

# Coherent interaction of spins induced by thermal bosonic environment

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Received 25 March 2006; accepted 1 June 2006

Available online 12 June 2006

Communicated by C.R. Doering

## Abstract

We obtain and analyze the indirect exchange interaction between two two-state systems, e.g., spins, in a formulation that also incorporates the quantum noise that they experience, due to a bosonic environment, for instance, phonons. We utilize a perturbative approach to obtain a quantum evolution equation for the two-spin dynamics. A nonperturbative approach is used to study the onset of the induced interaction, which is calculated exactly. We predict that for low enough temperatures the interaction is coherent over time scales sufficient to create entanglement, dominated by the zero-point quantum fluctuations of the environment. We identify the time scales for which the spins develop entanglement for various spatial separations.

PACS: 75.30.Et; 71.70.-d; 71.55.-i; 73.21.-b

Keywords: Exchange interaction; Entanglement; Thermal environment; Qubit

Studies of open quantum systems have a long history [1,2]. Recent experimental advances have allowed the observation of fundamental quantum mechanical phenomena in nanostructured systems in condensed matter and other fields. Promise of applications for quantum information processing has stimulated significant interest in theoretical studies of quantum coherence and entanglement in situations when the quantum system is subject to environmental noise [1–5]. In the present work, we consider two two-state systems: qubits, e.g., spins 1/2, in a thermal bosonic environment (bath). We study the emergence of the indirect exchange interaction between two localized spins and identify the regimes where entanglement generated by this interaction can be observed. We report results for geometries relevant for recent experiments [6–8].

Quantum computing schemes with qubits coupled by indirect exchange of excitons were proposed in [9–12], and these interactions are also of interest in studies of quantum phase transitions [13]. Traditionally, such RKKY-type interactions

were calculated perturbatively at zero temperature, without considering quantum noise, with conduction electrons [14] or excitons [15] acting as the “bath”. Recent works [16,17] considered the effects of noise for a thermalized bath of, respectively, noninteracting and interacting electrons. A bath of thermalized bosonic modes can cause decoherence and, for more than one qubit, disentanglement. These effects have been studied extensively in the recent literature [18]. It has also been anticipated [19] that such a thermalized bath of modes can induce entanglement under certain conditions.

In this Letter, we investigate both physical effects of a thermalized bosonic bath in which two qubits are immersed. Specifically, the induced interaction, which is effectively a zero-temperature effect, and the quantum noise, originating from the same bath modes, are derived within a uniform treatment. We study the dependence of the induced interaction (coherent) vs. noise (decoherence) effects on the parameters of the bath modes, qubits, and their coupling, as well as on the geometry of the qubit system.

We consider two localized spins separated by distance  $\mathbf{d}$  and identically coupled with the modes of a thermalized bosonic bath, described by  $H_B = \sum_{\mathbf{k}} \omega_{\mathbf{k}} a_{\mathbf{k}}^{\dagger} a_{\mathbf{k}}$ , where we set  $\hbar = 1$ . The external magnetic field will be represented by the Hamil-

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tonian  $H_S$ , corresponding to the energy gap  $\Delta$  between the up and down states for each spin. A natural example of the described model are spins of two localized electrons interacting via lattice vibrations (phonons) by means of the spin-orbit interaction [20]. For each type of phonons, the interaction can be assumed [1,2,20] linear in the bosonic variables. Without loss of generality, the calculational techniques can be developed for a coupling that involves a specific spin component, which will be the same for both spins and denoted by  $S_j$  for the  $j$ th spin, localized at  $\mathbf{r}_j$ . Thus we take  $H_{SB} = \sum_{j=1,2} S_j X_j$ , where  $X_j = \sum_{\mathbf{k}} g_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{r}_j} (a_{\mathbf{k}} + a_{-\mathbf{k}}^\dagger)$ . All our derivations, however, can be generalized to include all the projections of the spin [21], and we give some illustrative numbers for such a calculation below. Our emphasis here is on comparing the relative importance of the coherent vs. noise effects of a given bosonic bath in the two-qubit dynamics. We do not include possible other two-qubit interactions in such comparative calculation of dynamical quantities.

The overall system is described by the Hamiltonian  $H = H_S + H_B + H_{SB}$ . Let us point out that such a model is quite general and it also finds applications, for instance, in quantum optics [22] where the Hamiltonian  $H$  would describe atoms (regarded as the two level systems) interacting with an electromagnetic field. Our detailed expressions here are obtained for the one-dimensional case, relevant for recent experiments that involve channel geometries [6–8], when phonons propagate along  $\mathbf{d}$ , i.e.,  $\mathbf{k} \cdot \mathbf{d} \rightarrow k|\mathbf{d}|$ .

