(12) **Problem 1.** Consider the region between the curves $y = 2 - x$ and $y = x^2$ for $0 \leq x \leq 2$.

(a) Sketch the region. Label the axis scales and any points of intersection.

*ANSWER:*

(b) Write an expression which represents the area of the region. *Do not evaluate it.*

*ANSWER:*

$$A = \int_0^1 [(2 - x) - x^2] \, dx + \int_1^2 [x^2 - (2 - x)] \, dx$$

(12) **Problem 2.** Consider the solid obtained by rotating the region bounded by the curves $y = x^2$ and $y = 4$ for $0 \leq x \leq 2$ about the $y$-axis.

(a) Sketch the region and the solid.

*ANSWER:*

(b) Set up an integral for the volume *using disks or washers* and *evaluate the integral.*

*ANSWER:*

Slice perpendicular to the axis of rotation (see sketch above):

$$V = \pi \int_0^4 (\sqrt{y})^2 \, dy = \pi \int_0^4 y \, dy = \pi \left. \frac{y^2}{2} \right|_0^4 = 8\pi$$
Problem 3. Consider the solid obtained by rotating the region bounded by the curves $y = 4x - x^2$ and $y = -5$ about the line $x = -2$.

(a) Sketch the region and the solid.

![Sketch of the region and solid](image)

(b) Set up an integral for the volume using cylindrical shells. Do NOT evaluate the integral.

**Answer:** Slice parallel to the axis of rotation (see sketch above):

$$V = 2\pi \int_{-1}^{5} (2 + x)(4x - x^2 + 5) \, dx$$

Problem 4. Solve the initial value problem: $\frac{dy}{dx} = 2xy^2 + 2x, \quad y(1) = 0$.

**Answer:** Factor out $2x$ on the right side, divide both sides by $y^2 + 1$, multiply by $dx$, and integrate:

$$\int \frac{dy}{1 + y^2} = \int 2x \, dx,$$

$$\tan^{-1}(y) = x^2 + C,$$

$$y = \tan \left(x^2 + C\right).$$

From the initial condition, $y(1) = \tan(1 + C) = 0$ so $1 + C = 0$ and thus $C = -1$. Therefore,

$$y = \tan \left(x^2 - 1\right).$$
Problem 5. Consider the curve \( y = \sqrt{x} \) for \( 4 \leq x \leq 9 \).

(a) Set up an integral for the length of the curve. **Do NOT evaluate the integral.**

**ANSWER:** Since \( y' = 1/(2\sqrt{x}) \),

\[
1 + (y')^2 = 1 + \frac{1}{4x} = \frac{4x + 1}{4x}
\]

so the length of the curve is

\[
L = \int_4^9 \sqrt{1 + (y')^2} \, dx = \int_4^9 \sqrt{\frac{4x + 1}{4x}} \, dx
\]

(b) Set up an integral for the area of the surface obtained by rotating the curve about the \( x \)-axis. Then **evaluate the integral.** Leave your answer in terms of roots (or fractional powers).

**ANSWER:** Using the results in part (a),

\[
S = 2\pi \int_4^9 y\sqrt{1 + (y')^2} \, dx = \pi \int_4^9 \sqrt{4x + 1} \, dx = \frac{\pi}{6} (4x + 1)^{3/2} \bigg|_4^9 = \frac{\pi}{6} \left[ (37)^{3/2} - (17)^{3/2} \right].
\]

Problem 6. A 1600-lb. elevator is suspended by a 200-ft. cable that weighs 10 lbs./ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft.?

**ANSWER:** Break this into three pieces:

- Work to raise elevator: \( (1600 \text{ lbs}) \times (30 \text{ ft}) = 48,000 \text{ ft-lbs} \)
- Work to raise bottom part of cable: \( (170 \text{ ft}) \times (20 \text{ lbs/ft}) \times (30 \text{ ft}) = 51,000 \text{ ft-lbs} \)
- Work to raise top part of cable: \( W = \int_0^{30} 10x \, dx = 5x^2 \bigg|_0^{30} = 4,500 \text{ ft-lbs} \)

Total work: \( 48,000 + 51,000 + 4,500 = 103,500 \text{ ft-lbs} \)

Problem 7. Find the average value of the function \( f(t) = t \sin(t) \) on the interval \( 0 \leq t \leq \pi \).

**ANSWER:** Using integration by parts (with \( u = t \) and \( dv = \sin(t) \, dt \)):

\[
f_{\text{avg}} = \frac{1}{\pi} \left[ \frac{1}{0} \int_0^\pi t \sin(t) \, dt \right] = \frac{1}{\pi} \left[ -t \cos(t) \bigg|_0^\pi + \int_0^\pi \cos(t) \, dt \right] = \frac{1}{\pi} [(-\pi)(-1) - 0 + \sin(t) \bigg|_0^\pi] = 1.
\]
Problem 8. A bacteria culture grows with constant relative growth rate. Initially there are 200 bacteria, and after 2 hours the count is 1800. Find an expression for the bacteria count as a function of time $t$ in hours.

**ANSWER:** With constant relative growth rate the number $N$ of bacteria grows exponentially with time $t$, so $N(t) = 200e^{kt}$ for some constant $k$. We can use the condition $N(2) = 1800$ to find $k$; the arithmetic is simplified by first writing $N(t) = 200(e^k)^t$ and then solving for $e^k$ (rather than $k$ itself): $1800 = 200(e^k)^2$ so $9 = (e^k)^2$ and thus $e^k = 3$. Thus, $N(t) = 200 \cdot 3^t$.

Problem 9. Consider the curve defined by the parametric equations

$$x = e^{-t}, \quad y = e^{2t}, \quad -1 \leq t \leq 1.$$ 

(a) Eliminate the parameter $t$ to find a Cartesian equation of the curve.

**ANSWER:** Writing $x = e^{-t} = 1/e^t$, we can substitute this to write $y$ as

$$y = e^{2t} = (e^t)^2 = \left(\frac{1}{x}\right)^2 = \frac{1}{x^2}.$$ 

(b) Sketch the curve. Label with appropriate values of $t$ and indicate with an arrow the direction in which the curve is traced as $t$ increases.

**ANSWER:**