\nu}(0) \propto 1/u$, where the superscript $(s)$ denotes the “singular part”]. Our definition of the coefficient of the $\sigma^6$ term allows $f_{\nu}(h, t)$ to remain finite when $w \to 0^+$ at fixed $u > 0$. (Note that, in calculating the leading correction to scaling, $u$ and $w$ will be set to constants in the final expression.) Now we proceed in two steps: first isolate the “dangerous” dependence on $u$ for finite $V$,
\[ f(\nu) = \frac{1}{V} X_{\left( \frac{V}{u}, t, h, \sqrt{u}, w \right)}, \tag{2.8} \]
where (2.8) is verified by inspection of (2.7) [rescale $\sigma$ with $\sqrt{u}$; note that $X(x_1, x_2, x_3, x_4) \rightarrow x_1 \hat{X}(x_2, x_3, x_4)$ in the limit $x_1 \rightarrow \infty$]. Next we use the bulk RG transformation with rescaling factor $b$, linearized near the Gaussian fixed point to write

$$f_{\Psi}^0(h, t, u, u^2w) \approx b^{-d} f_{\Psi}^0(hb^{d+d/2}, tb^2, ub^{4-d}, (u^2w)b^{5-2d}).$$

(2.9)

(The artificial dimensionality $d > 4$ and volume $V \propto N$ are introduced so that the conventional form for the RG transformation$^{26,27}$ may be used.) Upon substituting $b \propto |t|^{-1/2}$ and using (2.8) for the RHS of (2.9), we obtain

$$f_{\Psi}^0 \approx t^2 G(h|t|^{-3/2}, |t|^2L_0^2u^{-1}, w|t|).$$

(2.10)

Finally, we restore the notation $L_0 \propto N$, replace $u$ and $w$ by their constant values at $T_c$ and $h = 0$ and "scale" everything with $N$, as is appropriate$^9$ for studying the convergence of finite-size results. We have

$$f_{\Psi}^0 \approx N^{-1} Y_f(hN^{3/4}, |t|N^{1/2}, N^{-1/2}).$$

(2.11)

giving

$$x_{N,c} \approx N^x,$$

as $N \rightarrow \infty,$

(2.13)

from the usual "phenomenological renormalization" relation

$$g \approx g_N \equiv \ln\left( \frac{x_{N,c}}{x_{N-1,c}} \right),$$

or

$$\ln (N) \quad \ln (N - 1).$$

(2.14)

From the general analysis of ref. 9 we expect [compare (1.1)]

$$g_N = g + \text{Const} \cdot N^{-y} + \cdots,$$

(2.15)

where $y = \frac{1}{2}$ because the argument $N^{-1/2}$ in (2.12) plays the role of $L^{-|\psi|}$ in the usual finite size scaling form$^9$. We also define $x_N^{(2)}$ by following (1.3),

$$x_N^{(2)} = -\ln\left( \frac{g_N - \tilde{g}}{g_{N-1} - \tilde{g}} \right).$$

(2.16)

In table I we list $g_2, \ldots, g_{11}$ and $x_N^{(2)}, \ldots, x_N^{(1)}$ (as well as the values for $N = 20, 40, \ldots, 200$, for illustration). We concentrate on $N \leqslant 11$ because $L = 11$ or 10 are the largest $L$ values in most "phenomenological renormalization" calculations. If no extrapolation is done, the values of $g_N$ with $N \leqslant 11$ provide a poor approximation to $g = \frac{1}{2}$. One may fit $g_N$ to the form (1.3) by choosing$^{11,15}$ an $x$ range so
that the plot of $g_N$ against $N^{-x}$ is more or less on a straight line (by visual inspection). The result is $x = 0.4 \pm 20\%$ and
\[
g = 0.502 \pm 0.004. \tag{2.17}
\]

As usual\textsuperscript{10-16}, the fitted $g$ value is an order of magnitude better than the best finite size value $g_{11} = 0.5278 \ldots$. Note that in this model $x < y$, in contrast to the situation for most of the reported "phenomenological renormalization" calculations.