The problem of describing the dynamics of the spin system and, thus, finding the reduced density matrix,  $\rho_S(t) = \text{Tr}_B \rho(t)$ , cannot be solved exactly in the general case. We first consider time scales longer than the thermalization times of the bath. Furthermore, we treat the interaction,  $H_{SB}$ , as a perturbation, and expand the equation for the density matrix,  $i\partial_t \rho(t) = [H, \rho(t)]$ , up to the second order in  $H_{SB}$ . In this regime, it is appropriate [2] to model the thermalization of the bath within a standard Markovian approximation which involves factoring out the environmental mode density matrix in the second order term, replacing it by the thermal one, as well as using [1] the initial condition  $\rho(0) = \rho_S(0) \otimes \rho_B^{\text{thermal}}$ . Here  $\rho_B^{\text{thermal}} = \prod_k Z_k^{-1} e^{-\omega_k a_k^\dagger a_k / k_B T}$ , and  $Z_k$  is the partition function for the oscillator  $k$ . The resulting equation [2] for the density matrix is

$$i\partial_t \rho_S(t) = [H_S, \rho_S(t)] + i \int_0^t dt' \Sigma(t' - t) \rho_S(t'), \quad (1)$$

where  $\Sigma(t' - t) \rho_S(t) = -\sum_{ij} \text{Tr}_B [S_j X_j, [S_i(t' - t) X_i(t' - t), \rho_B^{\text{thermal}} \rho_S(t)]]$  is the self-energy superoperator term.

This expression involves the correlation functions  $C_{ji}(t) = \text{Tr}_B [X_j X_i(t) \rho_B^{\text{thermal}}]$ , where  $i, j = 1, 2$  for the two spins, with the property  $C_{ji}^*(t) = C_{ji}(-t)$ . The present approximation assumes [2] that the bath has very short memory. One can argue that the correlation functions  $C_{ji}(t)$  are nonnegligible only up to times  $1/\omega_c$ , where  $\omega_c$  is the frequency cutoff for electron spins interacting with phonons. This cutoff comes either from the phonon density of states or from the localization of the electron wavefunctions. This time scale will be considered to be

significantly smaller than the system evolution times defined by the inverse of the energy gap  $\Delta$ .

In the resulting evolution equation,

$$i\partial_t \rho_S(t) = [H_{\text{eff}}, \rho_S(t)] + i\mathcal{L}_c \rho_S(t) + i\mathcal{L}_s \rho_S(t), \quad (2)$$

we separate out the coherent dynamics in the first term. The superoperators  $\mathcal{L}_c$  and  $\mathcal{L}_s$  will be addressed shortly. In the two-spin case, one can establish by a lengthy calculation [21], which is not reproduced here, that

$$H_{\text{eff}} = H_S + 2\chi_c(\mathbf{d}) S_1 S_2 + O(\chi_s(\mathbf{d}), \eta_s(0)), \quad (3)$$

which includes the effective coupling of the spin components. Expressions for the quantities  $\chi_{c,s}$  and  $\eta_{c,s}$  follow from, respectively, the imaginary and real parts of the correlation functions and will be given explicitly below. The last term in (3) introduces corrections of relative order  $\Delta/\omega_c$  in the induced interaction and is not of interest here.

The second term in (2) is

$$\begin{aligned} \mathcal{L}_c \rho_S(t) = & - \sum_{ij} \eta_c(\theta_{ij} \mathbf{d}) \{S_i S_j, \rho_S(t)\} \\ & + 2 \sum_{i \neq j} \eta_c(\theta_{ij} \mathbf{d}) S_i \rho_S(t) S_j, \end{aligned} \quad (4)$$

where  $\theta_{ij} \equiv 1 - \delta_{ij}$ ; it accounts for the dominant relaxation and decoherence processes. The third term in (2),  $\mathcal{L}_s \rho_S(t)$ , involves expressions proportional to  $\eta_s(\theta_{ij} \mathbf{d})$  or  $\chi_s(\theta_{ij} \mathbf{d})$  and may often be considered small compared to terms in (4), as will be shown below by analyzing the magnitudes of  $\eta_s(\theta_{ij} \mathbf{d})$  and  $\chi_s(\theta_{ij} \mathbf{d})$ .

