

A FETI-DP PRECONDITIONER WITH A SPECIAL SCALING FOR MORTAR DISCRETIZATION OF ELLIPTIC PROBLEMS WITH DISCONTINUOUS COEFFICIENTS*

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Abstract. We consider two-dimensional elliptic problems with discontinuous coefficients discretized by the finite element method on geometrically conforming nonmatching triangulations across the interface using the mortar technique. The resulting discrete problem is solved by a dual-primal FETI method.

In this paper we introduce and analyze a preconditioner with a special scaling of coefficients and step parameters and establish convergence bounds. We show that the preconditioner is almost optimal with constants independent of the jumps of coefficients and step parameters. Extensive computational evidence is presented that illustrates an almost optimal convergence for a variety of situations (distribution of subregions, grid assignment, grid ratios, number of subregions) for both continuous and discontinuous problems.

Key words. domain decomposition, mortar finite element method, dual-primal FETI preconditioner, nonmatching grids, saddle-point problem, elliptic problems with discontinuous coefficients

AMS subject classifications. 65N55, 65N30, 65F10

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1. Introduction. In this paper we discuss a second order elliptic problem with discontinuous coefficients defined on a polygonal region $\Omega \subset \mathbb{R}^2$ which is a union of many polygons Ω_i . The problem is discretized by the finite element method (FEM) on geometrically conforming nonmatching triangulations across $\bar{\Gamma} = \cup_i \partial\Omega_i \setminus \partial\Omega$ using the mortar technique; see [1]. The resulting discrete problem is solved by a dual-primal FETI (FETI-DP) method; see [5], [6], [7] for the matching triangulation and [3], [4] for the nonmatching one. The method is discussed under the assumption of continuity of the solution at vertices of Ω_i . We prove that the method is convergent and its rate of convergence is almost optimal and independent of the jumps of coefficients, provided that a mortar side is associated with the higher coefficient. Consequently, the method is well suited for parallel processors.

The presented results are a generalization of results obtained in [4] and [3] for problems with continuous and discontinuous coefficients, respectively. In [4] a modified mortar condition at the vertices of substructures is employed using the assumption that the solution at the vertices is continuous, while in [3] a standard approximation to the mortar condition is employed. The preconditioner in [3] which does not use the scaling of the coefficients was tested for the simplest case of four subregions. In general, however, the experiments show that for discontinuous coefficients the preconditioner without proper scaling of coefficients exhibits poor convergence.

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In this paper we introduce a preconditioner with special scaling of coefficients and step parameters. The theoretical analysis and experimental results show that the proposed preconditioner exhibits excellent properties for general cases considered here: its convergence is almost optimal with respect to the parameters of triangulations (it depends on a logarithmical factor only) and independent of the jumps of coefficients. Extensive numerical experiments on many subregions are reported.

The paper is organized as follows. In section 2, the differential and discrete problems are formulated. In section 3, a matrix form of the discrete problem is given. The preconditioner is described and analyzed in section 4. The implementation of the method and numerical experiments are presented in section 5.

2. Differential and discrete problem. We consider the following differential problems.

Find $u^* \in H_0^1(\Omega)$ such that

$$(1) \quad a(u^*, v) = f(v), \quad v \in H_0^1(\Omega),$$

where

$$a(u, v) = (\rho(x)\nabla u, \nabla v)_{L^2(\Omega)}, \quad f(v) = (f, v)_{L^2(\Omega)}.$$

We assume that Ω is a polygonal region and $\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$, Ω_i are disjoint polygonal subregions of diameter H_i , $\rho(x) = \rho_i$ is a positive constant on Ω_i , and $f \in L^2(\Omega)$. We solve (1) by the FEM on nonmatching triangulation across $\partial\Omega_i$. To describe a discrete problem the mortar technique is used; see [1] and [8] and the literature therein.

We impose on Ω_i a triangulation with triangular elements and parameter h_i . The resulting triangulation of Ω is nonmatching across $\partial\Omega_i$. We assume that the triangulation on each Ω_i is quasi-uniform. Let $X_i(\Omega_i)$ be a finite element space of piecewise linear continuous functions defined on the introduced triangulation. We assume that functions of $X_i(\Omega_i)$ vanish on $\partial\Omega_i \cap \partial\Omega$. Let

$$X^h(\Omega) = X_1(\Omega_1) \times \dots \times X_N(\Omega_N).$$

Note that $X^h(\Omega) \subset L^2(\Omega)$ but $X^h(\Omega) \not\subset H_0^1(\Omega)$. To formulate a discrete problem for (1) we use the mortar technique for the geometrically conforming case. For that the following notation is used. Let Γ_{ij} be a common edge of two substructures Ω_i and Ω_j , $\Gamma_{ij} = \partial\Omega_i \cap \partial\Omega_j$. Let $\Gamma = (\cup_i \partial\Omega_i) \setminus \partial\Omega$. We now select open edges $\gamma_m \subset \Gamma$, called *mortar*, such that $\bar{\Gamma} = \cup \bar{\gamma}_m$ and $\gamma_m \cap \gamma_n = \emptyset$ for $m \neq n$. Let Γ_{ij} as an edge of Ω_i be denoted by $\gamma_{m(i)}$ and called *mortar* (master), and let Γ_{ij} as an edge of Ω_j be denoted by $\delta_{m(j)}$ and called *nonmortar* (slave). The criterion for choosing $\gamma_{m(i)}$ as the mortar side is that $\rho_i \geq \rho_j$, the coefficients on Ω_i and Ω_j , respectively.

Let $M(\delta_{m(j)})$ be a subspace of $W_j(\delta_{m(j)})$, the restriction of $X_j(\Omega_j)$ to $\delta_{m(j)}$, $\delta_{m(j)} \subset \partial\Omega_j$. Functions of $M(\delta_{m(j)})$ are constants on elements of the triangulation on $\delta_{m(j)}$ which touch $\partial\delta_{m(j)}$. We say that $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ on $\delta_m \equiv \delta_{m(j)} = \gamma_{m(i)} = \Gamma_{ij}$, an edge common to Ω_i and Ω_j , satisfy the mortar condition if

$$(2) \quad \int_{\delta_m} (u_i - u_j)\psi \, ds = 0, \quad \psi \in M(\delta_m).$$

Note that for the given u_i on $\gamma_{m(i)}$ and u_j on $\partial\delta_{m(j)}$, denoted by $\text{Tr } u_j$, we can compute u_j at the interior nodal points of $\delta_{m(j)}$. Denoting the u_j computed in this

way by $\pi_m(u_i; \text{Tr } u_j)$ we have

$$\int_{\delta_m} \pi_m(u_i; \text{Tr } u_j) \psi \, ds = \int_{\delta_m} u_i \psi \, ds, \quad \psi \in M(\delta_m),$$

$$\pi_m(u_i; \text{Tr } u_j) = \text{Tr } u_j \text{ on } \partial\delta_m.$$

Note that $\pi_m(u_i; \text{Tr } u_j)$ is an element of X_j restricted to $\delta_{m(j)}$.

We are now in a position to introduce V^h , the space for discretization of (1). Let $V^h(\Omega)$ be a subspace of $X^h(\Omega)$ of functions which satisfy the mortar condition (2) for each $\delta_m \subset \Gamma$ and which are continuous at common vertices of the substructures. The discrete problem for (1) in V^h is defined as follows.

Find $u_h^* \in V^h$ such that

$$(3) \quad a_H(u_h^*, v_h) = f(v_h), \quad v_h \in V^h,$$

where $a_H(u, v) = \sum_{i=1}^N a_i(u, v)$, $a_i(u, v) = \rho_i(\nabla u, \nabla v)_{L^2(\Omega_i)}$.

The problem has a unique solution and the error bound is known; see [1]. Using the basis functions of V^h , $V^h = \text{span} \{ \Phi_k \}$, the problem (3) is rewritten as

$$A \underline{u}_h^* = \underline{f}.$$

The form of Φ_k can be found, for example, in [4]. The matrix A is symmetric positive definite and $\text{cond}(A) \leq \frac{C}{\min h_i^2}$, where C here depends on the ρ_i .

