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Wedges I

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The wedge problem, that is, the propagation of radiation or particles in the presence of a wedge, is examined in different contexts. Generally, the paper follows the historical order from Sommerfeld's early work to recent stochastic results—hindsights and new results being woven in as appropriate. In each context, identifying the relevant mathematical problem has been the key to the solution. Thus each section can be given both a physics and a mathematics title: Section 2: diffraction by reflecting wedge; boundary value problem of differential equations; solutions defined on mutiply connected spaces. Section 3: geometrical theory of diffraction; identification of function spaces. Section 4: path integral solutions; path integration on multiply connected spaces; asymptotics on the boundaries of function spaces. Section 5: probing the shape of the wedge and the roughness of its surface: stochastic calculus. Several propagators and Green functions are given explicitly, some old ones and some new ones. They include the knife-edge propagator for Dirichlet and Neumann boundary conditions, the absorbing knife edge propagator, the wedge propagators, the propagator for a free particle on a μ-sheeted Riemann surface, the Dirichlet and the Neumann wedge Green function.

To John Archibald Wheeler: Recent interest in the wedge problem (Schulman, 1982; Shiekh, 1985) has led us to review several of its aspects from Sommerfeld's calculation of diffraction of light by a wedge in 1896 (Sommerfeld, 1896) to current investigations of stochastic diffusions when

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sharp obstructions are present. The various facets of this problem touch upon your many interests and there is nobody more appropriate than you to whom we would like to dedicate this paper. There is also no occasion more appropriate than this Festschrift to offer it to you: In 1938 Sommerfeld celebrated his seventieth birthday, and for this occasion Pauli (1938) dedicated to his "old teacher"[8] a paper on wedges.[9]

It seems impossible to treat a subject with so many ramifications in a few pages. But we have all seen your colorful blackboards summarizing questions and answers of multifaceted issues; and we have all heard you saying at the end of a seminar, "Tell us in one sentence what we have learned today." Here are, in black and white, a few sentences on the wedge.

1. STATING THE WEDGE PROBLEM

Given a wedge, and a source of radiation or a source of particles, compute the intensity of the radiation or the density of the particles detected on the other side of the wedge. Solve (first) the problem in a plane, since many wedge problems in $\mathbb{R}^3$ can be reduced to problems in $\mathbb{R}^2$.

Notation. Points in $\mathbb{R}^3$ are given in cylindrical coordinates. The edge of the wedge is the z-axis. The angle of physical interest, i.e., the external angle for a solid wedge, is θ; if $\theta = 2\pi$ the wedge is a half plane barrier, or as it is so often called, a knife edge; rational wedges refer to wedges of angle $\theta = \mu\pi/\nu$, where μ and ν are positive integers. The source is located at a with coordinates (R', ϕ') and the detector is located at b with coordinates (R, ϕ). (See Fig. 1.)

The Green function G of the elliptic operator $\Delta + k^2$, where $\mathbf{k}$ is the wave vector of a monochromatic incident plane wave, satisfies $(\Delta + k^2)\, G(x, a) = -\delta(x-a)$. The elementary kernel of the Schrödinger equation for propagation from a to b in time t is labelled $K(b, t; a)$; it satisfies $(\partial/\partial t - i\hbar\Delta/2M)\, K(x, t; a) = \delta(t)\, \delta(x-a)$. Subscripts e and w on G and K denote the edge and wedge cases and superscripts $+$ and $-$ refer, respectively, to the Neuman and Dirichlet boundary conditions. The definitions of special functions are the ones used by Gradshteyn and Ryzhik (1980), and by Abramowitz and Stegun (1965). The Bessel function

[8] Not "old" then; this is a term used by Pauli, with affection, many years later, in a letter sent to Cecile Morette and Bryce DeWitt on the occasion of their marriage.

[9] Rubinowicz (1938) also writing at this occasion refers to Sommerfeld's 1896 paper as his "first great scientific achievement." Indeed, some years before, Lord Rayleigh in the article on "Wave Theory" in the *Encyclopaedia Britannica* had stated that the mathematical difficulties were so formidable that no successful attempt had yet been made to solve this problem.