In (3) and (4) we introduced the quantities

$$\chi_c(\mathbf{d}) = \int_{-\infty}^0 dt \int_0^{\infty} d\omega \mathcal{D}(\omega) |g(\omega)|^2 \sin \omega t \cos \frac{\omega|\mathbf{d}|}{c_s} \cos \Delta t \quad (5)$$

and

$$\begin{aligned} \eta_c(\mathbf{d}) = & \int_{-\infty}^0 dt \int_0^{\infty} d\omega \mathcal{D}(\omega) |g(\omega)|^2 \coth \frac{\omega}{2k_B T} \cos \omega t \\ & \times \cos \frac{\omega|\mathbf{d}|}{c_s} \cos \Delta t, \end{aligned} \quad (6)$$

where  $\mathcal{D}(\omega)$  is the density of states of the bosonic modes; this factor should also include the Debye cutoff at large frequencies. For definiteness, we have assumed the linear dispersion,  $\omega = c_s k$ , because details of the dispersion relation for larger frequencies usually have little effect on decoherence properties. Another reason to focus on the low-frequency modes is that an additional cutoff,  $\omega_c$ , resulting from the localization of the electron wave functions, typically much smaller than the Debye frequency, will be present due to the factors  $|g(\omega)|^2$ . The expressions for  $\eta_s(\mathbf{d})$  and  $\chi_s(\mathbf{d})$  can be obtained by replacing  $\cos \Delta t \rightarrow \sin \Delta t$  in  $\eta_c(\mathbf{d})$  and  $\chi_c(\mathbf{d})$ , respectively. The integration in (6) then yields

$$\eta_c(\mathbf{d}) = \frac{\pi}{2} \mathcal{D}(\Delta) |g(\Delta)|^2 \coth \frac{\Delta}{2k_B T} \cos \frac{\Delta|\mathbf{d}|}{c_s}. \quad (7)$$

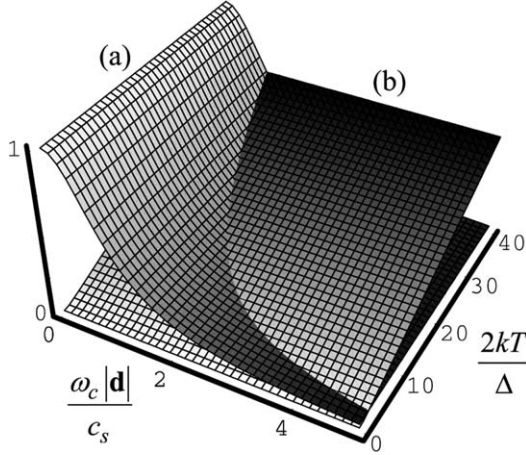


Fig. 1. The magnitude, in units of  $\alpha_1 \omega_c$ , of (a) the induced spin–spin interaction, and (b) the largest decoherence amplitude. Here we took  $\Delta/\omega_c = 0.01$ .

Similarly, we find

$$\chi_s(\mathbf{d}) = \frac{\pi}{2} \mathcal{D}(\Delta) |g(\Delta)|^2 \cos \frac{\Delta |\mathbf{d}|}{c_s}. \quad (8)$$

To derive explicit expressions for  $\chi_c(\mathbf{d})$ , one needs to specify the  $\omega$ -dependence in  $\mathcal{D}(\omega) |g(\omega)|^2$ . For purposes of modeling bosonic heat-bath effects, this product is usually approximated [1] by a power law with superimposed exponential cutoff,  $\mathcal{D}(\omega) |g(\omega)|^2 = \alpha_n \omega^n e^{-\omega/\omega_c}$ , with  $n \geq 1$ . Finally, we get

$$\chi_c(\mathbf{d}) = -\alpha_n \omega_c^n \left[ \xi^n \frac{\partial^{n-1}}{\partial \xi^{n-1}} \frac{(-1)^{n-1} \xi}{1 + \xi^2} \right]_{\xi=c_s/(\omega_c d)}. \quad (9)$$

This quantity gives the coefficient of the leading induced interaction; see (3). The individual Lamb shifts in (3) are defined by  $\eta_s(0)$  and are not important for our discussion.

