A general framework for non-equilibrium phenomena: the master equation and its formal consequences

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Abstract

We consider non-equilibrium systems defined by a state space, and by a stochastic dynamics and its stationary state. The dynamics need not satisfy detailed balance. In this abstract framework we do the following: (1) define and analyze “relative entropy”, (2) study dissipation in the relaxation to the stationary state, as well as the extra dissipation to maintain the system in its stationary state against some detailed balance dynamics, (3) extend the fluctuation–dissipation theorem and the Onsager relations, and (4) give a formula for the stationary state in terms of a summation over trees. © Elsevier Science B.V.

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1. Introduction

Most natural systems are out of equilibrium and are often far from equilibrium. Still, we do not know general laws for them comparable to those of equilibrium statistical mechanics, although many attempts have been made, for example, to define thermodynamic functions in non-equilibrium situations (see, e.g., Refs. [1–5]).

One reason for the difficulty in extending the equilibrium formalism to non-equilibrium situations is the fact that non-equilibrium situations are dynamical: things happen in non-equilibrium systems whether they be in stationary or non-stationary states, while the theory of equilibrium statistical mechanics and thermodynamics can be entirely deduced from the knowledge of the Boltzmann distribution.

Our approach [6] begins from the master equation, in effect assuming the system can be described by a Markovian stochastic process. This will be inadequate for systems in which quantum interference is important or for systems with infinite memory; nevertheless, the domain of applicability is still large. This approach shares features with many others, not surprising in view of the already wide application of the master equation. For example, an early paper with relative entropy results similar to ours is Ref. [7].

In this Letter we summarize and extend Ref. [6]. The system is represented by a state space $X$, which we take to be finite. The specification of a stationary state $\rho(x)$ (as a probability distribution on $X$) does not characterize the dynamics, because such a state can always be considered to be a Boltzmann state.
with energy proportional to $-\log \bar{p}(x)$. For the equilibrium case, $\bar{p}$ generally carries most of the information one wants (although equilibration rates may be of interest too), while for the non-equilibrium case $\bar{p}$ generally tells you only a small portion of the essential properties of the system.

In our framework, the dynamics are specified by a stochastic matrix $R_{xy}$ which is the probability of a transition from state-$y$ to state-$x$ in a time step $\Delta t$, so that

$$\sum_{x \in X} R_{xy} = 1, \quad \sum_{y \in X} R_{xy} \bar{p}(y) = \bar{p}(x). \quad (1.1)$$

We assume that $R$ is irreducible, but we do not assume that it satisfies detailed balance, i.e., we do not assume that $R_{xy} \bar{p}(y) = R_{yx} \bar{p}(x)$ for all $x$ and $y$. In fact, deviation from detailed balance seems to be essential for systems exhibiting organization and which, as a consequence, exhibit net currents, although they remain in their stationary state.

The matrix $R$ includes all dynamical effects, those taking place in the system, as well as those describing the exchange of matter and energy with external reservoirs.

2. Relative entropy of two probability distributions

Given two probability distributions $p$ and $q$ on the state space $X$, we define

$$S(p) = - \sum_{x \in X} p(x) \log p(x),$$

$$S(p \mid q) = - \sum_{x \in X} p(x) \log \frac{p(x)}{q(x)}. \quad (2.1)$$

$S(p)$ is the usual entropy and $S(p \mid q)$ (which we call relative entropy) was introduced in Ref. [8], and used in Ref. [9] and in Ref. [10] in information theory. $S(p \mid q)$ is always negative and is 0 if and only if $p = q$. For any distribution $p$, the product $Rp$ gives the evolution of $p$ in one time step. It can then be shown that

$$S(p \mid q) \leq S(Rp \mid Rq). \quad (2.2)$$

Let us now assume that $p_\delta(x)$ is a probability distribution close to the stationary state $\bar{p}$

$$p_\delta(x) = \bar{p}(x) \exp[\varphi(x, \delta)],$$

$$\varphi(x, \delta) = \delta \varphi_1(x) + \delta^2 \varphi_2(x) + \ldots. \quad (2.3)$$

Then

$$S(p_\delta \mid \bar{p}) \sim -\frac{1}{2} \langle (\delta \varphi_1^2) \rangle_\bar{p}, \quad (2.4)$$

where $\langle \cdot \rangle_\bar{p}$ denotes the average with respect to the distribution $\bar{p}$. If $\bar{p}$ is an equilibrium distribution (with free energy $F_{eq}$, i.e., $\bar{p}(x) = \exp[-\beta(E(x) - F_{eq})]$), then

$$S(q \mid \bar{p}) = \beta(F_{eq} - F_q),$$

$$F_q = \langle E \rangle_q - S(q) / \beta.$$ 

Thus $F_q$ can be interpreted as the free energy of the probability distribution $q$. In general, $S(q \mid p)$ represents the dissipation during the relaxation of the state $q$ to $\bar{p}$.