3. FETI-DP equation. To derive a FETI-DP method we first rewrite the problem (3) as a saddle-point problem using Lagrange multipliers; see, for example, [8] and the literature therein. For $u = \{u_i\}_{i=1}^N \in X^h(\Omega)$ and $\psi = \{\psi_p\}_{p=1}^P \in M(\Gamma) = \prod_m M(\delta_m)$, the mortar condition (2) can be rewritten as

$$b(u, \psi) \equiv \sum_{i=1}^N \sum_{\delta_{m(i)} \subset \partial\Omega_i} \int_{\delta_{m(i)}} (u_i - u_j) \psi_k \, ds = 0,$$

where $\delta_{m(i)} = \gamma_{m(j)} = \Gamma_{ij}$, $\psi_k \in M(\delta_{m(i)})$. Let $\tilde{X}^h(\Omega)$ denote a subspace of $X^h(\Omega)$ of functions which are common to the vertices of substructures.

The problem now consists of finding $(u_h^*, \lambda_h^*) \in \tilde{X}^h(\Omega) \times M(\Gamma)$ such that

$$(4) \quad a(u_h^*, v_h) + b(v_h, \lambda_h^*) = f(v_h), \quad v_h \in \tilde{X}^h(\Omega),$$

$$(5) \quad b(u_h^*, \psi_h) = 0, \quad \psi_h \in M(\Gamma).$$

It can be proved that u_h^* , the solution of (4)–(5), is the solution of (3) and vice versa. Therefore the problem (4)–(5) has a unique solution. This can be proved straightforwardly using the inf-sup condition, including the error bound; see [8] and the literature therein.

To derive a matrix form of (4)–(5) we first need a matrix formulation of (5). Using the nodal basis functions $\varphi_{\delta_{m(i)}}^{(l)} \in W_i(\delta_{m(i)})$, $\varphi_{\gamma_{m(j)}}^{(k)} \in W_j(\gamma_{m(j)})$, and $\psi_{\delta_{m(i)}}^{(p)} \in M_m(\delta_{m(i)})$ ($\delta_{m(i)} = \gamma_{m(j)} = \Gamma_{ij}$), (5) can be rewritten on $\tilde{\delta}_{m(i)}$ as

$$(6) \quad B_{\delta_{m(i)}} u_{i\delta_{m(i)}} - B_{\gamma_{m(j)}} u_{j\gamma_{m(j)}} = 0,$$

where $u_{i\delta_{m(i)}}$ and $u_{j\gamma_{m(j)}}$ are vectors which represent $u_i|_{\delta_{m(i)}} \in W_i(\delta_{m(i)})$ and $u_j|_{\gamma_{m(j)}} \in W_j(\gamma_{m(j)})$, and $(n_{\delta_{(i)}} \equiv n_{\delta_{m(i)}} \text{ and } n_{\gamma_{(j)}} \equiv n_{\gamma_{m(j)}})$:

$$B_{\delta_{m(i)}} = \{(\psi_{\delta_{m(i)}}^{(p)}, \varphi_{\delta_{m(i)}}^{(k)})_{L^2(\delta_{m(i)})}\}, \quad p = 1, \dots, n_{\delta(i)}, \quad k = 0, \dots, n_{\delta(i)} + 1,$$

$$B_{\gamma_{m(j)}} = \{(\psi_{\delta_{m(i)}}^{(p)}, \varphi_{\gamma_{m(j)}}^{(l)})_{L^2(\gamma_{m(j)})}\}, \quad p = 1, \dots, n_{\delta(i)}, \quad l = 0, \dots, n_{\gamma(j)} + 1.$$

Here $n_{\delta(i)}$, $n_{\delta(i)} + 2$, and $n_{\gamma(j)} + 2$ are the dimensions of $M_m(\delta_{m(i)})$, $W_i(\delta_{m(i)})$, and $W_j(\gamma_{m(j)})$, respectively. Note that $B_{\delta_{m(i)}}$ and $B_{\gamma_{m(j)}}$ are rectangular matrices. We split the vectors $u_{i\delta_{m(i)}}$ and $u_{j\gamma_{m(j)}}$ into vectors $u_{i\delta_{m(i)}}^{(r)}$, $u_{i\delta_{m(i)}}^{(c)}$ and $u_{j\gamma_{m(j)}}^{(r)}$, $u_{j\gamma_{m(j)}}^{(c)}$, respectively, where $u_{i\delta_{m(i)}}^{(c)}$ and $u_{j\gamma_{m(j)}}^{(c)}$ represent values of functions u_i and u_j at the end points of $\delta_{m(i)}$ and $\gamma_{m(j)}$, and $u_{i\delta_{m(i)}}^{(r)}$ and $u_{j\gamma_{m(j)}}^{(r)}$ represent values of u_i and u_j at the interior nodal points of $\delta_{m(i)}$ and $\gamma_{m(j)}$. Using this notation one can rewrite (6) as

$$(7) \quad (B_{\delta_{m(i)}}^{(r)} u_{i\delta_{m(i)}}^{(r)} + B_{\delta_{m(i)}}^{(c)} u_{i\delta_{m(i)}}^{(c)}) - (B_{\gamma_{m(j)}}^{(r)} u_{j\gamma_{m(j)}}^{(r)} + B_{\gamma_{m(j)}}^{(c)} u_{j\gamma_{m(j)}}^{(c)}) = 0.$$

Note that

$$B_{\delta_{m(i)}}^{(r)} = \{(\psi_{\delta_{m(i)}}^{(p)}, \varphi_{\delta_{m(i)}}^{(k)})_{L^2(\delta_{m(i)})}\}, \quad p, k = 1, \dots, n_{\delta(i)}$$

is a square tridiagonal matrix $n_{\delta(i)} \times n_{\delta(i)}$, symmetric and positive definite and $\text{cond}(B_{\delta_{m(i)}}^{(r)}) \sim 1$, while the remaining matrices $B_{\delta_{m(i)}}^{(c)}$, $B_{\gamma_{m(j)}}^{(c)}$, $B_{\gamma_{m(j)}}^{(r)}$ are rectangular with dimensions $n_{\delta(i)} \times 2$, $n_{\delta(i)} \times 2$, $n_{\delta(i)} \times n_{\gamma(j)}$, respectively.

Let $K^{(l)}$ be the stiffness matrix of $a_l(\cdot, \cdot)$. It is represented as

$$(8) \quad K^{(l)} = \begin{pmatrix} K_{ii}^{(l)} & K_{ic}^{(l)} & K_{ir}^{(l)} \\ K_{ci}^{(l)} & K_{cc}^{(l)} & K_{cr}^{(l)} \\ K_{ri}^{(l)} & K_{rc}^{(l)} & K_{rr}^{(l)} \end{pmatrix},$$

where the rows correspond to the interior unknowns $u_l^{(i)}$ of Ω_l , $u_c^{(l)}$ to its vertices and $u_l^{(r)}$ to its edges.

Using the above notation and the assumption of continuity of u_h^* at the vertices of $\partial\Omega_l$, (4)–(5) can be rewritten as

$$(9) \quad \begin{pmatrix} K_{ii} & K_{ic} & K_{ir} & 0 \\ K_{ci} & \tilde{K}_{cc} & K_{cr} & B_c^T \\ K_{ri} & K_{rc} & K_{rr} & B_r^T \\ 0 & B_c & B_r & 0 \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(c)} \\ u^{(r)} \\ \tilde{\lambda}^* \end{pmatrix} = \begin{pmatrix} f^{(i)} \\ f^{(c)} \\ f^{(r)} \\ 0 \end{pmatrix}.$$

Here $\tilde{\lambda}^* = \{B_{\delta_{m(i)}}^{(r)} \lambda_{\delta_{m(i)}}^*\}$, $\delta_{m(i)} \subset \Gamma$, u_h^* is the solution of (4)–(5) and is represented by the vectors $u^{(i)}$, $u^{(c)}$, and $u^{(r)}$, which are the values of u_h^* at the interior nodal points of Ω_l , the vertices of Ω_l , and the remaining nodal points of $\partial\Omega_l \setminus \partial\Omega$, respectively; matrices K_{ii} and K_{rr} are diagonal block-matrices of $K_{ii}^{(l)}$ and $K_{rr}^{(l)}$, respectively, while matrix \tilde{K}_{cc} is built from diagonal block matrices $K_{cc}^{(l)}$ taking into account that $u^{(c)}$ are the same at the common vertices of substructures. The remaining K -matrices represent coupling between the corresponding unknowns. The mortar condition is

represented by $B = (B_c, B_r)$, where these global matrices are represented by the local ones $((B_{\delta_{m(i)}}^{(r)})^{-1}B_{\delta_{m(i)}}^{(c)}, -(B_{\delta_{m(i)}}^{(r)})^{-1}B_{\gamma_{m(j)}}^{(c)})$, and $(I_{\delta_{m(i)}}^{(r)}, -(B_{\delta_{m(i)}}^{(r)})^{-1}B_{\gamma_{m(j)}}^{(r)})$, respectively, and $I_{\delta_{m(i)}}^{(r)}$ is an identity matrix of $n_{\delta(i)} \times n_{\delta(i)}$. The form of these matrices follows from (7) after multiplying it by $(B_{\delta_{m(i)}}^{(r)})^{-1}$.