For the commonly studied case of Ohmic dissipation [1],  $n = 1$ , the environment induces the following spin–spin interaction,

$$H_{\text{int}} = -\frac{2\alpha_1 \omega_c}{1 + \omega_c^2 \mathbf{d}^2 / c_s^2} S_1 S_2. \quad (10)$$

This induced interaction is temperature independent and is mediated by the zero-point fluctuations of the bosonic field. It is long-range and decays as a power-law for large  $|\mathbf{d}|$ . On the other hand, quantum noise terms, see (2)–(4), depend weakly on  $|\mathbf{d}|$ , and increase with temperature. In Fig. 1, we plot the magnitude of the interaction  $\chi_c(\mathbf{d})$  as a function of the spin separation. It is compared, for varying temperature, to the magnitude of the decoherence terms,  $\chi_s(\mathbf{d})$ ,  $\eta_s(\mathbf{d})$  and  $\eta_c(\mathbf{d})$ , among which  $\eta_c(\mathbf{d})$  is the dominant for the Ohmic case,  $n = 1$ , and the temperature scale of Fig. 1, i.e. for  $2kT/\omega_c$  from  $\sim 0.05$ . For lower temperatures evaluation of  $\eta_s(\mathbf{d})$  may be important. For the super-Ohmic case,  $n > 1$ , one obtains similar behavior, except that the interaction decays as a higher negative power of  $|\mathbf{d}|$ .

The amplitudes plotted in Fig. 1 provide qualitative information on the dynamics. For definiteness, let us consider the

case of  $H_S = \frac{1}{2} \Delta \sigma_z^{(1)} + \frac{1}{2} \Delta \sigma_z^{(2)}$  and  $H_{\text{int}}$  in (10) involving  $S_{1,2} = \sigma_x^{(1,2)}$ . If the induced interaction were the only effect of the bath, then the system’s dynamics would be coherent, with oscillations determined by the energy gaps of  $H_S + H_{\text{int}}$ . This Hamiltonian has the singlet state  $(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)/\sqrt{2}$ , with the energy  $E_2 = -2\chi_c(\mathbf{d})$ , and the (split) “triplet”  $C(|\uparrow\uparrow\rangle - \delta|\downarrow\downarrow\rangle)$ ,  $(|\uparrow\downarrow\rangle + |\downarrow\uparrow\rangle)/\sqrt{2}$ , and  $C(|\downarrow\downarrow\rangle + \delta|\uparrow\uparrow\rangle)$ , with the energies  $E_0 = -\sqrt{\Delta^2 + 4\chi_c^2(\mathbf{d})}$ ,  $E_1 = 2\chi_c(\mathbf{d})$ , and  $E_3 = \sqrt{\Delta^2 + 4\chi_c^2(\mathbf{d})}$ , respectively, where  $\delta \equiv 2\chi_c(\mathbf{d})/(\Delta - E_0)$ , and  $C$  is the normalization constant. The effect of the noise terms is to wash away the coherent behavior. The time scales of the coherent oscillations and of the noise-induced relaxation processes, defined by the corresponding frequencies in Fig. 1, become comparable at the intersection curve of the two surfaces in Fig. 1. For larger distances and/or temperatures the noise terms dominate.

The energy gap between  $E_1$  and  $E_2$  is defined by  $\chi_c(\mathbf{d})$ , whereas the effective width of each level due to decoherence will be determined by the magnitudes of  $\chi_s(\mathbf{d})$ ,  $\eta_s(\mathbf{d})$  and  $\eta_c(\mathbf{d})$ . Let us estimate the magnitude of this energy splitting. As an example, we consider two phosphorus donor impurities in Ge, in external magnetic field in the  $z$  direction. Due to the symmetry of the spin–phonon coupling (via spin–orbit interaction), only two components, namely,  $\sigma_x$  and  $\sigma_y$ , are important [23]. One can demonstrate that in this case the terms in (1) proportional to products of  $\sigma_x$  and  $\sigma_y$  vanish, while the contributions from the  $\sigma_x$  and  $\sigma_y$  terms are identical. Therefore, we can use the single-component results, with  $S_j \sim \sigma_x^{(j)}$ . For a donor impurity electron spin, the cutoff  $\omega_c \sim c_s/a_B$  comes from the donor electron wave function [23], which is localized on the scale of  $a_B \sim 4$  nm. The characteristic value of the phonon group velocity in Ge is about  $3 \times 10^3$  m/s. The strength of the spin–orbit coupling,  $\alpha_1 \omega_c$ , is [23] of the order of  $10^7$  s $^{-1}$ . Utilizing these data in (7), (10), with  $n = 1$  which, strictly speaking should be only valid for one-dimensional channel for phonon propagation, we obtain the estimate for the  $E_1 \leftrightarrow E_2$  energy splitting of about 10 MHz, whereas the noise level for mK temperatures varies from 0.1 to 1 MHz, depending on the magnitude of the Zeeman splitting. This coherence/noise “measure” can be further improved for different cases of the phonon spectrum, the shape of the wave function, etc. [21].

