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A Projector Path Integral for the Dirac Equation and the Spin Derivation of Space

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We construct a new kind of path integral for the Dirac equation propagator, intended as an extension to 3 space dimensions of the Feynman "checkerboard" propagator. One form of this path integral is a "projector path" summation, out of which one can reconstruct standard 3D space and chirality. Other forms allow the particle velocity along the path to be adjusted.

I. INTRODUCTION

The Dirac equation and its propagator are more fundamental concepts than the Schrödinger equation and its propagator. It is natural to attempt to write a path integral expression for the Dirac propagator similar to the Feynman path integral expression for the Schrödinger propagator and which would show clearly how the non-relativistic limit is attained at the level of paths. The purpose of this work is to give such a formula for the Dirac path integral, using a space-time formulation as in Feynman's original papers and book [1, 2].

Many authors, including Feynman, have obtained expressions for the Dirac propagator as a path integral and we recall briefly the main attempts. The first was by Feynman [2] for the Dirac equation in 1 space dimension and is known as the checkerboard path integral. Another attempt, elaborating on Feynman's checkerboard was done in [3] where it was shown that the knowledge of the propagator of the Weyl equation would generate a path integral for the Dirac equation. It was also implicit in that paper that the checkerboard path integral was a complex measure (in the mathematical sense) on the space of paths, which is not the case for the Feynman path integral formula for the Schrödinger equation. In fact, this approach has been used recently to regularize the path integral for the Schrödinger

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equation [4], providing justification for an idea of Nelson [5]. The path integral for the Schrödinger equation appears as a limit of complex measures on smooth paths in any number of spatial dimensions, although it is not a measure, as is known [5]. Another approach, using a momentum representation and Poisson processses in p-space has been introduced in [6]. Completely different approaches using Grassmann variables have been developed, in particular using differential forms on SU(2) (see [7]) but these attempts are completely formal, and in a sense tautological. Moreover they do not generate paths in space-time. A non-Grassmanniann approach allowing canonical quantization is [8]. Finally, a path integral using two-component spinors was constructed in [9], following suggestions of [10]. The strange feature of this last approach is that the paths that are constructed propagate at a velocity which is 3 times the velocity of light. In the method of [9], a spinor-chain integral is constructed for the Weyl equation propagator. Using the Poisson process of [3], it allows one to write a path integral formula for the Dirac propagator.

The present work is intended as an extension to 3 dimensions of the original checkerboard path integral. We will construct several formulas for the Dirac propagator in the spirit of [9], mostly without a Poisson process. The velocity of the path is the velocity of light. Our formalism is more relativistically covariant than in [9], because the Dirac matrices are treated on the same footing and four-component spinors are used. Moreover, this new formulation of the path integral shows that it is really a projector path integral in the 4-component spinor space. The standard path in 3D space and the propagation of the wave function are derived concepts emerging from the projector path integral. In an appendix we develop a corresponding formalism for "between flips" propagator, i.e., the zero-mass-particle propagation in 3 dimensions corresponding to the speed-of-light propagation in Feynman's orginal formulation.

II. THE PATH INTEGRAL FOR THE FREE PROPAGATOR

The Hamiltonian formulation of the free Dirac equation can be written

$$i\frac{\partial\psi}{\partial t} = H\psi \qquad (\hbar = 1), \tag{1}$$

where

$$H = \boldsymbol{\alpha}\cdot\mathbf{p} + \beta m \qquad \text{(see, e.g., [11])}. \tag{2}$$

Here ψ is a 4-component spinor, $\mathbf{p} = -i\nabla$, m is the rest mass of the particle, and $\boldsymbol{\alpha}$, β are the Dirac matrices

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \qquad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

where the $\boldsymbol{\sigma}$ are the standard Pauli matrices and I is the 2×2 identity matrix. We have set $\hbar=1$ and $c=1$. The propagator is then the continuous product,

$$\exp(-iHt)=\prod \exp[(-i\boldsymbol{\alpha}\mathbf{p}-i\beta m)\,\Delta t].$$

For small Δt, we can replace the exponential by

$$\exp[(-i\boldsymbol{\alpha}\mathbf{p}-i\beta m)\,\Delta t]=I-(i\boldsymbol{\alpha}\mathbf{p}+i\beta m)\,\Delta t. \tag{3}$$