In the system (9) we eliminate the unknowns $u^{(i)}$ and $u^{(c)}$ to obtain

$$(10) \quad \begin{pmatrix} \tilde{S} & \tilde{B}^T \\ \tilde{B} & \tilde{S}_{cc} \end{pmatrix} \begin{pmatrix} u^{(r)} \\ \tilde{\lambda}^* \end{pmatrix} = \begin{pmatrix} \tilde{f}_r \\ \tilde{f}_c \end{pmatrix},$$

where

$$(11) \quad \begin{cases} \tilde{S} = K_{rr} - (K_{ri}, K_{rc}) \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} K_{ir} \\ K_{cr} \end{pmatrix}, \\ \tilde{f}_r = f^{(r)} - (K_{ri}, K_{rc}) \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} f^{(i)} \\ f^{(c)} \end{pmatrix}, \\ \tilde{B} = B_r - (0, B_c) \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_c^T \end{pmatrix}, \\ \tilde{S}_{cc} = -(0, B_c) \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ B_c^T \end{pmatrix}, \\ \tilde{f}_c = -(0, B_c) \begin{pmatrix} K_{ii} & K_{ic} \\ K_{ci} & \tilde{K}_{cc} \end{pmatrix}^{-1} \begin{pmatrix} f^{(i)} \\ f^{(c)} \end{pmatrix}. \end{cases}$$

Note that \tilde{S} is invertible since u_h^* is continuous at the vertices of Ω_l and vanishes on $\partial\Omega$.

We next eliminate the unknown $u^{(r)}$ to get for $\tilde{\lambda}^* \in M(\Gamma)$

$$(12) \quad F\tilde{\lambda}^* = d,$$

where

$$(13) \quad F = \tilde{B}\tilde{S}^{-1}\tilde{B}^T - \tilde{S}_{cc} \quad \text{and} \quad d = \tilde{B}\tilde{S}^{-1}\tilde{f}_r - \tilde{f}_c.$$

This is the FETI-DP equation for the Lagrange multipliers. Since F is positive definite the problem has a unique solution. This problem can be solved by conjugate gradient iterations with a preconditioner discussed in the next section.

4. FETI-DP preconditioner. In this section we define a preconditioner for the problem (12). For that let $S^{(l)}$ denote the Schur complement of $K^{(l)}$, see (8), with respect to unknowns at the nodal points of $\partial\Omega_l$. This matrix is represented as

$$(14) \quad S^{(l)} = \begin{pmatrix} S_{rr}^{(l)} & S_{rc}^{(l)} \\ S_{cr}^{(l)} & S_{cc}^{(l)} \end{pmatrix},$$

where the second row corresponds to unknowns at the vertices of $\partial\Omega_l$ while the first one corresponds to the remaining unknowns of $\partial\Omega_l$. Note that B_r is a matrix obtained from B defined on functions with zero values at the vertices of Ω_l and let

$$(15) \quad \begin{aligned} S &= \text{diag} \{S^{(l)}\}_{l=1}^N, & S_{rr} &= \text{diag} \{S_{rr}^{(l)}\}_{l=1}^N, \\ S_{cc} &= \text{diag} \{S_{cc}^{(l)}\}_{l=1}^N, & S_{cr} &= (S_{cr}^{(1)}, \dots, S_{cr}^{(N)}). \end{aligned}$$

The standard preconditioner developed for continuous problems in [4] is defined as

$$(16) \quad \bar{M}^{-1} = B_r \widehat{S}_{rr} B_r^T,$$

where $\widehat{S}_{rr} = \text{diag} \{ \widehat{S}_{rr}^{(i)} \}_{i=1}^N$, $\widehat{S}_{rr}^{(i)} = S_{rr}^{(i)}$ for $\rho_i = 1$.

We employ a special scaling to generalize \bar{M} to problems with discontinuous coefficients. The preconditioner M for (12) is defined as

$$(17) \quad M^{-1} = \widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T,$$

where

$\widehat{B}_r|_{\delta_{m(i)}} = (\rho_i^{1/2} I_{\delta_{m(i)}}, -\frac{h_{\delta_{m(i)}} \rho_i}{h_{\gamma_{m(j)}} \rho_j} \rho_i^{1/2} B_{\delta_{m(i)}}^{-1} B_{\gamma_{m(j)}})$ for $\delta_{m(i)} \subset \partial\Omega_i$, $i = 1, \dots, N$; $h_{\delta_{m(i)}}$ and $h_{\gamma_{m(j)}}$ are the step parameters on $\delta_{m(i)}$ and $\gamma_{m(j)}$, $\delta_{m(i)} = \gamma_{m(j)}$, respectively.

An ordering of substructures Ω_l is called mortar-nonmortar (M-N) ordering if all sides of a fixed Ω_l are mortar while all sides of the neighboring substructures of Ω_l are nonmortar.

THEOREM 4.1. *Let the mortar side be chosen where the coefficient ρ_i is larger. Then for $\lambda \in M(\Gamma)$ the following holds:*

$$(18) \quad c_0 \left(1 + \log \frac{H}{h} \right)^\alpha \langle M\lambda, \lambda \rangle \leq \langle F\lambda, \lambda \rangle \leq c_1 \left(1 + \log \frac{H}{h} \right)^2 \langle M\lambda, \lambda \rangle,$$

where $\alpha = 0$ for M-N ordering of substructures and $\alpha = -2$ in the general case; c_0 and c_1 are positive constants independent of h_i, H_i , and the jumps of ρ_i ; and $h = \min_i h_i, H = \max_i H_i$.

Proof. To prove Theorem 4.1 we need some additional facts. We first reformulate the process of reaching (12) from (9). For that we eliminate $u^{(i)}$ from the system (9). Using the notation (14) and (15) we get

$$(19) \quad S_{rr}u^{(r)} + S_{rc}u^{(c)} + B_r^T \tilde{\lambda}^* = g_r,$$

$$(20) \quad S_{cr}u^{(r)} + \bar{S}_{cc}u^{(c)} + B_c^T \tilde{\lambda}^* = g_c,$$

$$(21) \quad B_r u^{(r)} + B_c u^{(c)} = 0.$$

Here S_{rr} and S_{cr} ($S_{cr} = S_{cr}^T$) are defined in (15) while \bar{S}_{cc} is defined by $S_{cc}^{(l)}$ (see (14)), taking into account that $u_l^{(c)}$ are the same at the common vertices of substructures.

We now eliminate $u^{(r)}$ and $u^{(c)}$ in (19)–(21). This leads to (12) with F and d of the form

$$(22) \quad F = F_{rr} + F_{rc}F_{cc}^{-1}F_{cr}, \quad d = d_r + F_{rc}F_{cc}^{-1}d_c.$$

Here

$$(23) \quad F_{rr} = B_r S_{rr}^{-1} B_r^T$$

and

$$F_{rc} = B_c - B_r S_{rr}^{-1} S_{rc}, \quad F_{cc} = \bar{S}_{cc} - S_{cr} S_{rr}^{-1} S_{rc},$$

$$d_c = g_c - S_{cr} S_{rr}^{-1} g_r, \quad d_r = B_r S_{rr}^{-1} g_r.$$

In the proof of Theorem 4.1 we will also need two lemmas.