Let us also consider the transpose $^t R$ of $R$. Defining the scalar product $\langle \cdot | \cdot \rangle$ by

$$\langle v | \omega \rangle_\bar{p} = \sum_{x \in X} \bar{p}(x) v(x) \omega(x),$$

the adjoint of $^t R$ is

$$(^t R)^* = \text{diag} \left( \frac{1}{\bar{p}} \right) R \text{diag}(\bar{p}).$$

Call $\psi_1$ the eigenvector of $(^t R)^* (^t R)$ associated with the maximal eigenvalue $\mu_{max} = \mu_1 < 1$ of this operator. For any distribution, $p$, that is close to the stationary state $\bar{p}$, as in (2.3), we have

$$S(Rp \mid \bar{p}) - S(p \mid \bar{p}) > \frac{\delta^2}{2} \langle \psi_1^2 \rangle \bar{p} (1 - \mu_{max}). \quad (2.5)$$

When $R$ satisfies detailed balance, $(^t R)^* = ^t R$, and $\mu_{max}$ is the leading eigenvalue of $R^2$.

To prove the inequality (2.5), define $p \equiv p_\delta$ as in Eq. (2.3). Then

$$R_{\delta}(x) = \bar{p}(x) \exp[\delta \varphi_1(x) + \ldots].$$

\footnote{In general $R$ can have complex eigenvalues of norm 1, and these can be useful for studying the same sort of dynamical situations that interest us here. We expect that our dynamical statements will ultimately be phrased in terms of the \textit{currents}, namely $J_\delta = R_{\delta}(x) \bar{p}(y) - R_{\delta}(y) \bar{p}(x)$ It then becomes convenient to redefine $R$ by $R \rightarrow (1 - \epsilon)R + \epsilon I$ for some small positive $\epsilon$. This pushes the norm-one eigenvalues down a bit, while leaving the essential spectrum intact. The periodic behavior that one might otherwise have studied using the norm-1 (non-real) eigenvalues can now be examined by looking at $J_\delta$, which fully reflects the periodic behavior, and which is essentially unchanged by the transformation.}
Comparison of order $\delta$ terms in the preceding equation leads to $\psi_i = \text{diag}(1/\bar{p}) R(\text{diag} \bar{p}) \varphi_i$. Using Eq. (2.4) it follows that

$$S(R p | \bar{p}) - S(p | \bar{p}) \sim -\frac{\delta^2}{2} \left[ \left( \frac{1}{\text{diag}(\bar{p})} R(\text{diag} \bar{p}) \varphi_i \right)_\beta \right]^2 - \langle \varphi^2_i \rangle_\beta.$$

One maximizes the second member under the conditions $\langle \varphi^2_i \rangle_\beta = 1$, $\langle \varphi_i \rangle_\beta = 0$ (the last condition follows from $\sum \varphi_j(x) \bar{p}(x) = 0$). Details are in Ref. [6].

3. Path entropy

(a) Absolute path entropy. A path up to time $T = n \Delta t$ is a sequence $\gamma = (x_0, \ldots, x_T)$ of points in $X$. If $p$ is an initial distribution, the $R$-dynamics induces a probability measure $\mu^{(R, p)}(\gamma)$ on the space of paths by

$$\mu^{(R, p)}(\gamma) = R_{x_T x_{T-1}} R_{x_{T-1} x_{T-2}} \cdots R_{x_1 x_0} p(x_0).$$

(3.1)

We define an absolute entropy

$$\sigma(T | R, p) = -\sum_{\gamma \text{ up to } T} \mu^{(R, p)}(\gamma) \log \mu^{(R, p)}(\gamma).$$