Let us now consider the unit sphere S^3 in the space R^4. Its volume is $2\pi^2$. If $n=(n_0,\mathbf{n})$ is a unit vector, one can easily compute

$$\int_{S^3} n_k^2\,dS=\frac{\pi^2}{2},\quad k=0,1,2,3,\qquad \int_{S^3} n_k\,dS=0,\quad \int n_k n_l dS=0\quad \text{for}\quad k\neq l.$$

From this, one can verify the identity

$$I-i(\boldsymbol{\alpha}\mathbf{p}+\beta m)\,\Delta t=\frac{1}{2\pi^2}\int_{S^3}(I-\mathbf{n}\boldsymbol{\alpha}-n_0\beta)(I+i(\mathbf{n}\mathbf{p}+n_0 m)\,4\,\Delta t)\,dS$$

and more generally, we can introduce a number ω that will be fixed later and verify

$$I-i(\boldsymbol{\alpha}\mathbf{p}+\beta m)\,\Delta t=\frac{1}{2\pi^2}\int_{S^3}\left(I-\frac{4}{\omega}(\mathbf{n}\boldsymbol{\alpha}+n_0\beta)\right)(I+i\omega(\mathbf{n}\mathbf{p}+n_0 m)\,\Delta t)\,dS.$$

For small Δt, we obtain

$$\exp(-iH\,\Delta t)=\frac{1}{2\pi^2}\int_{S^3}\left(I-\frac{4}{\omega}(\mathbf{n}\boldsymbol{\alpha}+n_0\beta)\right)\exp[i\omega(\mathbf{n}\mathbf{p}+n_0 m)\,\Delta t]\,dS. \tag{4}$$

In these formulas, the spin-chirality rotation has been factored out from the spatial translation and shift of the phase. As a consequence the full propagator can be written

$$\begin{aligned}\exp(-iHt)&\\ &=\lim_{N\to\infty}\int_{(S^3)^N}\frac{dS_N}{(2\pi^2)^N}\prod_{k=1}^{N}\left(I-\frac{4}{\omega}(\mathbf{n}^{(k)}\boldsymbol{\alpha}+n_0^{(k)}\beta)\right)\\ &\quad\times\exp\left(i\omega m\sum n_0^{(k)}\Delta t\right)\exp\left(i\omega\mathbf{p}\sum_k \mathbf{n}^{(k)}\Delta t\right).\end{aligned} \tag{5}$$

In these formulas, $\Delta t=\frac{t}{N}$ and the integration is on the product of N spheres, S^3, the variables being $(n_0^{(1)},\mathbf{n}^{(1)})$, ..., $(n_0^{(k)},\mathbf{n}^{(k)})$, ..., $(n_0^{(N)},\mathbf{n}^{(N)})$. The products $\prod_{k=1}^{N}$ are time ordered products over a sequence of 4×4 matrices.

We see that this integral defines a path in 3 space by

$$\mathbf{x}_k = \omega \sum_{j=1}^{k} \mathbf{n}^{(j)} \Delta t \tag{6}$$

and it also defines a phase shift by

$$\varphi_k = m\omega \sum_{j=1}^{k} n_0^{(j)}. \tag{7}$$

The operator

$$\exp\left(i\omega \mathbf{p} \sum \mathbf{n}^{(k)} \Delta t \right) = \exp(\mathbf{x}_N \cdot \nabla) \tag{8}$$

is the translation operator by the path value at the final time. The emergence of sums of **n**'s as position is purely a consequence of the defintion of momentum as translation generator and of our ability to factor $\mathbf{p}\boldsymbol{\alpha}$ into separate **p** and $\boldsymbol{\alpha}$ expressions in the propagator.

Remark. The factoring of space and spin by means of Eq. (4) is the essential technical step in this work. In Feynman's one dimensional checkerboard formulation one has simply $\exp(icp\,\Delta t)$, which automatically supplies real space translations (since $p = -i\partial/\partial x$). But in three space dimensions one has $\boldsymbol{\alpha} \cdot \mathbf{p}$ (as in (3)). This makes the translation in a sense be $\boldsymbol{\alpha}\,\Delta t$; however, that "sense" is far from clear, due to the action of $\boldsymbol{\alpha}$ on spin coordinates. The factoring of (4) allows **p** to generate translations in an ordinary 3 space, which as we point out, becomes identified with physical space.