LEMMA 4.2. *For $w \in X_1(\partial\Omega_1) \times \cdots \times X_N(\partial\Omega_N)$ with the same values at the vertices of Ω_i , the following holds:*

$$(24) \quad |\widehat{B}_r^T B_r z|_{\widehat{S}_{rr}}^2 \leq C \left(1 + \log \frac{H}{h}\right)^2 |w|_S^2,$$

provided that ρ_i on the mortar side is larger than on the nonmortar side, where $z = w - I_H w$ and $I_H w$ is a linear interpolant of w on edges of $\partial\Omega_i$ with values w at the end points of the edges.

Proof. A proof of this estimate is a modification of the proof of Lemma 1 from [4]. We have

$$|\widehat{B}_r^T B_r z|_{\widehat{S}_{rr}}^2 = \langle \widehat{S}_{rr} \widehat{B}_r^T B_r z, \widehat{B}_r^T B_r z \rangle.$$

Hence

$$(25) \quad |\widehat{B}_r^T B_r z|_{\widehat{S}_{rr}}^2 = \sum_{i=1}^N |\widehat{B}_r^T B_r z|_{\widehat{S}^{(i)}}^2.$$

Note that $\widehat{B}_r^T B_r z = 0$ at the vertices. Using that, we get

$$(26) \quad |\widehat{B}_r^T B_r z|_{\widehat{S}^{(i)}}^2 \leq C \left(\sum_{\delta_{m(i)} \subset \partial\Omega_i} |\widehat{B}_r^T B_r z|_{\widehat{S}_{\delta_{m(i)}}}^2 + \sum_{\gamma_{m(i)} \subset \partial\Omega_i} |\widehat{B}_r^T B_r z|_{\widehat{S}_{\gamma_{m(i)}}}^2 \right),$$

where $\widehat{S}_{\delta_{m(i)}}$ and $\widehat{S}_{\gamma_{m(i)}}$ are matrix representations of the $H_{00}^{1/2}$ -norm on $\delta_{m(i)}$ and $\gamma_{m(i)}$, respectively. From the structure of \widehat{B}_r follows

$$(27) \quad |\widehat{B}_r^T B_r z|_{\widehat{S}_{\delta_{m(i)}}}^2 \leq 2 \left(\rho_i |z_i|_{\widehat{S}_{\delta_{m(i)}}}^2 + \rho_i |B_{ij} z_j|_{\widehat{S}_{\delta_{m(i)}}}^2 \right),$$

where here and below $z = \{z_i\}_{i=1}^N \in X^h(\Gamma)$, the restriction of $X^h(\Omega)$ to Γ , $B_{ij} \equiv (B_{\delta_{m(i)}}^{(r)})^{-1} B_{\gamma_{m(j)}}^{(r)}$, and $\delta_{m(i)} = \gamma_{m(j)}$, $\gamma_{m(j)} \subset \partial\Omega_j$;

$$(28) \quad |\widehat{B}_r^T B_r z|_{\widehat{S}_{\gamma_{m(i)}}}^2 \leq 2 \left(\rho_k \left(\frac{\rho_k}{\rho_i} \right)^2 |\widehat{B}_{ki}^T z_k|_{\widehat{S}_{\gamma_{m(i)}}}^2 + \rho_k \left(\frac{\rho_k}{\rho_i} \right)^2 |B_{ki}^T \widehat{B}_{ki} z_i|_{\widehat{S}_{\gamma_{m(i)}}}^2 \right),$$

where $B_{ki} \equiv (B_{\delta_{m(k)}}^{(r)})^{-1} B_{\gamma_{m(i)}}^{(r)}$, $\widehat{B}_{ki} = \alpha_{ki} B_{ki}$, $\alpha_{ki} = \frac{h_{\delta_{m(k)}}}{h_{\gamma_{m(i)}}}$, $\gamma_{m(i)} = \delta_{m(k)}$, and $\delta_{m(k)} \subset \partial\Omega_k$. We now estimate each term of (27) and (28).

We estimate the first term of (27) as in [4]:

$$(29) \quad \begin{aligned} \rho_i |z_i|_{\widehat{S}_{\delta_{m(i)}}}^2 &\leq C \rho_i (1 + \log \frac{H}{h})^2 |w_i|_{H^{1/2}(\partial\Omega_i)}^2 \\ &\leq C \rho_i (1 + \log \frac{H}{h})^2 |w_i|_{\widehat{S}^{(i)}}^2 \leq C (1 + \log \frac{H}{h})^2 |w_i|_{\widehat{S}^{(i)}}^2. \end{aligned}$$

To estimate the second term of (27) we use the stability of the mortar projection. Let $\pi_{\delta_{m(i)}}(z_j, 0)$ correspond to $B_{ij}(z_j|_{\gamma_{m(j)}})$ for z_j restricted to $\gamma_{m(j)}$. Using that, we have

$$(30) \quad \begin{aligned} \rho_i |B_{ij} z_j|_{\widehat{S}_{\delta_{m(i)}}}^2 &\leq C \rho_i \|\pi_{\delta_{m(i)}}(z_j, 0)\|_{H_{00}^{1/2}(\delta_{m(i)})}^2 \\ &\leq C \rho_i \|z_j\|_{H_{00}^{1/2}(\gamma_{m(j)})}^2 \leq C \rho_i (1 + \log \frac{H}{h})^2 |w_j|_{\widehat{S}^{(j)}}^2 \leq C (1 + \log \frac{H}{h})^2 |w_j|_{\widehat{S}^{(j)}}^2. \end{aligned}$$

We now estimate the terms of (28). It has been shown in [4, proof of Lemma 1 and (28)] that the following holds:

$$|B_{ki}^T z_k|_{\widehat{S}_{\gamma_{m(i)}}}^2 \leq C |z_k|_{\widehat{S}_{\delta_{m(k)}}}^2 \leq C(1 + \log \frac{H}{h})^2 |w_k|_{\widehat{S}^{(k)}}^2,$$

under the assumption that $h_{\delta_{m(k)}} \sim h_{\gamma_{m(i)}}$. This assumption can be removed by introducing the scaling $\alpha_{ki} = \frac{h_{\delta_{m(k)}}}{h_{\gamma_{m(i)}}$ in B_{ki} . Thus, this estimate is valid for $\alpha_{ki} B_{ki}$ without assuming that $h_{\delta_{m(k)}} \sim h_{\gamma_{m(i)}}$; for details see the proof of Lemma 1 in [4].

Thus, the first term of (28) can be estimated as

$$(31) \quad \begin{aligned} \alpha_{ki}^2 \rho_k \left(\frac{\rho_k}{\rho_i}\right)^2 |B_{ki}^T z_k|_{\widehat{S}_{\gamma_{m(i)}}}^2 &\leq \alpha_{ki}^2 \rho_k |B_{ki}^T z_k|_{\widehat{S}_{\gamma_{m(i)}}}^2 \\ &\leq \rho_k |z_k|_{\widehat{S}_{\delta_{m(k)}}}^2 \leq C(1 + \log \frac{H}{h})^2 |w_k|_{\widehat{S}^{(k)}}^2. \end{aligned}$$

It remains to estimate the second term of (28). It has been shown in [4, proof of Lemma 1] that the following holds under the assumption that $h_{\delta_{m(k)}} \sim h_{\gamma_{m(i)}}$:

$$|B_{ki}^T B_{ki} z_i|_{\widehat{S}_{\gamma_{m(i)}}}^2 \leq C \left(1 + \log \frac{H}{h}\right)^2 |w_i|_{\widehat{S}^{(i)}}^2.$$

Thus, using the scaling α_{ki} in \widehat{B}_{ki} we get

$$(32) \quad \begin{aligned} \alpha_{ki}^2 \rho_k \left(\frac{\rho_k}{\rho_i}\right)^2 |B_{ki}^T B_{ki} z_i|_{\widehat{S}_{\gamma_{m(i)}}}^2 &\leq \alpha_{ki}^2 \rho_k |B_{ki}^T B_{ki} z_i|_{\widehat{S}_{\gamma_{m(i)}}}^2 \\ &\leq C \rho_k (1 + \log \frac{H}{h})^2 |w_i|_{\widehat{S}^{(i)}}^2 \leq C(1 + \log \frac{H}{h})^2 |w_i|_{\widehat{S}^{(i)}}^2, \end{aligned}$$

without the assumption that $h_{\delta_{m(k)}} \sim h_{\gamma_{m(i)}}$.