It is easy to verify that

$$\sigma(T | R, p) \leq \sum_{n=0}^T S(R^n p).$$

(b) Relative entropy. The relative entropy is a measure of the extra dissipation needed to maintain a non-equilibrium state in a larger environment that is at thermal equilibrium. We represent the action of this environment on our system by a stochastic matrix $W$. Under the influence of $W$, the system $X$ would relax to an equilibrium state $p_{eq}$ and the "thermal equilibrium" condition is taken to mean that $W$ satisfies detailed balance

$$W_{yx} p_{eq}(x) = W_{xy} p_{eq}(y)$$

for all $x$ and $y$. The relative path entropy is defined by

$$\mathcal{S}(T | (R, p), (W, q)) = -\sum_{\gamma} \mu^{(R, p)}(\gamma) \log \frac{\mu^{(R, p)}(\gamma)}{\mu^{(W, q)}(\gamma)}.$$  

(3.2)

It is always negative. It is easy to see that one has

$$\mathcal{S}(T | (R, p), (W, q)) = S(p | q) + \sum_{i=0}^{T-1} \Delta_i \mathcal{S}(R, R^i p, W),$$

(3.4)

where we have denoted

$$\Delta_i \mathcal{S}(R, R^i p, W) = -\sum_{x,y} R_{xy} q(y) \log \frac{R_{xy}}{W_{xy}}.$$  

(3.5)

and $R^t$ denotes $R$ to the power $t$.

(c) Analysis near equilibrium. Assume now that $\bar{p}(x)$ is close to equilibrium and that $R$ is close to $W$. We write

$$\bar{p}(x) = p_{eq}(x) \exp[\varphi(x, \delta)]$$

$$\equiv p_{eq}(x) \exp[\delta \varphi_1(x) + \delta^2 \varphi_2(x) + \ldots].$$

(3.6)

$$R_{xy} = W_{xy} \exp[f(x, y, \delta)]$$

$$= W_{xy} \exp(\delta f^{(1)} + \delta^2 f^{(2)} + \ldots).$$

(3.7)

Then we have

$$|\Delta_i \mathcal{S}(R, \bar{p}, W)| = \frac{\delta^2}{2} \sum_{x,y} W_{xy} p_{eq}(y) (f^{(1)}_{xy})^2.$$  

(3.8)

To prove Eq. (3.8), we derive the following identities,

$$\sum_y W_{yx} f^{(1)}_{yx} = 0, \quad \sum_y W_{yx} \left[ f^{(2)}_{yx} + \frac{1}{2} (f^{(1)}_{yx})^2 \right] = 0,$$
which are obtained by writing \( \Sigma_x R_{xy} = 0 \) and expanding Eq. (3.7) to order \( \delta^2 \). Next, using Eq. (3.5),

\[
| \Delta_1 S(R, \tilde{\rho}, W) | = \sum_{xy} W_{xy} \exp \left( \delta f_{xy}^{(1)} + \delta^2 f_{xy}^{(2)} \right) \times \tilde{\rho}(x) \left( \delta f_{xy}^{(1)} + \delta^2 f_{xy}^{(2)} \right).
\]

We write

\[
\tilde{\rho} = p_{eq}(x) + \delta \rho'(x) + O(\delta^3).
\]

(The term of \( O(\delta^3) \) is not needed.) Then

\[
| \Delta_1 S(R, \tilde{\rho}, W) | = \delta \sum_{xy} p_{eq}(y) W_{xy} f_{xy}^{(1)} + \delta^2 \left( \sum_{xy} W_{xy} p_{eq}(y) \left[ f_{xy}^{(2)} + (f_{xy}^{(1)})^2 \right] + \sum_{xy} W_{xy} \rho'(y) f_{xy}^{(1)} \right) + \ldots.
\]

Using the previous identities, this reduces to Eq. (3.8).

Now consider the system \( X \) in its \( R \)-dynamics stationary state, \( \tilde{\rho} \). A measure of the extra dissipation per unit time step to maintain it there is given by \( | \Delta_1 S(R, \tilde{\rho}, W) | \). If we switch off the external reservoirs that produce the \( R \) dynamics, then the state \( \tilde{\rho} \) starts relaxing under the \( W \) dynamics and the rate of dissipation is measured by \( S(W \tilde{\rho} \mid \tilde{\rho}) \). One has the inequality

\[
\frac{1}{2} \left[ S(W \tilde{\rho} \mid p_{eq}) - S(\tilde{\rho} \mid p_{eq}) \right] \leq | \Delta_1 \mathcal{S}(R, \tilde{\rho}, W) |. \tag{3.9}
\]

This inequality is obtained by a lengthy variational calculation [6].