III. PATH INTEGRAL IN THE PRESENCE OF EXTERNAL FORCES

The hamiltonian of the Dirac equation for a particle in a vector potential **A** and a scalar potential V is

$$H = \boldsymbol{\alpha}(\mathbf{p} - e\mathbf{A}) + \beta m + eV, \tag{9}$$

where e is the electric charge.

It is easy to modify the path integral formulation Eq. (5) for the propagator to take into account the influences of **A** and V. This is done with a Feynman Kac type factor using the method of [7]. In fact, we have in Eq. (5) instead of $\exp(\omega i\mathbf{p} \sum_k \mathbf{n}^{(k)} \Delta t)$,

$$\exp\left(\omega i(\mathbf{p} - e\mathbf{A}) \left(\sum_k \mathbf{n}^{(k)} \right) \Delta t - ieV\,\Delta t \right).$$

We separate the action of the translation operator and the action of $\mathbf{A}$ and V as

$$\exp\left[-\omega i e \sum_{k=1}^{N-1} \mathbf{A}(\mathbf{x}_k)\, \mathbf{n}^{(k+1)}\, \Delta t - ie \sum_{k=1}^{N-1} V(\mathbf{x}_k)\, \Delta t\right] \exp(\mathbf{x}_N \cdot \nabla), \tag{10}$$

where the path $\mathbf{x}_k$ has been defined in Eqs. (6). In Eq. (10) we have put on the right the translation operator, and on the left the multiplication operators. Moreover, in this equation $\mathbf{A}(\mathbf{x}_k)$ and $V(\mathbf{x}_k)$ denote multiplication operators by the functions of $\mathbf{x}$ which are $\mathbf{x} \to \mathbf{A}(\mathbf{x}_k + \mathbf{x})$ and $\mathbf{x} \to V(\mathbf{x}_k + \mathbf{x})$.

If we denote the increment of the paths by

$$\delta_k \mathbf{x} = \mathbf{x}_{k+1} - \mathbf{x}_k \tag{11}$$

the full propagator is given by

$$\exp(-iHt) = \lim_{N\to\infty} \int_{(S^3)^N} \frac{dS_N}{(2\pi^2)^N} \prod_{k=1}^{N} \left(I - \frac{4}{\omega}(\mathbf{n}^{(k)}\boldsymbol{\alpha} + n_0^{(k)}\beta)\right) e^{i\varphi_N}$$
$$\times \exp\left[-ie \sum_{k=1}^{N-1} \mathbf{A}(\mathbf{x}_k)\, \delta_k \mathbf{x} - ie \sum_{k=1}^{N-1} V(\mathbf{x}_k)\, \Delta t\right] \exp(\mathbf{x}_N \cdot \nabla). \tag{12}$$

We now consider the matrix element of $\exp(-iHt)$ between an initial state $u_i\, \delta(\mathbf{x}_i - \bullet)$ and a final state $u_f \delta(\mathbf{x}_f - \bullet)$, where u_i, u_f are constant spinors. Specifically,

$$\langle u_f \delta(\mathbf{x}_f - \bullet)\, | e^{-iHt} |\, u_i\, \delta(\mathbf{x}_i - \bullet)\rangle$$
$$= \lim_{N\to\infty} \int \frac{dS_N}{(2\pi^2)^N} \left\langle u_f \left| \prod_{k=1}^{N} \left(I - \frac{4}{\omega}(\mathbf{n}^{(k)} \cdot \boldsymbol{\alpha} + n_0^{(k)}\beta)\right) \right| u_i \right\rangle e^{i\varphi_N}$$
$$\times \exp\left[-ie \sum_{k=1}^{N-1} \mathbf{A}(\mathbf{x}_k + \mathbf{x}_f)\, \delta_k \mathbf{x}' - ie \sum_{k=1}^{N-1} V(\mathbf{x}_k + \mathbf{x}_f)\, \Delta t\right] \delta(\mathbf{x}_N - \mathbf{x}_i + \mathbf{x}_f). \tag{13}$$

In both cases we integrate on paths $(\mathbf{x}_k)_{k=1,\dots,N}$ such that the total amount of space between initial and final times $\mathbf{x}_N$ is $-\mathbf{x}_f + \mathbf{x}_i$.

IV. PROJECTOR PATH INTEGRALS AND QUANTUM SPACE

In this section, we interpret Eqs. (5) and (12).