Substituting these four estimates (29)–(32) into (27)–(28) and the resulting estimates into (26) gives

$$|\widehat{B}_r^T B_r z|_{\widehat{S}^{(i)}}^2 \leq C \left(1 + \log \frac{H}{h}\right)^2 \left(|w_i|_{\widehat{S}^{(i)}}^2 + \sum_j |w_j|_{\widehat{S}^{(j)}}^2 \right),$$

where the sum is taken over $\partial\Omega_j$, which intersects $\partial\Omega_i$ by an edge. Using this in (25) provides (24). This completes the proof of Lemma 4.2. \square

LEMMA 4.3. For F_{rr} defined in (23) and $\lambda \in M(\Gamma)$,

$$(33) \quad C \left(1 + \log \frac{H}{h}\right)^\alpha \langle M\lambda, \lambda \rangle \leq \langle F_{rr}\lambda, \lambda \rangle,$$

where $\alpha = 0$ for a M-N ordering of substructures Ω_l and $\alpha = -2$ in the general case, and C is independent of h, H , and the jumps of ρ_i .

Proof. A proof of this estimate is a modification of the proof of Theorems 2 and 3 from [4]. We first prove it for the M-N ordering of substructures. In this case \widehat{B}_r can be represented as (see (17))

$$(34) \quad \widehat{B}_r = (\widehat{I}_N, -\widehat{B}_M),$$

where \widehat{I}_N and \widehat{B}_M are block diagonal matrices with blocks $\rho_i^{1/2} I_{\delta_{m(i)}}$ and $\alpha_{ij} \frac{\rho_i}{\rho_j} \rho_i^{1/2} (B_{\delta_{m(i)}}^{(r)})^{-1} B_{\gamma_{m(j)}}^{(r)}$, $\alpha_{ij} = \frac{h_{\delta_{m(i)}}}{h_{\gamma_{m(j)}}$, corresponding to the N (nonmortar) and

M (mortar) substructures Ω_i , respectively. Matrix B_r is decomposed in the same way. For this ordering we can reorder matrices (15) as

$$(35) \quad S_{rr} = \begin{pmatrix} S_{rr}^N & 0 \\ 0 & S_{rr}^M \end{pmatrix}, \quad \widehat{S}_{rr} = \begin{pmatrix} \widehat{S}_{rr}^N & 0 \\ 0 & \widehat{S}_{rr}^M \end{pmatrix},$$

where the first row corresponds to the nonmortar subregions and $S_{rr}^N = \text{diag}_{i \in N} \{S_{rr}^{(i)}\}$, while the second one corresponds to mortar subregions and $S_{rr}^M = \text{diag}_{i \in M} \{S_{rr}^{(i)}\}$. Then using (34) we can write preconditioner M (see (17)) in the form

$$(36) \quad M^{-1} = \widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T = S_{rr}^N + \widehat{B}_M \widehat{S}_{rr}^M \widehat{B}_M^T.$$

Note, since both terms are positive definite, that

$$\langle S_{rr}^N \lambda, \lambda \rangle \leq \langle M^{-1} \lambda, \lambda \rangle,$$

and as a consequence

$$\langle M \lambda, \lambda \rangle \leq \langle (S_{rr}^N)^{-1} \lambda, \lambda \rangle.$$

Using this and

$$\langle S_{rr}^{-1} B_r^T \lambda, B_r^T \lambda \rangle = \langle (S_{rr}^N)^{-1} \lambda, \lambda \rangle + \langle (S_{rr}^M)^{-1} B_M^T \lambda, B_M^T \lambda \rangle,$$

we obtain (see (23))

$$\lambda_{\min}(M^{-1/2} F_{rr} M^{-1/2}) = \min_{\lambda} \frac{\langle S_{rr}^{-1} B_r^T \lambda, B_r^T \lambda \rangle}{\langle M \lambda, \lambda \rangle} \geq 1,$$

which completes the proof for the M-N ordering.

In the case of a general ordering (non-M-N) of substructures, we have

$$\widehat{B}_r = (\widehat{I}_r^{(n)}, -\widehat{B}_r^{(m)}),$$

where $\widehat{I}_r^{(n)}$ and $\widehat{B}_r^{(m)}$ are block diagonal matrices with blocks $\rho_i^{1/2} I_{\delta_{m(i)}}$ and $\alpha_{ij} \frac{\rho_i}{\rho_j} \rho_i^{1/2} (B_{\delta_{m(i)}}^{(r)})^{-1} B_{\gamma_{m(j)}}^{(r)}$ corresponding to the nonmortar and mortar sides, respectively. In this general case matrix (15) is not block diagonal and is of the form

$$(37) \quad S_{rr} = \begin{pmatrix} S_{rr}^{nn} & S_{rr}^{nm} \\ S_{rr}^{mn} & S_{rr}^{mm} \end{pmatrix},$$

where the first row corresponds to the nonmortar sides and the second to the mortar sides. We introduce an auxiliary matrix

$$(38) \quad \text{diag}\{S_{rr}\} = \begin{pmatrix} S_{rr}^{nn} & 0 \\ 0 & S_{rr}^{mm} \end{pmatrix}.$$

Using the fact that $S_{rr} = S_{rr}^T > 0$ we get

$$\pm \begin{pmatrix} 0 & S_{rr}^{nm} \\ S_{rr}^{mn} & 0 \end{pmatrix} \leq \begin{pmatrix} S_{rr}^{nn} & 0 \\ 0 & S_{rr}^{mm} \end{pmatrix},$$

from which follows that for w with zero values at the vertices of Ω_i we have

$$(39) \quad \langle S_{rr} w, w \rangle \leq 2 \langle \text{diag}\{S_{rr}\} w, w \rangle.$$

Additionally, the following holds (see Lemma 2 in [4]):

$$(40) \quad \langle \text{diag}\{S_{rr}\}w, w \rangle \leq C \left(1 + \log \frac{H}{h}\right)^2 \langle S_{rr}w, w \rangle.$$

The proof of Lemma 4.3 reduces to showing that

$$\lambda_{\min}(M^{-1/2}F_{rr}M^{-1/2}) = \min_{\lambda} \frac{\langle S_{rr}^{-1}B_r^T \lambda, B_r^T \lambda \rangle}{\langle (\widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T)^{-1} \lambda, \lambda \rangle} \geq \frac{C}{(1 + \log \frac{H}{h})^2}.$$

(This fact has been proved in Lemma 1 of [4] for $\rho_i = 1$; the generalization for $\rho_i \neq 1$ is straightforward.) We have

$$(41) \quad \lambda_{\min}(M^{-1/2}F_{rr}M^{-1/2}) = \min_{\lambda} \frac{\langle F_{rr} \lambda, \lambda \rangle}{\langle M \lambda, \lambda \rangle} = \min_{\lambda} \frac{\langle (S_{rr})^{-1} B_r^T \lambda, B_r^T \lambda \rangle}{\langle (\widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T)^{-1} \lambda, \lambda \rangle}.$$

Using (40) we obtain the following estimate:

$$\begin{aligned} \langle S_{rr}^{nn} \lambda, \lambda \rangle &= \langle \widehat{S}_{rr}^{nn} \widehat{I}_r^{(n)} \lambda, \widehat{I}_r^{(n)} \lambda \rangle \leq \langle \widehat{S}_{rr}^{nn} \widehat{I}_r^{(n)} \lambda, \widehat{I}_r^{(n)} \lambda \rangle + \langle \widehat{S}_{rr}^{mm} (\widehat{B}_r^{(m)})^T \lambda, (\widehat{B}_r^{(m)})^T \lambda \rangle \\ &= \langle \text{diag}\{\widehat{S}_{rr}\} \widehat{B}_r^T \lambda, \widehat{B}_r^T \lambda \rangle \leq C \left(1 + \log \frac{H}{h}\right)^2 \langle \widehat{S}_{rr} \widehat{B}_r^T \lambda, \widehat{B}_r^T \lambda \rangle, \end{aligned}$$

where $\widehat{I}_r^{(n)} = \rho_i^{1/2} I_{\delta_{m(i)}}$ on $\delta_{m(i)} \subset \partial\Omega_i$. Hence,

$$(42) \quad \langle (\widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T)^{-1} \lambda, \lambda \rangle \leq C \left(1 + \log \frac{H}{h}\right)^2 \langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle.$$