The dissipation necessary to maintain \( \tilde{\rho} \) can also be measured by \( S(W \tilde{\rho} \mid p_{eq}) \). This quantity is the dissipation, during the first time step, of the relaxation from \( \tilde{\rho} \) under the \( W \)-dynamics. One verifies that, up to higher order terms in \( \delta \),

\[
S(W \tilde{\rho} \mid \tilde{\rho}) = S(\tilde{\rho} \mid W \tilde{\rho}) = -\frac{\delta^2}{2} \sum_x p_{eq}(x) \left( \sum_y W_{xy} f_{xy}^{(1)} \right)^2.
\tag{3.10}
\]

The proof of Eq. (3.10) was not included in Ref. [6], so that we now give an indication of its derivation. Write

\[
W_{xy} = R_{xy} + \delta V_{xy} + \ldots, \quad \sum_x V_{xy} = 0.
\]

Then

\[
S(W \tilde{\rho} \mid p_{eq}) = -\sum_x \left[ \tilde{\rho}(x) + \delta \tilde{\rho}'(x) \right] \lim \left( 1 + \frac{\delta \tilde{\rho}(x)}{\tilde{\rho}} \right) \times \left[ \frac{\delta \tilde{\rho}'(x)}{\tilde{\rho}} - \frac{1}{2} \delta^2 \left( \frac{\tilde{\rho}'(x)}{\tilde{\rho}} \right)^2 \right].
\]

The term in \( \delta \) drops out because \( \sum_x V_{xy} \tilde{\rho}(y) = 0 \). It then follows that

\[
S(W \tilde{\rho} \mid p_{eq}) = -\frac{\delta^2}{2} \sum_x \left[ \frac{\tilde{\rho}'(x)}{\tilde{\rho}} \right]^2.
\]

But \( W = R \exp(-\delta f^{(1)}) = R - \delta R f^{(1)} + \ldots, \) so that \( \tilde{\rho}'(x) = \sum_x R_{xy} f_{xy}^{(1)} \tilde{\rho}(y) \). It follows that up to terms of order \( \delta \)

\[
\tilde{\rho}'(x) = \sum_y W_{xy} f_{xy}^{(1)} \tilde{\rho}(y) = \sum_y W_{xy} f_{xy}^{(1)} p_{eq}(y)
\]

and using detailed balance this is \( \tilde{\rho}'(x) = p_{eq}(x) \sum_y W_{xy} f_{xy}^{(1)} \). Therefore, up to higher order terms

\[
S(W \tilde{\rho} \mid \tilde{\rho}) \sim -\frac{\delta^2}{2} \sum_x p_{eq}(x) \left( \sum_y W_{xy} f_{xy}^{(1)} \right)^2.
\]

This is Eq. (3.10). The convexity inequality

\[
\sum_y W_{xy} \left[ f_{xy}^{(1)}(y) \right]^2 \geq \left( \sum_y W_{xy} f_{xy}^{(1)}(y) \right)^2
\]

(valid because the square function is convex and \( \sum_x W_{xx} = 1 \)) implies the inequality

\[
| S(W \tilde{\rho} \mid \tilde{\rho}) | \leq | \Delta_1 \mathcal{S}(R, \tilde{\rho}, W) |.
\]

Finally, comparing Eq. (3.9) and Eq. (3.10), we obtain the inequality

\[
| S(W \tilde{\rho} \mid \tilde{\rho}) | \leq | \Delta_1 \mathcal{S}(R, \tilde{\rho}, W) |. \tag{3.11}
\]
4. Reduced description

Usually, a system with a large state space $X$ is characterized by a small number of slow variables $A_1,\ldots,A_n$ (under the $R$-dynamics). The $A_j$ are in general complex valued functions on $X$, and we assume

$$ \langle A_j \rangle_R = 0, \quad j = 1,\ldots,n. $$

A point $x$ in $X$ is thus characterized by

$$ x = (A_1(x),\ldots,A_n(x),u), \quad (4.1) $$

where $u$ denotes the other coordinates (the fast variables). Essentially, the slow variables $A_1,\ldots,A_n$ are left eigenvectors of $R$ with eigenvalues close to 1 (but not equal to 1). For given values $a_1,\ldots,a_n$ of $A_1,\ldots,A_n$, we define

$$ \tilde{P}(a_1,\ldots,a_n) = \sum_{x \in X} \tilde{p}(x) \prod_{j=1}^n \delta(A_j(x) - a_j), \quad (4.2) $$

$$ \Sigma(a_1,\ldots,a_n) = -\lim \tilde{P}(a_1,\ldots,a_n). \quad (4.3) $$

The function $\Sigma$ is the generalization of the Einstein entropy to a non-equilibrium situation.