$K_\nu(z)$ of imaginary argument is given in terms of the Bessel functions of the first kind $J_\nu(z)$ by

$$K_\nu(-ix) = \pi \exp(\tfrac{1}{2}i\pi\nu)\,\frac{J_{-\nu}(x) - \exp(-i\pi\nu)\,J_\nu(x)}{2\sin(\nu\pi)}$$

The special functions used in the references are related to the ones we use as follows:

$$U_\nu(x)\ \text{(in Sommerfeld, 1896)} := \exp(-\tfrac{1}{2}i\pi\nu)\,K_\nu(-ix)$$

$$K_\nu(x)\ \text{(in Sommerfeld, 1896)} := -\frac{\pi}{2}\,Y_\nu(x)$$

In reviewing the problem, we generally follow historical order, but bring hindsight and describe new results as we proceed.

2. EXACT AND ASYMPTOTIC SOLUTIONS OF THE DIFFRACTION OF LIGHT BY A WEDGE

In 1896, A. Sommerfeld, then a *Privat-dozent*, solved the problem of the diffraction of light from a perfectly reflecting knife edge by recognizing that Fresnel diffraction is a well-defined mathematical boundary value problem,[10] and making a "decisive"[11] *ansatz* suggested by Riemann's solution for a similar problem.

Riemann had computed the potential χ of a point charge at (R', ϕ') outside a conducting wedge of external angle $\theta = \mu\pi/\nu$, illustrated in Fig. 1, as follows. The potential χ satisfies Laplace's equation

$$\Delta\chi(r, \psi) = 0 \qquad \text{at } (r, \psi) \neq (R', \phi') \tag{1}$$

and the boundary condition $\chi = 0$ on the surface of the wedge. The wedge problem for a monochromatic incident plane wave (**k** perpendicular to the edge of the wedge) reduces to the solution of the two-dimensional scalar wave equation (Helmholtz equation)

$$\Delta u + k^2 u = 0 \qquad \text{at } (r, \psi) \neq (R', \phi') \tag{2}$$

[10] Sommerfeld (1954), p. 249, referring to this 1896 paper "With this problem it was first demonstrated that Fresnel diffraction constitutes a well-defined mathematical boundary value problem (Fraunhofer diffraction cannot be treated directly by this method but only as a limiting case of Fresnel diffraction)."

[11] Pauli (1983), p. 925.

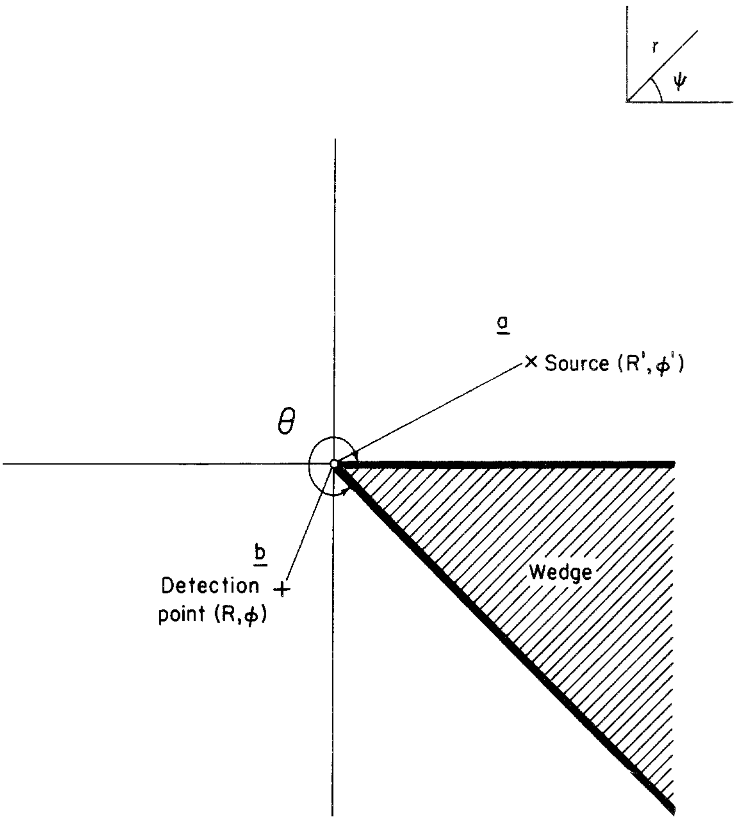

Fig. 1. Diffraction from a wedge.

together with either of the following boundary conditions:

$u = 0$ for a transverse magnetic wave (Dirichlet)
$\partial u/\partial n = 0$ for a transverse electric wave (Neumann)
where n is the normal to the surface.