On the other hand, by (39)

$$(43) \quad \begin{aligned} \langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle &\leq \langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle + \langle (S_{rr}^{mm})^{-1} \lambda, \lambda \rangle \\ &= \langle \text{diag}\{S_{rr}^{-1}\} \lambda, \lambda \rangle \leq 2 \langle S_{rr}^{-1} \lambda, \lambda \rangle. \end{aligned}$$

Using (42) and (43) in (41) we get

$$\begin{aligned} \lambda_{\min}(M^{-1/2}F_{rr}M^{-1/2}) &\geq \min_{\lambda} \frac{\langle (S_{rr})^{-1} B_r^T \lambda, B_r^T \lambda \rangle}{C(1 + \log \frac{H}{h})^2 \langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle} \\ &\geq \min_{\lambda} \frac{\langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle}{C(1 + \log \frac{H}{h})^2 \langle (S_{rr}^{nn})^{-1} \lambda, \lambda \rangle} = \frac{1}{C(1 + \log \frac{H}{h})^2}. \end{aligned}$$

This completes the proof of Lemma 4.3. \square

Proof of Theorem 4.1. To prove the right-hand side (RHS) of Theorem 4.1 we proceed as follows. For $-\lambda \in M(\Gamma)$ we compute $w = (w^{(r)}, w^{(c)})$ by solving (19) and (20) with $g_r = 0$ and $g_c = 0$. Note that this problem has a unique solution under the assumption that $u^{(c)}$ is continuous at the cross points. Using this we get

$$(44) \quad \begin{aligned} \langle F \lambda, \lambda \rangle &= \langle (F_{rr} + F_{rc} F_{cc}^{-1} F_{cr}) \lambda, \lambda \rangle \\ &= \langle (B_r S_{rr}^{-1} B_r^T + (B_c - B_r S_{rr}^{-1} S_{rc}) F_{cc}^{-1} F_{cr}) \lambda, \lambda \rangle = \langle B_r w^{(r)} + B_c w^{(c)}, \lambda \rangle = \langle B w, \lambda \rangle. \end{aligned}$$

Let $I_H w$ be a linear interpolant of w on edges with values w at the end points of each edge. Note that

$$Bw = B(w - I_H w) \equiv B_r z_r$$

since $z_r \equiv w - I_H w = 0$ at the end points of the edges. Using that in (44), we get

$$(45) \quad \langle F\lambda, \lambda \rangle = \langle Bw, \lambda \rangle = \langle B_r z_r, \lambda \rangle.$$

On the other hand, using that $Sw = B^T \lambda$ (see (19) and (20)), we have

$$(46) \quad \begin{aligned} \langle Bw, \lambda \rangle &= \frac{\langle Bw, \lambda \rangle^2}{\langle Bw, \lambda \rangle} = \frac{\langle B_r z_r, \lambda \rangle^2}{\langle Sw, w \rangle} \\ &= \frac{\langle M^{1/2} \lambda, M^{-1/2} B_r z_r \rangle^2}{|S^{1/2} w|^2} \leq \frac{|M^{1/2} \lambda|^2 |M^{-1/2} B_r z_r|^2}{|w|_S^2}. \end{aligned}$$

Note that by Lemma 4.2 we get

$$|M^{-1/2} B_r z_r|^2 = \langle \widehat{B}_r \widehat{S}_{rr} \widehat{B}_r^T B_r z_r, B_r z_r \rangle = |\widehat{B}_r^T B_r z_r|_{\widehat{S}_{rr}}^2 \leq C \left(1 + \log \frac{H}{h}\right)^2 |w|_S^2.$$

Substituting this into (46) we have

$$\langle Bw, \lambda \rangle \leq C \left(1 + \log \frac{H}{h}\right)^2 |M^{1/2} \lambda|^2.$$

Using this in (45) we get the RHS estimate of (18).

To prove the left-hand side (LHS) of Theorem 4.1 we first note that

$$(47) \quad \langle F\lambda, \lambda \rangle \geq \langle F_{rr} \lambda, \lambda \rangle, \quad \lambda \in M(\Gamma)$$

since $F_{cc}^{-1} > 0$. By Lemma 4.3

$$\langle F_{rr} \lambda, \lambda \rangle \geq c_0 \left(1 + \log \frac{H}{h}\right)^\alpha \langle M\lambda, \lambda \rangle,$$

where $\alpha = 0$ for M-N ordering of substructures and $\alpha = -2$ in the general case. Using this in (47) we get the LHS of (18). \square

5. Implementation and numerical results. The test example for all our experiments is the weak formulation, see (1), of

$$(48) \quad -\operatorname{div}(\rho(x)\nabla u) = f(x) \text{ in } \Omega,$$

with the homogenous Dirichlet boundary conditions on $\partial\Omega$, where $\Omega = (0, 1) \times (0, 1)$ is a union of disjoint square subregions Ω_i , $i = 1, \dots, N$, and $\rho(x) = \rho_i$ is a positive constant in each Ω_i . The diffusion function $\rho(x)$ is chosen larger on the mortar sides of the interfaces; see Theorem 4.1.

The region Ω is cut into N regular subregions. Below we indicate the distribution of 4 coefficients ρ_i and 4 grids h_i in Ω_i , $i = 1, \dots, 4$ with a maximum mesh ratio 8 : 1 used in our tests (for larger number of subregions, this pattern of coefficients is repeated).

For the M-N subregion ordering test case we have

$$(49) \quad \begin{pmatrix} 1e6 & 1 \\ 1e2 & 1e4 \end{pmatrix}, \quad \begin{pmatrix} h/8 & h \\ h/2 & h/4 \end{pmatrix}.$$

For the arbitrary (other than M-N) ordering of subregions test case we have

$$(50) \quad \begin{pmatrix} 1e6 & 1e4 \\ 1e2 & 1 \end{pmatrix}, \quad \begin{pmatrix} h/8 & h/4 \\ h/2 & h \end{pmatrix}.$$

Additionally, we test a 4×4 subregions case (denoted by * in the tables) that employs coefficients of the following form without a repetitive pattern:

$$(51) \quad \begin{pmatrix} 1e6 & 1 & 1 & 1e3 \\ 1e4 & 1e2 & 1e6 & 1 \\ 1e2 & 1e5 & 1e4 & 1e2 \\ 10 & 1e3 & 10 & 1e6 \end{pmatrix}, \quad \begin{pmatrix} h/8 & h & h & h/4 \\ h/4 & h/2 & h/8 & h \\ h/2 & h/8 & h/4 & h/2 \\ h & h/4 & h/2 & h/8 \end{pmatrix}.$$

5.1. Implementation. The discrete solution u_h^* of (48) is obtained as follows. Random solution at the nodal points is expressed as $(u^{(i)}, u^{(c)}, u^{(r)})$. Mortar condition on each side of the interface $\delta_{m(i)} = \gamma_{m(j)}$ is represented by (7). This gives on $\delta_{m(i)}$

$$(52) \quad u_{i\delta_{m(i)}}^{(r)} = (B_{\delta_{m(i)}}^{(r)})^{-1} (B_{\gamma_{m(j)}}^{(r)} u_{j\gamma_{m(j)}}^{(r)} + B_{\gamma_{m(j)}}^{(c)} u_{j\gamma_{m(j)}}^{(c)} - B_{\delta_{m(i)}}^{(c)} u_{i\delta_{m(i)}}^{(c)}).$$

The solution u_h^* is obtained from $(u^{(i)}, u^{(c)}, u^{(r)})$ by replacing $u^{(r)}$ on each nonmortar side by values computed by (52) and taking into account the continuity at the cross points: $u_{i\delta_{m(i)}}^{(c)} = u_{j\gamma_{m(j)}}^{(c)}$. For the given u_h^* the discrete RHS $(f^{(i)}, f^{(c)}, f^{(r)})$ is then computed.