If we prepare an ensemble in the stationary state $\tilde{p}$ and observe only those samples in the ensemble that exhibit a fluctuation $a = (a_1,\ldots,a_n)$ of the slow variables $A_1,\ldots,A_n$, we can consider this sub-ensemble to define a probability distribution $q_a(x)$ given by

$$ q_a(x) = \frac{\tilde{p}(x)}{P(a_1,\ldots,a_n)} \prod_{j=1}^n \delta(A_j(x) - a_j). \quad (4.4) $$

It is then easy to see that the relative entropy as defined in Section 2 is exactly the opposite of the Einstein entropy

$$ S(q_a \mid \tilde{p}) = -\Sigma(a). \quad (4.5) $$

This justifies taking $\Sigma(a)$ to be a Lyapunov function for the evolution of the $a_1,\ldots,a_n$ [1,3]. Namely, if we wait a long enough time so that the $a$ vary from $a$ to $a + \delta a$, while the fast variables $u$ recover their conditional stationary distribution, the state $q_a$ would become $q_{a+\delta a}$ in the $R$-dynamics (to a very good approximation) so that

$$ S(q_a+\delta a \mid \tilde{p}) - S(q_a \mid \tilde{p}) \geq 0 $$

and then

$$ \Sigma(a + \delta a) \geq \Sigma(a). $$

5. Fluctuation–dissipation theorem

Call $\alpha = (\alpha_1,\ldots,\alpha_n)$ the conjugate variables to the $A_j$ and define a state

$$ p_\alpha(x) = \frac{1}{Z_n} \tilde{p}(x) \exp\left(\sum_{i=1}^n \alpha_i A_i + O(\alpha^2)\right). \quad (5.1) $$

so that, in the linear response approximation (i.e., modulo $O(\alpha^2)$),

$$ \langle A_j \rangle_{p_\alpha} = \sum_{j=1}^n \langle A_j A_j \rangle_{p_\alpha} \alpha_j \equiv (\chi \alpha)_j, \quad (5.2) $$

where $\chi_{ij} = \langle A_i A_j \rangle_{p_\alpha}$ and

$$ \langle A_i(\Delta t) - A_i(0) \rangle_{p_\alpha} = \sum_{j=1}^n \alpha_j \sum_{x,y} A_j(x) ((R_{xy} - \delta_{xy}) \tilde{p}(y) A_j(y)) \quad (5.3) $$

where $\langle \cdot \rangle_{p_\alpha}$ denotes the average in the state $p_\alpha$ and $A_i(\Delta t)$ is the evolute $A_i R$ of $A_i$ in one time step $\Delta t$. Moreover,

$$ \langle [A_i(\Delta t) - A_i(0)] [A_j(\Delta t) - A_j(0)] \rangle_{p_\alpha} = \sum_{x,y} A_i(x) ((R_{xy} - \delta_{xy}) \tilde{p}(y) A_j(y) + O(\alpha A^3)). \quad (5.4) $$

We deduce easily

$$ \sum_{i,j} \alpha_i \alpha_j \langle [A_i(\Delta t) - A_i(0)] [A_j(\Delta t) - A_j(0)] \rangle_{p_\alpha} = -2 \sum_i \alpha_i \langle A_i(\Delta t) - A_i(0) \rangle_{p_\alpha}. \quad (5.5) $$

The elimination of the $\alpha_i$ in terms of $\langle A_i \rangle_{p_\alpha}$, using Eq. (5.2) and the susceptibility $\chi$, gives the fluctuation–dissipation theorem in our context. In particular, we can choose for the $A_i$ left eigenvectors of $R$, $A_i R = \lambda_i A_i$. In this case, the susceptibility matrix
becomes diagonal. We then obtain the following statement of the fluctuation-dissipation theorem
\[ \langle [ A_i(\Delta t) - A_i(0)] [ A_j(\Delta t) - A_j(0)] \rangle_{p_a} \]
\[ = -2 \frac{\partial^2}{\partial \langle A_i \rangle_{p_a}} \langle A_i(\Delta t) - A_i(0) \rangle_{p_a} \delta_{ij} \quad (5.6) \]
(see Ref. [12] for the classical equilibrium statement).