(i) First we choose $\omega = 4$ in Eq. (5) or (12); the increments of the paths in time Δt are $4\mathbf{n}^{(k)}\, \Delta t$ where $\mathbf{n}^{(k)}$ are independent random vectors of length $\leqslant 1$. The average of $|\mathbf{n}^{(k)}| = \sqrt{1 - n_0^{(k)2}}$ is $\int_{S^3} |\mathbf{n}^{(k)}|\, \frac{dS}{2\pi^2} = \frac{8}{3\pi}$. In fact

$$dS = \sin^2\theta_1\, d\theta_1 \sin\theta_2\, d\theta_2\, d\theta_3$$
$$|\mathbf{n}^{(k)}| = \sin\theta_1.$$

So the average velocity of the path $(\mathbf{x}_k)$ is $\frac{32}{3\pi} > 1$. The operator $P(k)$ defined as

$$P^{(k)} = \tfrac{1}{2}\,(I - \mathbf{n}^{(k)}\boldsymbol{\alpha} - n_0^{(k)}\beta) \tag{14}$$

is obviously a projector (due to the commutativity properties of $\boldsymbol{\alpha}$, β). In fact, it is a projector on a two-dimensional subspace of the space of 4-components spinors, namely the space S_k of spinors of type

$$u = \begin{pmatrix} -n_0^{(k)} w \\ (I + \mathbf{n}^{(k)}\boldsymbol{\sigma})\, w \end{pmatrix}; \tag{15}$$

where w is any 2-component spinor (normalized by $|w|^2 = 1$) and $\boldsymbol{\sigma}$ are the Pauli matrices. This result is easily obtained using the remark that formally $\mathbf{n}^{(k)}\boldsymbol{\alpha} + n_0^{(k)}\beta$ is a free Hamiltonian as in Eq. (2) with $\mathbf{n}^{(k)}$ instead of $\mathbf{p}$, and $n_0^{(k)}$ instead of m. Its energy is therefore ± 1. Then the 4-component spinors u of Eq. (15) are the traditional eigenstates of this hamiltonian of energy $+1$ (see [11]).

Thus we see that in Eq. (5) the product $\prod_{k=1}^{N} (I - \mathbf{n}^{(k)}\boldsymbol{\alpha} - n_0^{(k)}\beta) = 2^N \prod_{k=1}^{N} P^{(k)}$ is a product over projectors on two dimensional spaces S_k of 4-component spinors of the type Eq. (15).

The square norm of u of Eq. (15) is found to be

$$\langle u \mid u \rangle = 2(\langle w \mid w \rangle + \langle w \mid \mathbf{n}\boldsymbol{\sigma} w \rangle). \tag{16}$$

Moreover

$$\frac{\langle u |\, \boldsymbol{\alpha}\, | u \rangle}{\langle u \mid u \rangle} = -\mathbf{n}^{(k)}, \qquad \frac{\langle u |\, \beta\, | u \rangle}{\langle u \mid u \rangle} = -n_0^{(k)}. \tag{17}$$

In other words, the projector $P^{(k)}$ projects on the space of spinors u such that the expectation of the velocity operator $\boldsymbol{\alpha}$ on these spinors is exactly $\mathbf{n}^{(k)}$ which is precisely the velocity of the path at time $k\,\Delta t$ (recall that the path is such that $\mathbf{x}_N = \mathbf{x}_i - \mathbf{x}_f$, so by our convention it runs backwards in time, whence the minus sign in Eq. (17)). In other words, we reconstruct the space translations, using the 4-component spinors u, as it should be. But our path has an average velocity which is greater than the velocity of light. The quantum expectation of the velocity is less than the velocity of light. We can compare our result with that of Jacobson [9]. In this work, Jacobson uses 2-dimensional spinors and he obtains a spinor chain integral.

In our case, which is more relativistically covariant, although not fully because we write a time propagator, the spinor chain integral becomes a *projector path integral* in the space of 4-component spinors. The expectation of the velocity operator $\boldsymbol{\alpha}$ on any state in the image S_k of the kth projector of the projector path becomes, up to a factor of 4, the local velocity of a path in space (the average velocity of this path being $\frac{32}{3\pi}$). But what is important is that space is no longer primary. What is primary is the path of projectors on spinors, out of which space is reconstructed and spinors are propagated. In other words, the projector path integral on 4 component spinors, defines at once, *the propagator and the medium of propagation.* The defect of this method is that the average velocity of the path $\mathbf{x}_k$ is larger than the speed of light, although the quantum expectation of the velocity is smaller than the speed of light.