To study the onset of the exchange interaction, we consider the case when the Zeeman splitting  $\Delta$  is negligible. Then one can actually derive an exact solution for  $H_S + H_B + H_{SB}$ , with the bath modes traced over without the Markovian assumption, and demonstrate the emergence of the effective interaction Hamiltonian as in (3), (9), (10). For  $\Delta = 0$ , a lengthy calculation utilizing bosonic operator techniques yields

$$\rho_S(t) = \sum_{\lambda, \lambda'} P_\lambda \rho_S(0) P_{\lambda'} e^{\mathcal{L}_{\lambda\lambda'}(t)}, \quad (11)$$

where  $|\lambda_j\rangle$  is an eigenstate of  $S_j$  labeled by its eigenvalue  $\lambda_j$ , and we introduced the projection operator  $P_\lambda = |\lambda_1 \lambda_2\rangle \langle \lambda_1 \lambda_2|$ . The exponent consists of the real part, which represents deco-

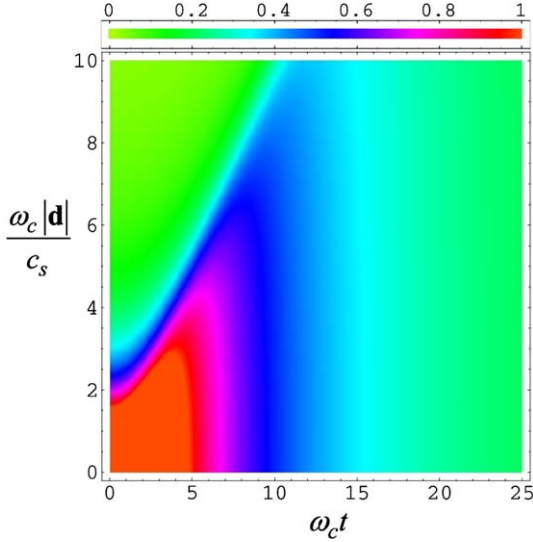


Fig. 2. The onset of the indirect spin–spin interaction is measured by the decay of the correction term  $F(t)$ . The values of  $F(t)$  are in units of  $\alpha_1 \omega_c$ , and are color-coded according to the top bar.

herence,

$$\text{Re } \mathcal{L}_{\lambda\lambda'}(t) = - \sum_k G_k(t; T) [(\lambda'_1 - \lambda_1)^2 + (\lambda'_2 - \lambda_2)^2 + 2 \cos(\omega_k |\mathbf{d}|/c_s) (\lambda'_1 - \lambda_1) (\lambda'_2 - \lambda_2)], \quad (12)$$

and imaginary part, which describes the coherent evolution,

$$\text{Im } \mathcal{L}_{\lambda\lambda'}(t) = \sum_k C_k(t) \cos(\omega_k |\mathbf{d}|/c_s) (\lambda_1 \lambda_2 - \lambda'_1 \lambda'_2). \quad (13)$$

Since for  $\Delta = 0$ ,  $H_S$  commutes with  $H_{SB}$ , quantum noise in this case only affects the off-diagonal matrix elements. Eventually, this destroys quantum correlations between the spins. In (12) and (13) the standard functions [1–3] were introduced,

$$G_k(t; T) = 2 \frac{|g_k|^2}{\omega_k^2} \sin^2 \frac{\omega_k t}{2} \coth(\omega_k/2k_B T), \quad (14)$$

$$C_k(t) = 2 \frac{|g_k|^2}{\omega_k^2} (\omega_k t - \sin \omega_k t). \quad (15)$$

Our focus here is on the imaginary part (13). One can demonstrate that if the noise terms (the real part) were completely absent, the resulting evolution would be coherent with the evolution operator given by  $e^{-i[H_{\text{int}} + F(t)]t}$ , where

$$F(t) = 2S_1 S_2 \int_0^\infty d\omega \frac{\mathcal{D}(\omega) |g(\omega)|^2}{\omega} \frac{\sin \omega t}{\omega t} \cos(\omega |\mathbf{d}|/c_s) \quad (16)$$

and

$$H_{\text{int}} = - \frac{2\alpha_n \Gamma(n) \omega_c^n}{(1 + \omega_c^2 \mathbf{d}^2 / c_s^2)^{n/2}} \cos \left[ n \arctan \left( \frac{\omega_c |\mathbf{d}|}{c_s} \right) \right] S_1 S_2. \quad (17)$$