(1) This problem had been solved by the method of images for $\theta = \pi/\nu$ for integral ν. The boundary conditions had been implemented through an algebraic sum of contributions from the source and its images placed symmetrically with respect to the wedge surfaces. When $\theta = \pi/\nu$ the images are all inside the wedge, which is not a region of physical interest. When $\theta \neq \pi/\nu$ some of the images (finitely many if θ is a rational multiple of π, infinitely many otherwise) are in the physical region outside the wedge. Riemann, in his doctoral thesis, developed a technique to overcome this difficulty in the rational case. In the complex plane a wedge of angle

$\theta=\mu\pi/\nu$ with μ, ν integral may be mapped into a wedge of angle $\theta=\pi/\nu$ by the transformation

$$z \rightarrow Z := T(z) = z^{1/\mu}$$

The solution of the Laplace equation (1) in the z-variable is an analytic function

$$w(z) = u(z) + iv(z), \qquad z = x + iy$$

The solution in the Z-variable

$$W(Z) = U(Z) + iV(Z) = W(T(z)) = w(z)$$

is also an analytic function. The boundary conditions in the Z-variable are implemented by images inside the wedge, and the problem can be solved in the previously described manner. The transformation back to the z-variable yields a multivalued function $w(z)$. The values of physical interest can be obtained by restricting z to $T^{-1}(Z)$ for Z outside the wedge. Alternatively a many-valued function may be made single valued by considering it defined on a many-sheeted Riemann surface. In this picture the images appear on the lower folds of the many-sheeted Riemann surface. They are "hidden" and act only to implement the boundary conditions.

(2) Recognizing the mathematical similarity of the wedge problem in electrostatic theory and in diffraction theory, Sommerfeld looked for a many-valued solution of the Helmholtz equation (2) satisfying the appropriate boundary conditions, or "in the language introduced by Riemann,"[12] for a solution defined on a many-sheeted Riemann surface.

There are two classes of boundary conditions:

(i) The radiation condition at infinity[13], for which the field must behave like an outgoing spherical wave at infinity. This condition is satisfied[14] by the following Green's function for the Helmholtz equation. When $R<R'$

$$G(R, \phi; R', \phi') = \frac{1}{2\pi} \sum_{n=0}^{\infty} \gamma_n J_{n/\mu}(kR) \exp\left(\frac{-i\pi n}{2\mu}\right) K_{n/\mu}(-ikR') \times \cos\left(\frac{n}{\mu}(\phi-\phi')\right) \tag{3}$$

where $\gamma_0 = 1/\mu$ and $\gamma_n = 2/\mu$ for $n>0$. If $R'<R$, interchange R and R'.

[12] Sommerfeld (1954), p. 251.

[13] It seems that Sommerfeld was the first person to identify the radiation condition in scattering problems.

[14] Sommerfeld (1896), Eqs. (12) and (13), p. 356.

(ii) The wedge boundary condition (Dirichlet or Neumann) which is implemented by the contributions of images placed appropriately on a μ-sheeted Riemann surface.

In his 1896 paper Sommerfeld fully developed only the solution for the knife edge but showed how his method can be used for an arbitrary wedge. In 1897, fully occupied with other work, Sommerfeld urged Carslaw to continue this investigation and guided him in the early stages of his work. Carslaw (1899) solved the Helmholtz equation (2) and the diffusion equation

$$\frac{\partial u}{\partial t} = \kappa \Delta u$$

with the knife-edge boundary in the form of a definite Fresnel integral on the real line instead of a series expansion.

$$K_e{}^{\mp} = \frac{1}{4\pi\kappa\sqrt{\pi t}} \left\{ \exp\left[-\frac{r^2}{4\kappa t}\right] \int_{-\infty}^{(R'R/\kappa t)^{1/2}\cos\frac{1}{2}(\phi-\phi')} \exp[-\lambda^2]\, d\lambda \right.$$
$$\left. \mp \exp\left[-\frac{r'^2}{4\kappa t}\right] \int_{-\infty}^{(R'R/\kappa t)^{1/2}\cos\frac{1}{2}(\phi+\phi')} \exp[-\lambda^2]\, d\lambda \right\} \tag{4}$$

where

$$r^2 := R^2 + R'^2 - 2R'R\cos(\phi - \phi'), \qquad r'^2 := R^2 + R'^2 - 2R'R\cos(\phi + \phi')$$

This classic solution has recently been obtained by path integral techniques independently by Schulman (1982, 1984), Wiegel *et al.*,[15] and Shiekh (1986).