We now introduce a generalized version of the transformation used in ref. 9. Suppose that an estimate of $g$, say $g_0$, is available. In this exercise we use $g_0 = 0.502$ from (2.17). (Also, for generality, we use the notation $v_L$, $v$, $v_0$ in place of $g_N$, $g$, $g_0$.) Following ref. 9, we define
\[
r_L = v_0 + \frac{v_L - v_0}{L} = v_0 + \frac{v - v_0}{L} + \text{Const} \cdot L^{-(y+1)} + \cdots, \tag{2.18}
\]
and construct a sequence
\[ a_0 = 1/v_0, \quad a_j = a_{j-1}/r_{j+1} \quad (\text{for} \quad j \geq 1), \]
and a generating function
\[ f(z) = \sum_{n=0}^\infty a_n z^n = (v_0 - z)^{-\gamma} [1 + A(v_0 - v)^\gamma + D(z)], \]
which has a divergence with exponent
\[ \Gamma = 2 - \frac{\gamma}{\nu_0} \quad \left( \text{or} \quad 2 - \frac{g}{g_0} \right) \]
and also a confluent term with exponent \( \gamma \). Additional background, \( D(z) \), is present due to higher order terms in (1.1) and also as a result of the transformation. This background will normally interfere with the analysis of the \( A(v_0 - v)^\gamma \) term mainly through a linear piece, \( D_1(v_0 - v) \), and this "interference" will be appreciable when \( \gamma \) is close to 1. The series analysis method of refs. 18 and 19 will be used to estimate \( \Gamma \) and \( \gamma \). This method involves a transformation (with a trial \( \gamma \) value) which makes the confluent term in (2.20) analytic at the correct \( \gamma \); \( \Gamma \) is estimated, as a function of the trial \( \gamma \), by forming several central Padé approximants. A region of "confluence" of the several \( \Gamma(\gamma) \) curves thus obtained is normally present and close to the correct \((\gamma, \Gamma)\) (see ref. 19 for details).

Fig. 1 shows the \( \Gamma(\gamma) \) curves produced from the sequence \( g_2, \ldots, g_{11} \). A clear "confluence" region is observed suggesting
\[ \gamma = 0.47 \pm 0.04, \]
\[ \Gamma = 1.006 \pm 0.004 \rightarrow g = (2 - \Gamma)g_0 = 0.499 \pm 0.002. \]
In this test case both \( \gamma \) and \( g \) estimates are improved. More generally, if \( \gamma \) is well away from 1 (and \( v_0 \) is close to \( v \)) we expect a correct \( \gamma \) range to be estimated. In general, the value of \( \nu \) will show no appreciable improvement over the \( \nu \) estimated from, say, a fit of (1.3), because the sequences are very short and this may introduce systematic errors into any extrapolation method (see, e.g., section 4). Notice, however, that non-monotonic exponent sequences can be analyzed by this method (see, e.g., section 3).

3. Lattice animals

A central question of the lattice animal problem\(^{28,35}\) is to understand how the number of animals of size \( n \), \( a_n \), grows with \( n \). For large \( n \), \( a_n \) is given by
\[ a_n \approx \mu^n n^{-\gamma} (1 + bn^{-\theta} + \cdots), \]
where the prefactor $n^{-1}$ is a consequence\(^{(39)}\) of the dimensionality $d = 2$. The correction-to-scaling exponent $\theta = \nu \gamma$ has been estimated by Guttmann\(^{(34)}\) (who employed the enumeration of Redelmeier\(^{(36)}\)) to be $\theta = 0.87 \pm 0.06$ for the square lattice, while Margolina et al.\(^{(35)}\) found $\theta = 0.86 \pm 0.05$ for the triangular lattice\(^{(35,37)}\). Guttmann\(^{(34)}\) carried out his analysis while exponentiating the confluent terms so that

$$1 + bn^{-\theta} + \cdots \to \exp(bn^{-\theta} + \cdots).$$

We will instead estimate $\theta$ (for the square lattice\(^{(36)}\)) by the following method. Define

$$c_n = \frac{na_n}{(n-1)a_{n-1}} = \mu(1 + b'n^{-(\theta+1)} + \cdots)$$

and calculate

$$\mu_n(\theta) = \frac{n^{\theta+1}c_n - (n-1)^{\theta+1}c_{n-1}}{n^{\theta+1} - (n-1)^{\theta+1}}$$