Expression (16) represents the initial, time-dependent correction present only during the onset of the induced interaction. Specifically,  $F(0) = -H_{\text{int}}$ , but  $F(t)$  decays for large times. In Fig. 2, we plot  $F(t)$  for the Ohmic case,  $n = 1$ , as a function of time and spin separation. The right side of the plot in Fig. 2 cor-

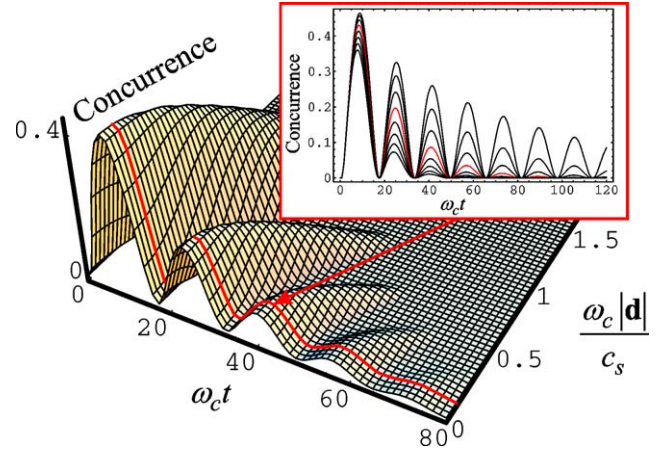


Fig. 3. The concurrence calculated for Ohmic dissipation with  $\alpha_1 = k_B T / \omega_c = 1/20$ . The inset gives the time dependence for different temperatures:  $80k_B T / \omega_c = 1, 2, 3, 4, 5, 6, 7, 8$ , from the top curve to the bottom one, respectively: the top curve corresponds to the lowest temperature.

responds to the constant coupling regime with the interaction Hamiltonian (17). Note that the Hamiltonian (17) with  $n \geq 1$  is identical to that obtained within the perturbative Markovian approach, cf. (3), (9), (10).

Let us investigate the role of the decoherence resulting from the real part of  $\mathcal{L}_{\lambda\lambda'}$  in (12). Note that within the exact solution the bath is thermalized only initially, at  $t = 0$ . However, it is expected that the effects of the quantum noise are represented *qualitatively* similarly to the Markovian approximation, see (2)–(4), which implies re-thermalization of the bath after each infinitesimal time step. To analyze the dynamics of quantum correlations between the spins, we have used the concurrence [24]—a measure of entanglement which is widely used in quantum information theory. For a mixed state of two qubits it is defined [24] via the eigenvalues  $\lambda_j$  of  $\sqrt{\sqrt{\rho_S} \sigma_y^1 \sigma_y^2 \rho_S^* \sigma_y^1 \sigma_y^2 \sqrt{\rho_S}}$  as  $\max\{0, 2 \max_i \lambda_i - \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4\}$ . For  $\Delta = 0$ , and as long as the effects of the quantum noise are small, the induced interaction will split the system energies into two degenerate pairs,  $E_0 = E_2$  and  $E_1 = E_3$ . As a result, the dynamics will involve coherent oscillations with the frequency defined by the gap  $E_1 - E_0 = 4\chi_c(\mathbf{d})$ . In Fig. 3, we plot the concurrence for the density matrix given by (11), as a function of time and spin–spin separation, for the (initially unentangled) state  $|\uparrow\uparrow\rangle$ , and  $n = 1$ . The inset in Fig. 3 shows the time dependence of the concurrence for different temperatures. Fig. 3 demonstrates that the system can develop and maintain entanglement over several coherent-dynamics oscillations before the noise-induced effects take over and the concurrence decays to zero.

In summary, we studied the induced indirect exchange interaction due to a bosonic bath of environmental modes which also introduce the noise. We demonstrated that it can create *observable* two-spin entanglement. For an appropriate choice of the system parameters, specifically, the spin–spin separation, this entanglement can be maintained and the system can evolve approximately coherently for many cycles of its internal dynamics. However, for large times the quantum noise effects will eventually dominate and the entanglement will be erased.

## Acknowledgements

The authors acknowledge useful discussions with and helpful suggestions by J. Eberly, L. Fedichkin and D. Mozyrsky. This research was supported by the NSF under grant DMR-0121146.

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