Problem 1. Evaluate the integral: \( \int \tan^2(x) \sec^4(x) \, dx \)

**ANSWER:**
\[
\int \tan^2(x) \sec^4(x) \, dx = \int \tan^2(x) [\tan^2(x) + 1] \sec^2(x) \, dx \quad [u = \tan(x), \, du = \sec^2(x) \, dx]
\]
\[
= \int u^2(u^2 + 1) \, du = \int (u^4 + u^2) \, du
\]
\[
= \frac{1}{5}u^5 + \frac{1}{3}u^3 + C = \frac{1}{5}\tan^5(x) + \frac{1}{3}\tan^3(x) + C
\]

Problem 2. For each of the following, give the form of the partial fraction decomposition. Do not solve for the unknown constants. *Hint: Factor where possible.*

(a) \[
\frac{x^3 + 3x^2 - 5}{(x-2)(x+5)^2(x^2 + 3)^2} = \frac{A}{x-2} + \frac{B}{x+5} + \frac{C}{(x+5)^2} + \frac{Dx + E}{x^2 + 3} + \frac{Fx + G}{(x^2 + 3)^2}
\]

(b) \[
\frac{x^4 - 2x^2 + 17}{x^5(x^2 - 1)(x^2 + x - 6)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x - 1} + \frac{E}{x + 1} + \frac{F}{x - 2} + \frac{G}{x + 3}
\]

Problem 3. Evaluate the integral: \( \int \frac{1}{\sqrt{x^2 - 2x + 5}} \, dx \)

**ANSWER:** First, complete the square: \( x^2 - 2x + 5 = x^2 - 2x + 1 + 4 = (x - 1)^2 + 2^2 \). Then use the trig substitution

\[
\begin{align*}
  x - 1 &= 2\tan(\theta) \\
  dx &= 2\sec^2(\theta) \, d\theta \\
  \sqrt{(x - 1)^2 + 2^2} &= 2\sec(\theta)
\end{align*}
\]

Then
\[
\int \frac{1}{\sqrt{x^2 - 2x + 5}} \, dx = \int \frac{1}{\sqrt{(x - 1)^2 + 2^2}} \, dx = \int \frac{2\sec^2(\theta)}{2\sec(\theta)} \, d\theta = \int \sec(\theta) \, d\theta
\]
\[
= \ln |\sec(\theta) + \tan(\theta)| + C = \ln \left| \frac{\sqrt{x^2 - 2x + 5} + x - 1}{2} \right| + C
\]
\[
= \ln \left| \sqrt{x^2 - 2x + 5} + x - 1 \right| + C
\]
Problem 4. The speed \( v(t) \) of a crawling slug\(^1\) for certain values of time \( t \) is given in the table:

<table>
<thead>
<tr>
<th>( t ) (minutes)</th>
<th>0</th>
<th>2</th>
<th>4</th>
<th>6</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( v(t) ) (cm/minute)</td>
<td>2</td>
<td>4</td>
<td>5</td>
<td>3</td>
<td>1</td>
</tr>
</tbody>
</table>

Estimate the distance traveled by the slug from \( t = 0 \) to \( t = 8 \) minutes.

(a) Use the Trapezoidal rule with \( n = 4 \) subintervals.

ANSWER:
\[
T_4 = \frac{\Delta t}{2} [v(0) + 2v(2) + 2v(4) + 2v(6) + v(8)] = \frac{2}{2} [2 + 2(4 + 5 + 3) + 1] = 27 \text{ cm}
\]

(b) Use the Midpoint rule with \( n = 2 \) subintervals.

ANSWER:
\[
M_2 = \Delta t [v(2) + v(6)] = 4 (4 + 3) = 28 \text{ cm}
\]

Problem 5. Evaluate the integral:
\[
\int \frac{x}{x + 3} \, dx
\]

ANSWER:
\[
\int \frac{x}{x + 3} \, dx = \int \frac{(x + 3) - 3}{x + 3} \, dx = \int \left( 1 - \frac{3}{x + 3} \right) \, dx = x - 3 \ln |x + 3| + C
\]
(or use the substitution \( u = x + 3 \)).