(ii) We next consider a different value for the constant ω appearing in Eqs. (15) and (12). We choose ω so that

$$\frac{8\pi\omega}{3} = 1.$$

In this case the velocity of the path $\mathbf{x}_k$, that is, $\mathbf{n}^{(k)}$, is bounded from above by the velocity of light. The price to be paid is that the chronological product is no longer a product over projectors in the 4-component spinor space. This seems to be less natural. After all, in the previously stated view, space can be reconstructed as derived from a quantum mechanism, where the projector is primary. The true velocity is not the path velocity, but the expectation of the $\boldsymbol{\alpha}$ matrices which is indeed less than the velocity of light.

(iii) The usual Feynman integral is also a projector path integral. The projectors are on the eigenstates of position operators (and so commute with each other). In our case, there is no position a priori, only spin and momentum (which, relativistically are no longer related to velocity and space). The path integral is an integral over projectors which do not commute with one another.

V. CONCLUDING REMARKS

We have written a path integral formula for the propagator of the Dirac equation (in hamiltonian form), in presence of external fields. This path integral is *not* a standard one, over paths in a given 3D space. Rather, 3D space emerges as a by-product of the path integral, similar in spirit, if not in precise implementation to the work in [9, 10]. In our picture with $\omega = 1$ the path integral is really an integral over projector paths in the space of 4-component spinors. This approach points towards an extension of the concept of path integral as projector valued paths.

Other values of ω allow adjustment of the velocity of the particle. We intentionally leave the choice of ω open. As an elementary object in quantum mechanics the Dirac equation arises in many contexts and one should not eliminate potential applications by too strongly advocating one or another choice.

APPENDIX A: AN ALTERNATIVE FORM OF THE PROPAGATOR

In our previous work on the propagator for the Dirac equation there is a concatenation of two processes. One is a Poisson-distributed mass-induced flipping between propagation as a left handed massless particle and a right handed massless particle. The second process describes between-flip propagation and is the stage at which Grassmann variables entered.

Using the approach of the present paper we next present a real-space process for the between-flips propagation. In this way a complete real-space Dirac equation propagator can be realized in another way. (It also can provide an alternative construction of space, along the lines of that described above.)

We will not review the entire formalism of [7]. Suffice to say that the between-flips propagator is a 2×2 matrix of the form $\exp(it\ \mathbf{p}\cdot\boldsymbol{\sigma})$ (or the same thing with opposite sign in the exponent), where $\boldsymbol{\sigma}$ are the Pauli spin matrices and $\mathbf{p}=-i\boldsymbol{\nabla}$. Following the usual Trotter product approach (which allows all we do here to be generalized to the case where external fields are present), the problem becomes the factorization into space and spin operators of $\exp(i\,dt\ \mathbf{p}\cdot\boldsymbol{\sigma})$. (Of course this is the objective of Eq. (4) above, but for 4×4 matrices.) We make use of the formula

$$\exp(i\,dt\,\mathbf{p}\cdot\boldsymbol{\sigma})=\frac{1}{4\pi}\int d\hat{n}\left[1+\frac{3}{\omega}\hat{n}\boldsymbol{\sigma}\right]\exp[i\omega\mathbf{p}\hat{n}\,dt], \tag{A.1}$$

where $d\hat{n}$ means integration on the 2-sphere and the equality holds to first order in dt (which is all that is necessary). In proving this equation we made use of $\int d\hat{n}\,n_i n_j=\delta_{ij}/3$.

Recognizing $\exp[i\omega\mathbf{p}\hat{n}\,dt]$ as a translation operator in the direction $\hat{n}$ at velocity ω, the interpretation of Eq. (A.1) is clear. If ω is taken to be 3, one has a spin projection and propagation with velocity $3c$, much as in Jacobson's approach. If ω is taken to be 1, then one obtains the generalization to 3 dimensions of the Feynman checkerboard prescription. Thus the particle travels at the speed of light, with Poisson-distributed flips. The action of the propagator on the spin coordinates is multiplication by the operator $1+3\hat{n}\boldsymbol{\sigma}$, which, because of the 3, is not a projection. There is also a choice of how to interpret the flipping. The minus sign (in $\exp(-i\,dt\ \mathbf{p}\cdot\boldsymbol{\sigma})$) for the opposite-handed massless particle propagation can be put in either factor in Eq. (A.1), so that one can think of the flip as changing the nature of the spin operation or the direction of propagation.

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