Solutions for other geometries of the source and the wedge and simplified versions of the calculations can be found in Macdonald (1902) and Carslaw (1916–1919). We note here a solution (5) derived by Macdonald which has also been obtained recently by path integration (Shiekh, 1985). For a Dirichlet wedge, and $R < R'$

$$G_w^- = \frac{2}{\theta} \sum_{n=1}^{\infty} \exp\left(\frac{-in\pi^2}{2\theta}\right) J_{n\pi/\theta}(kR)\, K_{n\pi/\theta}(-ikR')$$
$$\times \sin\left(\frac{n\pi\phi}{\theta}\right) \sin\left(\frac{n\pi\phi'}{\theta}\right) \tag{5}$$

For $R > R'$ interchange R and R'.

[15] Wiegel and Boersma (1983) were the first to notice that the path integral calculation gave a formula previously computed by Carslaw.

For a Neumann wedge with $R<R'$,

$$G_w^+ = \frac{1}{\theta}\left\{J_0(kR)\,K_0(-ikR') + 2\sum_{n=1}^{\infty}\exp\left(\frac{-in\pi^2}{2\theta}\right) J_{n\pi/\theta}(kR)\,K_{n\pi/\theta}(-ikR')\cos\left(\frac{n\pi\phi}{\theta}\right)\cos\left(\frac{n\pi\phi'}{\theta}\right)\right\} \tag{6}$$

For $R>R'$ interchange R and R'.

The series in (3), (5), and (6) converge slowly for large arguments of the Bessel function. Asymptotic solutions were developed by Sommerfeld; they are valid only away from the boundary of the geometric shadow. Pauli developed alternative asymptotic solutions valid on the shadow boundary.

3. GEOMETRICAL THEORY OF DIFFRACTION. JWKB APPROXIMATIONS

(i) Keller, in 1953, extended the geometric theory of optics to include diffraction effects.[16] This makes it possible to solve problems such as the diffraction of waves by wedges at the JWKB level of approximation without having to solve the full partial differential equation. The power of the method comes from the fact that one can solve a wide class of problems at the JWKB level by solving a number of comparatively simple canonical problems in the immediate neighborhood of a diffracting edge or vertex. A justification for the localization of the problem will be presented in the section on stochastic diffusion where it will be shown to be a consequence of Kac's "principle of imperceptibility" (Kac, 1959).

The geometric theory of optics models the propagation of light by rays which satisfies Fermat's principle. [See, for instance, Baker and Copson (1950), Longhurst (1957), Felsen (1976), Kouyoumjiam (1975), and Wait (1960).] This principle states that the path q taken by a ray extremizes the optical length L which is a function(al) on the space $\mathscr{C}(a, b)$ of continuous paths c from a to b,

$$L\colon \mathscr{C}(a,b) \to \text{by } c \to \int_a^b n(c(s))\,ds \tag{7}$$

In this expression, the paths c are the maps

$$c\colon \mathbb{R}_+ \to M\colon s \mapsto c(s)$$

[16] Keller's generalization also incorporated effects due to caustics and "creeping" or grazing rays. As only diffractional effects are considered here, the interested reader is referred to Keller's original work. The extensive literature that it has stimulated is reviewed in the work of Lewis *et al.* (1967) and Wait (1960).

where s is the arc length of the path and M is a smooth manifold (which may have a boundary) in which the rays propagate. In this discussion M will be assumed to be three dimensional; $a, b \in M$. The index of refraction n is a function on M.

The usual theory of geometrical optics considers only direct, reflected and refracted rays. Direct rays, if any, are *smooth* paths q which extremize the optical length for paths c in the function space $\mathscr{C}(a, b)$. Reflection and refraction occur when there are two-dimensional subspaces of M—called surfaces of discontinuity—on which n is discontinuous or which form the boundary of M. For clarity, these surfaces of discontinuity will be assumed to be smooth except along "edges," one-dimensional subspaces of the surface along which the curvature of the surface is discontinuous. In turn, the edges will be assumed to be smooth except at "vertices" where the curvature of the edge is discontinuous.