Since $K_{ic} = 0 = K_{ci}$ in the case of triangular elements and a piecewise linear continuous finite element space, in the numerical experiments we implement somewhat simplified formulas (11):

$$\begin{aligned} \tilde{S} &= K_{rr} - K_{ri}K_{ii}^{-1}K_{ir} - K_{rc}\tilde{K}_{cc}^{-1}K_{cr}, & \tilde{f}_r &= f^{(r)} - K_{ri}K_{ii}^{-1}f^{(i)} - K_{rc}\tilde{K}_{cc}^{-1}f^{(c)}, \\ \tilde{B} &= B_r - B_c\tilde{K}_{cc}^{-1}K_{cr}, & \tilde{S}_{cc} &= -B_c\tilde{K}_{cc}^{-1}B_c^T, & \text{and} & \tilde{f}_c = -B_c\tilde{K}_{cc}^{-1}f_c. \end{aligned}$$

Computing the RHS of the Schur complement system $d = \tilde{B}\tilde{S}^{-1}\tilde{f}_r - \tilde{f}_c$ (see (13)) is equivalent to solving N coupled Neumann problems (those with Neumann boundary conditions at the interfaces and, if a subregion is adjacent to the boundary of Ω , with zero Dirichlet conditions at $\partial\Omega$) connected through the cross points and with the only nonzero values at the interfaces:

$$(53) \quad \begin{pmatrix} K_{ii} & 0 & K_{ir} \\ 0 & \tilde{K}_{cc} & K_{cr} \\ K_{ri} & K_{rc} & K_{rr} \end{pmatrix} \begin{pmatrix} v_i \\ v_c \\ v_r \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \tilde{f}_r \end{pmatrix}.$$

Note that this step is implemented using the capacitance matrix approach employing solvers on the subregions only. Note also that computing \tilde{f}_r requires solving N uncoupled Dirichlet problems $K_{ii}w_i = f^{(i)}$. The final result is then multiplied by \tilde{B} and corrected by $-\tilde{f}_c$.

The preconditioned conjugate gradient (PCG) iterations to solve (12) are terminated when the norm of the residual has decreased 10^6 times in the norm generated

by the inverse of the preconditioner M^{-1} . In each PCG iteration, there are two main operations:

1. multiplication by $F = \tilde{B}\tilde{S}^{-1}\tilde{B}^T - \tilde{S}_{cc}$ (see (13)) and
2. multiplication by $M^{-1} = \hat{B}_r\hat{S}_{rr}\hat{B}_r^T$ (see (17)).

Their implementation is as follows.

1. Given the search directions $p_\delta^k \in R^{n_\delta}$ at all nonmortar sides of the interfaces, we compute $r_\delta^k = Fp_\delta^k = (\tilde{B}\tilde{S}^{-1}\tilde{B}^T - \tilde{S}_{cc})p_\delta^k$ as follows: we first compute $p^k = \tilde{B}^T p_\delta^k$; then solve for $(v_i, v_c, v_r)^T$ the N coupled Neumann problems connected through the cross points as in (53) but with the RHS $(0, 0, p^k)^T$; and finally compute $r_\delta^k = \tilde{B}v_r - \tilde{S}_{cc}p_\delta^k$.

2. Given the residual $r_\delta^k \in R^{n_\delta}$ at all nonmortar sides of the interfaces we compute $z_\delta^k = M^{-1}r_\delta^k = \hat{B}_r\hat{S}_{rr}\hat{B}_r^T r_\delta^k$, where $\hat{S}_{rr} = \text{diag} \{ \hat{S}_{rr}^{(j)} \}$, $\hat{S}_{rr}^{(j)} = S_{rr}^{(j)}$ for $\rho_i = 1$, $S_{rr}^{(j)} = K_{rr}^{(j)} - K_{ri}^{(j)}(K_{ii}^{(j)})^{-1}K_{ir}^{(j)}$ as follows: we compute $z = \hat{B}_r^T r_\delta^k$; $v_j = (K_{ii}^{(j)})^{-1}K_{ir}^{(j)}z_j$, $z_j = z|_{\partial\Omega_j}$, which is equivalent to solving N uncoupled Dirichlet problems $K_{ii}^{(j)}v_j = K_{ir}^{(j)}z_j$ for v_j ; and finally $z_\delta^k = \hat{B}_r\tilde{v}$, where $\tilde{v} = \{ \tilde{v}_j \}$, $\tilde{v}_j = K_{rr}^{(j)}z - K_{ri}^{(j)}v_j$.

After solving (12) for $\tilde{\lambda}^*$ the final solution is obtained by solving the N coupled Neumann problems connected through the cross points (see (9))

$$\begin{pmatrix} K_{ii} & 0 & K_{ir} \\ 0 & \tilde{K}_{cc} & K_{cr} \\ K_{ri} & K_{rc} & K_{rr} \end{pmatrix} \begin{pmatrix} u^{(i)} \\ u^{(c)} \\ u^{(r)} \end{pmatrix} = \begin{pmatrix} f^{(i)} \\ f^{(c)} - B_c^T \tilde{\lambda}^* \\ f^{(r)} - B_r^T \tilde{\lambda}^* \end{pmatrix}.$$

All the experiments were performed with the complete scaling of the preconditioner as in (17), including the scaling involving step parameters. In the tables, $\max \frac{H}{h_i}$ is the largest number of mesh steps on each subregion interface, “dim” is the dimension of the reduced (Schur) matrix, “# it” is the number of the PCG iterations, “ $\kappa(Q)$ ” is the condition number estimate of the iteration matrix, and “error” is the normalized L_2 error. In all the examples the max grid ratio is 8 : 1. The criterion for choosing $\gamma_{m(i)}$ as the mortar side is that $\rho_i \geq \rho_j$, the coefficients on Ω_i and Ω_j , and, if equal, where the grid is finer, $h_{\gamma_{m(i)}} \leq h_{\delta_{m(j)}}$, unless indicated otherwise.

5.2. Continuous problems. These examples serve as a comparison with the discontinuous problems investigated in further detail.

Table 1 shows that the preconditioner M of (17) employed for the continuous problem and grids (49) (with the M-N ordering of substructures) is well scalable and gives convergence logarithmically dependent on the step sizes. The exhibited dependence $\kappa(Q) = (1 + \log(H/h_{\min}))^p$ with about $p = 1$ is better than the theoretical value of $p = 2$.

Table 2 shows the results for the arbitrary ordering on grids (50). Performance results for M-N ordering (Table 1) and for arbitrary ordering (Table 2) are very similar. In the latter case the computed value in the logarithmic dependence also is about $p = 1$, which is superior to the theoretical estimate of $p = 4$.

If one violates the above-mentioned recommendation and chooses $h_{\delta_{m(i)}} < h_{\gamma_{m(j)}}$, then the rate of convergence deteriorates somewhat; compare Table 3 with the upper part of Table 2.

It should be noted that results presented in Tables 1 to 3 are significantly better than those when the standard preconditioner (16) without the scaling involving step parameters is employed.

TABLE 1

Continuous coefficients. Mortar-nonmortar ordering of subregions for grids as in (49).

$\max \frac{H}{h_i}$	4 × 4 subregions			8 × 8 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	120	14	5.36	560	15	5.33
64	264	14	5.62	1232	15	5.74
128	552	14	6.27	2576	16	6.50
256	1128	15	7.17	5264	17	7.55
$\max \frac{H}{h_i}$	12 × 12 subregions			16 × 16 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	1320	15	5.31	2400	15	5.30
64	2904	15	5.76	5280	15	5.77
128	6072	16	6.54	11040	16	6.55
256	12408	17	7.62	22560	17	7.18

TABLE 2

Continuous coefficients. Arbitrary ordering of subregions for grids as in (50).