(3.3)
as functions of the trial $\theta$ value. When $\theta$ is equal to the correct value, the $b'n^{-\theta+1}$ term of (3.2) cancels in (3.3), so $\mu_n(\theta)$ should be almost independent of $n$ (a similar transformation was used in a different context in ref. 9). The functions $\mu_{24}(\theta)$, $\mu_{23}(\theta)$, . . . , $\mu_{18}(\theta)$ are plotted in fig. 2, and their region of intersection falls well within the range of Guttmann’s estimates (which are marked in fig. 2). The apparent stability of these $\theta$ and $\mu$ estimates from intersection points may be misleading. Similarly stable results follow from an analysis of the triangular lattice enumeration data. Thus the exponentiation of the correction terms is not crucial: an ordinary “confluent-term” form (3.1) may be used. Finally, we combine the $\theta$ value of ref. 34 with the $\nu$ estimate ($\nu \approx 0.64$) of refs. 15 and 29 to obtain a reference $y$ range

$$y = \frac{\theta}{\nu} = 1.36 \pm 0.09,$$

which will be used in the following study of “phenomenological renormalization” results.

Finite size sequences for $\nu_L$ of (1.2) were calculated by Derrida and DeSeze for the lattice animal problem on a square lattice with periodic and helical
boundary conditions (b.c.). For periodic b.c. the sequence is rather short, however, it is monotonic and smooth and the fit (1.3) gives $v = 0.6408 \pm 0.0003$.  

For helical b.c., $v_2, \ldots, v_{10}$ are available, however this sequence is not monotonic so cannot be analyzed using (1.3). We applied the procedure described in section 2, using the initial estimate $v_0 = 0.6408$ from (3.5). The $\Gamma(y)$ curves are plotted in fig. 3; there are two regions of confluence; one at $y \approx 1.0$ and another at $y \approx 1.3$. These may be attributed to the analytic term, $D_1(v_0 - z)$, in (2.20), and to the leading irrelevant-variable term, respectively, because only the second "confluence" is consistent with (3.4). (This $y$ range is shown on the $y$-axis of fig. 3.) In order to obtain a reliable $\Gamma$ estimate, we choose a range which includes both "confluences", namely $0.9966 < \Gamma < 1.0020$ (marked in fig. 3), giving

$$v = 0.6412 \pm 0.0035.$$  

(3.6)
We also analyzed the abovementioned $v_L$ sequence\(^1\)) for periodic b.c. (again with $v_0 = 0.6408$). In this case only the “confluence” at $y \approx 1$, attributed to an analytic background, was found. We do not present a plot here, but mention that the $v$ estimate from this “confluence region” is

$$v = 0.6394 \pm 0.0067,$$

which should be compared with the estimate\(^1\)) (3.5). We believe that (3.6) and (3.7) represent realistic confidence limits provided no systematic errors are present due to the shortness of the sequences (see section 4).

4. Self-avoiding random walks

Several estimates of the bulk correction to scaling exponent for SAW suggest $\theta > 1$: (the $\epsilon$-expansion gives\(^3\)) $\theta \approx 1.15$, which value is supported by the recent Monte Carlo estimate\(^3\)) $\theta = 1.2 \pm 0.1$. In a series analysis\(^4\)), Grassberger did not find evidence for confluent correction terms with exponents less than 1. Recently, Nienhuis has obtained\(^6\)) the presumably exact value of $\nu = 3/4$. (He also found\(^7\)) a confluent term with $\theta = 3/2$. This correction, however, is apparently not the leading one.) We now apply a method similar to that used in section 3 to obtain a biassed (with $\nu = 3/4$) series estimate for $\theta$. We concentrate on the triangular lattice data\(^8\)) for $\rho_n = \langle R^n \rangle$ because for that lattice alone, $\Sigma_n \rho_n x^n$ seems to have no appreciable nonconfluent singularities near the origin. We have

$$\rho_n = r n^2 (1 + b n^{-\theta} + \cdots),$$

from which we construct the estimators

$$r_n(\theta) = \frac{n^{\theta - 2} \rho_n - (n - 1)^{\theta - 2} \rho_{n-1}}{n^\theta - (n - 1)^\theta},$$