6. Onsager reciprocity relations for non-equilibrium states

The relative entropy of the state \( p_a \) is
\[ S(p_a | \bar{p}) = -\frac{1}{2} \sum_{k,l} \langle A_k A_l \rangle_{p_a} \bar{P}^{-1} \langle A_k \rangle_{p_a} \quad (6.1) \]
so that the corresponding forces are
\[ F_k = -\frac{\partial S(p_a | \bar{p})}{\partial \langle A_k \rangle_{p_a}} \]
\[ = \sum_l \langle A_k A_l \rangle_{\bar{p}}^{-1} \langle A_l \rangle_{p_a} = \alpha_k. \quad (6.2) \]
The current \( J_k \) for \( A_k \) is defined by
\[ J_k = \langle A_k(\Delta t) - A_k(0) \rangle_{p_a} \]
\[ = \sum_j \alpha_j \sum_{x,y} A_k(x)(R_{xy} - \delta_{xy}) \bar{p}(y) A_j(y). \quad (6.3) \]
We can then write
\[ J_k = \sum L_{kj} F_j, \quad (6.4) \]
with
\[ L_{kj} = \sum_{x,y} A_k(x)(R_{xy} - \delta_{xy}) \bar{p}(y) A_j(y). \quad (6.5) \]
In general, the matrix \( L_{kj} \) is not symmetric. It becomes symmetric if we assume that \( R \) satisfies detailed balance, \( R_{xy} \bar{p}(y) = R_{yx} \bar{p}(x) \) for all \( x, y. \)

7. A tree summation formula for the stationary state

In this section, we give a tree integral formula for the stationary state of a stochastic matrix \( R \) (see also Refs. [13,14]). Consider a stochastic \( N \times N \) matrix and denote by \( \{1, \ldots, N\} \) the set of states. Let us mark one point, say \( j \), of this set and call a spanning tree of root \( j \) an oriented tree of root \( j \) (the orientation going from the root to the leaves) so that any \( 1 \leq k \leq N \) is a vertex of the tree.

Consider an edge \((k, l)\) of such a tree, \( T \). We assign to this edge the weight \( R_{kl} \) and we assign to the tree \( T \) the weight
\[ W(T) = \prod_{(k, l) \text{ an edge of } T} R_{kl}. \quad (7.1) \]
The main statement is that the minor \( M_{jj} \) of \(-1 + R_{jj}\) in \(-1 + R\) is given by
\[ M_{jj} = (-1)^{N-j} \sum_{T_j} W(T_j), \quad (7.2) \]
where the sum is taken over all spanning trees \( T_j \) with root \( j \) as defined above. In particular, \( M_{jj} \) is a sum of positive quantities, although it was defined as a determinant.

The stationary state is given by the formula
\[ \bar{p}_i = \frac{M_{ji}}{\sum_{j=1}^N M_{jj}}. \quad (7.3) \]
This replaces the Boltzmann distribution of equilibrium statistical mechanics.

8. Conclusion

In this Letter we have presented a summary of a general framework for non-equilibrium phenomena. The basic concept is the stationary state, given as a probability distribution on the state space of the system, together with a stochastic dynamics. In this abstract setting we obtain, in a straightforward manner, extension to non-equilibrium situations of many of the familiar identities and inequalities of equilibrium theory (decrease of potential, fluctuation-dissipation, Onsager relations). In some cases we find additional new relations.

In particular, we can extend all the usual results to dynamics that do not necessarily satisfy detailed balance. Such an extension is almost surely needed if one is to understand systems with emergent organization (see, e.g., Ref. [15]). Moreover, this abstract point of view is justified by particular models that
have been previously analyzed [16] in various contexts.

Although our efforts, as described here, have been mostly devoted to providing parallels in the non-equilibrium case of equilibrium structures, there is one set of quantities which, by definition, do not appear at all in the equilibrium case. These are the currents, \( J_{yx} \equiv R_{xy} \tilde{p}(y) - R_{yx} \tilde{p}(x) \). It is these that allow the stationary state to nevertheless be dynamical. Interestingly, it is analogous quantities that one generally uses in the description of biological, chemical, meteorological, and many other processes. We expect that these markers of non-equilibrium, the currents, will be important for the description of complex phenomena and even in the elucidation of the elusive concept of complexity itself.

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