$\max \frac{H}{h_i}$	4 × 4 subregions			8 × 8 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	168	13	4.45	784	14	4.70
64	360	13	4.76	1680	14	5.06
128	744	14	5.38	3472	15	5.70
256	1512	14	6.24	7056	16	6.65
$\max \frac{H}{h_i}$	12 × 12 subregions			16 × 16 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	1848	13	4.75	3360	13	4.75
64	3960	14	5.12	7200	14	5.15
128	8184	15	5.81	14880	15	5.84
256	16632	16	6.77	30240	16	6.84

TABLE 3

The effect of choosing sides: $h_\delta < h_\gamma$. Continuous coefficients. Arbitrary ordering of subregions with grids as in (50).

$\max \frac{H}{h_i}$	4 × 4 subregions			8 × 8 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	504	15	10.50	2352	18	10.88
64	1032	15	13.97	4816	19	14.71
128	2088	16	18.03	9744	21	19.20
256	4200	17	22.74	19600	23	24.41

5.3. Discontinuous problems. For discontinuous problems the standard preconditioner (16) which does not employ scaling of coefficients exhibits poor convergence, often worse than the conjugate gradient iterations without preconditioning. Fortunately, this preconditioner allows for a multitude of scalings to be employed.

It should be pointed out that in the simplest case of M-N ordering of 2×2 subregions with only two grids and two coefficients $\rho_i : 1 = \rho_N < \rho_M$ investigated in [3], the standard preconditioner (16) displayed convergence almost independent of the ratio H/h_i (although the condition number and the number of iterations were quite high).

Several other scalings have been tried and tested. For example, for the M-N ordering as in (49) the preconditioner $M^{-1} = S_{rr}^N + B_M \widehat{S}_{rr}^M B_M^T$ gives convergence

TABLE 4

Discontinuous coefficients. Mortar-nonmortar ordering of subregions and grids as in (49).

$\max \frac{H}{h_i}$	4 × 4 subregions			8 × 8 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	120	3	1.03	560	3	1.03
64	264	3	1.04	1232	3	1.04
128	552	3	1.05	2576	3	1.05
256	1228	3	1.07	5264	3	1.07
$\max \frac{H}{h_i}$	12 × 12 subregions			16 × 16 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	1320	3	1.03	2400	3	1.03
64	2904	4	1.04	5280	4	1.04
128	6072	3	1.05	11040	4	1.05
256	12408	3	1.07	22560	3	1.07

TABLE 5

Discontinuous coefficients. Arbitrary ordering of subregions and grids as in (50).

$\max \frac{H}{h_i}$	4 × 4 subregions			8 × 8 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	168	8	3.27	784	9	3.40
64	360	9	4.28	1680	11	4.46
128	744	10	5.45	3472	12	5.65
256	1512	11	6.77	7056	14	7.00
$\max \frac{H}{h_i}$	12 × 12 subregions			16 × 16 subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	1848	9	3.38	3360	9	3.38
64	3960	11	4.45	7200	11	4.45
128	8184	12	5.65	14880	12	5.65
256	16632	14	7.00	30240	14	7.00

almost independent of the ratio H/h_i and the iteration count is a fraction of that obtained with preconditioner (16). However, none of these simple preconditioner scalings is satisfactory in the case of arbitrary (other than M-N) ordering, in which case a scaling that acts only on the nonmortar sides of the interfaces is required.

The preconditioner M of (17) is one of possible choices of such a scaling, and one that is exhibiting good convergence properties both in the continuous case and the discontinuous one, as we shall demonstrate.

Table 4 shows that in the case of M-N ordering of the subregions the preconditioner M gives convergence independent of the step sizes (the ratio H/h_i), the jump of coefficients, and the number of subregions.

Table 5 shows that for arbitrary ordering of subregions convergence is only logarithmically dependent of the step size, independent of the jump of coefficients, and well scalable (independent on the number of subregions). The exhibited logarithmic dependence $\kappa(Q) = (1 + \log(H/h_{\min}))^p$ with $p = 1.8$ is better than the theoretical estimate $p = 4$.

Viewing Tables 1–2 and 4–5 we can compare performances of our preconditioner for continuous and discontinuous problems. For M-N ordering we observe a much faster rate of convergence in the discontinuous case over the continuous one, while for the arbitrary ordering the rates of convergence do not differ significantly.

TABLE 6

*Discontinuous coefficients. Performance comparison for arbitrary ordering with a repetitive pattern of grids as in (50) versus that of nonrepetitive grids as in (51) (denoted by *).*

$\max \frac{H}{h_i}$	4×4 subregions			$4 \times 4^*$ subregions		
	dim	# it	$\kappa(Q)$	dim	# it	$\kappa(Q)$
32	168	8	3.27	160	11	4.13
64	360	9	4.28	344	12	4.44
128	744	10	5.45	712	13	4.91
256	1512	11	6.77	1448	14	5.71

TABLE 7

Discontinuous coefficients. Performance comparison for the random solution versus that of (54) on arbitrary ordering of subregions $4 \times 4^$ and grids as in (51).*

$\max \frac{H}{h_i}$	Random solution		Solution as in (54)		
	# it	$\kappa(Q)$	# it	$\kappa(Q)$	Error
32	11	4.13	10	4.16	8.57e-5
64	12	4.41	12	4.42	1.74e-5
128	13	4.91	13	5.33	4.04e-6
256	14	5.71	14	6.33	9.73e-7

Table 6 presents the comparison in performance for arbitrary ordering of 4×4 subregions between the case when the pattern of coefficients and grids is repetitive as in (50) and when it is nonrepetitive as in (51). The differences are not pronounced, which allows us to conclude that the results of experiments elsewhere in this paper with larger numbers of subregions give a reasonable representation.

We have also tested problems with extreme variations of coefficients where coefficients, ρ_i in (49) were replaced by

$$\begin{pmatrix} 1e+6 & 1e+2 \\ 1e-2 & 1e-6 \end{pmatrix}.$$

The differences in performance were only slight.

For discontinuous problems with large jump of coefficients the question of choosing sides, i.e., $h_{\gamma_{m(j)}} < h_{\delta_{m(i)}}$ versus $h_{\delta_{m(i)}} < h_{\gamma_{m(j)}}$, has virtually no effect on the convergence rate, in contrast with the continuous problems.

The variational formulation of the problem with discontinuities at the interfaces automatically imposes the continuity of the flux condition in the weak sense. The following solution (that is nonzero at the interfaces) was designed to satisfy this condition in the classical sense:

$$(54) \quad u(x, y) = v(x)(1 - v(x))v(y)(1 - v(y)),$$

$$v(z) = z - \frac{\sin(2m\pi z)}{2m\pi},$$

where $m = 2^k$, $k = 1$ to 4.

Choosing (54) as the exact solution allows us to test the accuracy of our solver. The results in Table 7 show that the accuracy is clearly $O(h^2)$. One needs to stress, however, that the rate of convergence remains virtually the same as with the random solution; see Table 7.

It should be mentioned that a violation of the theoretical requirement that mortar sides should be chosen where the coefficients are larger leads to a very slow convergence when preconditioner M of (17) is used.

The largest tests reported here (16×16 subregions case in Table 5) were run with the dimension of the reduced (Schur) matrix of 30,240 and about 5,500,000 grid points (degrees of freedom) in the whole domain.

6. Conclusions. In this paper we introduced and analyzed a preconditioner with special scaling involving discontinuous coefficients and step parameters, and established convergence bounds.

Extensive computational evidence presented illustrates an excellent performance of the preconditioner: its convergence is almost optimal for a variety of situations (distribution of subregions, grid assignment, grid ratios, number of subregions) and independent of the jumps of coefficients and the parameter of triangulation. This holds for both continuous and discontinuous problems (in the latter case under the theoretical assumption that a mortar side is associated with the higher coefficient).

The experiments using the proposed preconditioner also show that for discontinuous problems the choice of mortar versus nonmortar sides has little influence on convergence rate. The scaling involving step parameters removes the assumption that $h_{\delta_m(k)} \sim h_{\gamma_m(i)}$ and, for continuous problems, significantly improves the rate of convergence.

Recent experiments show that the method exhibits almost linear parallel scalability properties; see [2].

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