Problem 6. Evaluate the integral:
\[
\int \theta^2 \sin(5\theta) \, d\theta
\]

ANSWER: Integrate by parts with \( u = \theta^2 \) and \( dv = \sin(5\theta) \, d\theta \):
\[
\int \theta^2 \sin(5\theta) \, d\theta = -\frac{1}{5} \theta^2 \cos(5\theta) + \frac{2}{5} \int \theta \cos(5\theta) \, d\theta
\]

Then integrate by parts again with \( u = \theta \) and \( dv = \cos(5\theta) \, d\theta \):
\[
\int \theta^2 \sin(5\theta) \, d\theta = -\frac{1}{5} \theta^2 \cos(5\theta) + \frac{2}{5} \left[ \frac{\theta}{5} \sin(5\theta) - \frac{1}{5} \int \sin(5\theta) \, d\theta \right]
\]
\[
= -\frac{1}{5} \theta^2 \cos(5\theta) + \frac{2}{25} \theta \sin(5\theta) + \frac{2}{125} \cos(5\theta) + C
\]

\(^1\)measured with a radar gun
Problem 7. Evaluate the following limits. If you use L’Hospital’s Rule, show where you used it and that it applies.

(a) \( \lim_{x \to \infty} x \sin \left( \frac{1}{x} \right) \)

**ANSWER:** This limit has the form \( \infty \cdot 0 \), which is indeterminate, so convert it to a quotient of the form \( 0/0 \) and use L’Hospital’s Rule:

\[
\lim_{x \to \infty} x \sin \left( \frac{1}{x} \right) = \lim_{x \to \infty} \frac{\sin(1/x)}{1/x} = \lim_{x \to \infty} \frac{\cos(1/x)(-1/x^2)}{-(1/x^2)} = \lim_{x \to \infty} \cos \left( \frac{1}{x} \right) = 1
\]

(b) \( \lim_{x \to \infty} x^{2/x} \)

**ANSWER:** This limit has the form \( \infty^0 \), which is indeterminate, so let \( y = x^{2/x} \), take the logarithm, and use L’Hospital’s Rule:

\[
\lim_{x \to \infty} \ln(y) = \lim_{x \to \infty} \frac{2\ln(x)}{x} = \lim_{x \to \infty} \frac{2/x}{1} = 0
\]

Thus

\[
\lim_{x \to \infty} x^{2/x} = \lim_{x \to \infty} y = \lim_{x \to \infty} e^{\ln(y)} = e^0 = 1
\]

Problem 8. Evaluate the integral: \( \int \frac{x - 6}{x^3 - 2x^2} \, dx \)

**ANSWER:** Use partial fractions:

\[
\int \frac{x - 6}{x^3 - 2x^2} \, dx = \int \frac{x - 6}{x^2(x - 2)} \, dx = \int \left( \frac{1}{x} + \frac{3}{x^2} - \frac{1}{x - 2} \right) \, dx = \ln |x| - \frac{3}{x} - \ln |x - 2| + C
\]

Problem 9. Determine whether the following integral is convergent or divergent. If it is convergent, evaluate it.

\( \int_{1}^{\infty} \frac{1}{(2x + 1)^2} \, dx \)

**ANSWER:** Using the substitution \( u = 2x + 1 \) gives the antiderivative

\[
\int \frac{1}{(2x + 1)^2} \, dx = \frac{1}{2} \int \frac{1}{u^2} \, du = -\frac{1}{2u} + C = -\frac{1}{2(2x + 1)} + C
\]

Then the improper integral is

\[
\int_{1}^{\infty} \frac{1}{(2x + 1)^2} \, dx = \lim_{t \to \infty} \int_{1}^{t} \frac{1}{(2x + 1)^2} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{2(2x + 1)} \right]_{1}^{t} = \lim_{t \to \infty} \left[ -\frac{1}{2(2t + 1)} - \left( -\frac{1}{6} \right) \right] = \frac{1}{6}
\]

(so the integral is convergent).
Problem 10. The integral \[\int_{-1}^{2} f(x) \, dx\] of the function \(f(x)\) shown in the graph below is to be estimated by numerical integration. Rank the following five numbers in order of size from smallest to largest:

- \(L_n\): left endpoint (left sum) approximation
- \(R_n\): right endpoint (right sum) approximation
- \(M_n\): midpoint approximation
- \(T_n\): trapezoidal approximation
- \(I\): the exact value of the integral \[\int_{-1}^{2} f(x) \, dx\]

Note: each approximation is computed using the same number \(n\) of subintervals.

\[\begin{align*}
&L_n < T_n < I < M_n < R_n
\end{align*}\]

\[\textbf{ANSWER:} \text{ Since the function is increasing, } L_n < I < R_n. \text{ Since the function is concave down, } T_n < I < M_n. \text{ Since the function is both increasing and concave down, } L_n < T_n \text{ and } M_n < R_n. \text{ Thus: } \]

\[L_n < T_n < I < M_n < R_n\]