as functions of the trial $\theta$ value. The curves $r_6(\theta), \ldots, r_{10}(\theta)$ are plotted in fig. 4. In the manner described in section 3, the curves $r_i(\theta)$ intersect in a region which estimates the true value of $\theta$. However, the size of this region and of the corresponding spread in estimates of $r$, cannot be accepted as a guide to the reliability of the estimation. We have instead chosen the “error brackets” in fig. 4 so that the spread in $r$ equals a typical spread of $r$ estimates $r \simeq n^{-2\nu} \rho_n$ as obtained without cancelling the confluent term. With this somewhat artificial prescription we have $r = 0.707 \pm 0.006$ and

$$\theta = 0.65 \pm 0.08.$$

This $\theta < 1$ value is not consistent with previous results. The reason for this discrepancy is not clear. We will see below that (4.3) is consistent with finite-size
estimates, which will be compared with

\[ y = \theta / v = 0.87 \pm 0.11 \]  \hspace{1cm} (4.4)

The available \( v_L \) SAW sequences\(^{11,16} \) indicate that the amplitudes of correction terms in this problem are especially large and that the data is far from the asymptotic \( L \rightarrow \infty \) behavior. Derrida\(^{11} \) calculated the sequence \( v_L \) for the square lattice with free b.c., which is non-monotonic, and the similar sequence with periodic b.c., which does increase monotonically (for \( L \leq 11 \)), but the fit of (1.3) gives \( v > 0.75 \). A similar monotonic \( v_L \) sequence, for \( L < 10 \), which increases above the expected value (\( v_{10} > 0.75 \)), was also found in ref. 16 using the \( N \rightarrow 0 \) limit\(^{14} \) of the "Quantum Hamiltonian" version of the \( N \)-vector model. For the last two sequences, non-monotonic behavior for larger \( L \) may be expected\(^{16} \).

Let us analyze the "Quantum Hamiltonian" \( v_L \) sequence with the procedure described in section 2. (We use the exact \( v_0 = v = 0.75 \) and replace \( r_{j+1} \) by \( r_{j+2} \) in (2.19) because \( v_2 \) was not calculated in ref. 16.) The curves \( \Gamma(y) \) are plotted in fig. 5 and the central estimate of (4.4) is marked on the \( y \)-axis. There is a well-defined "confluence region" which is consistent with (4.4). Notice, however, that the corresponding \( \Gamma \) range is not at the expected (exact) \( \Gamma = 1 \) but rather at \( \Gamma \approx 0.991 \). This \( \Gamma \) value would give \( v \approx 0.757 \). The discrepancy suggests the presence of background terms of appreciable amplitude and illustrates a danger which is common to series analysis methods when applied to short series: apparently stable but inaccurate results.
The $v_L$ sequences of ref. 11, when analyzed in a similar manner (plots not presented here) gave "confluence regions" which are wider in $y$: they covered the ranges of $0.75 < y < 1.15$ (free b.c.) and $0.78 < y < 1.09$ (periodic b.c.). These $y$-ranges are consistent with (4.4), but they probably result both from the leading correction term and from the analytic background (see (2.20)). The $\Gamma$-ranges, as defined by these "confluences", are $\Gamma = 0.9894 \pm 0.0031$ (free b.c.) and $\Gamma = 0.9993 \pm 0.0013$ (periodic b.c.), which give $v = 0.7580 \pm 0.0023$ and $v = 0.7505 \pm 0.0010$, respectively. Again, one of the $v$ ranges (for free b.c.) does not include the correct $v$ value, reflecting the presence of an appreciable background. Various discrepancies and systematic errors observed in our study of the SAW problem suggest that both "bulk" and finite-size calculations should be extended to larger systems (longer series).

Note added in proof

We mention here some results for SAW which were announced after this paper had been submitted. Djordjević et al.\textsuperscript{42}) estimated $\theta \approx 2/3$ and $r \approx 0.708 \approx 1/\sqrt{2}$, from the analysis of extended triangular lattice $\rho_n$ series\textsuperscript{43}), in excellent agreement with (4.3). However, other recent series analyses\textsuperscript{44,45}), of the number of SAW and of self-avoiding rings, for several lattices, did not provide consistent evidence for $\theta < 1$.

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