The reflected and refracted rays are relative extrema of the optical length for paths c restricted to the subspace $\mathscr{D}_\rho(a, b)$ of $\mathscr{C}(a, b)$ of paths which touch a point on the interior (i.e., the point is not on an edge) of a surface of discontinuity ρ times.

Keller realized that this theory could be generalized to include diffracted rays (and, as noted in a previous footnote, grazing and caustics) by decomposing $\mathscr{C}(a, b)$ into the disjoint subspaces $\mathscr{D}_{\rho,\sigma,\tau}(a, b)$. The integers ρ, σ, τ label, respectively, the number of smooth arcs (which may be simply points) the curves c have in the interior of the surfaces of discontinuity, on the edges and vertices.

Then, the variation problem of L restricted to the subspace $\mathscr{D}_{\rho,\sigma,\tau}(a, b)$ yields the rays which have a total of ρ reflections, refractions, and/or grazings along the surface, σ diffractions by an edge, and τ diffractions by a vertex.

(ii) Congruence of rays normal to a given surface S_0 (possibly a point) can be used to construct approximate monochromatic solutions $u(x)\exp(-i\omega t)$ to the wave equation called JWKB approximations:

$$u(x) = \sum_i A(q_i(s)) \exp(ik\psi(q_i(s))), \qquad k = \omega/c \tag{8}$$

for all q_i such that $q_i(s) = x$. The phase function $\psi(q(s))$ is the optical length along the ray q from S_0 to $q(s)$. The surfaces of constant phase are the wave fronts. $\psi(x)$ is a possibly multivalued, real or complex solution of the eikonal equation. The amplitude $A(q_i(s))$ is a function of the Jacobi field along q_i determined by the congruence of rays. The energy flux per unit area is proportional to nA^2, and the law of conservation of energy determines the amplitude. In the presence of caustics, which are characterized by Jacobi fields with vanishing boundary conditions, or in the presence of diffraction when one ray splits into infinitely many rays, the JWKB approximation takes a more complicated form. In this paper we consider only wedges. By analogy with the reflection and transmission coefficient which apportion the amplitude attached to an incident ray between its reflected and transmitted rays, Keller proposes to multiply the amplitude by an "edge diffraction" coefficient, or a "vertex diffraction" one as the case may be. Of interest for the wedge problem is the "straight edge diffraction" coefficient which has been computed by Keller as follows. Given an incident plane wave u_i of amplitude A_i propagating in a direction normal to the edge, with phase ψ_i at the edge, the diffracted wave fronts are cylinders with the edge as their axis, and Keller writes the approximate field constructed from the diffracted rays

$$u(r, \psi) = \sum_i DA_i r^{-1/2} \exp(i(kr + \psi_i)) \tag{9}$$

The straight edge diffraction coefficient is determined by comparing (9) with the exact solution (5), (6) obtained by Sommerfeld—or rather with the asymptotic value of the exact solution for large values of kr. It is found to be

$$\begin{aligned} D = \frac{\exp(i\pi/4)\sin(\pi^2/\theta)}{\theta(2k/\pi)^{1/2}} \Bigg[&\left(\cos\frac{\pi^2}{\theta} - \cos\frac{\pi(\phi - \phi')}{\theta}\right)^{-1} \\ &\mp \left(\cos\frac{\pi^2}{\theta} - \cos\frac{\pi(\phi + \phi' - 4\pi)}{\theta}\right)^{-1}\Bigg] \end{aligned} \tag{10}$$

ϕ' and ϕ are the incident and diffracted ray angles given in Fig. 1 and θ is the exterior angle of the wedge.

Keller's theory then asserts that this coefficient may be used to solve a wide class of diffraction problems in which the surface of discontinuity locally looks like a wedge.

The question remains of solving these canonical problems without recourse to known exact solutions. Functional integration provides methods of directly obtaining approximate solutions of the form suggested by Keller. In certain cases, as we shall see in the next section, it does even more; it provides exact solutions to diffraction problems.

4. PATH INTEGRAL SOLUTIONS

Diffraction plays an important role in the experimental verification of quantum mechanics. It is therefore surprising that edges and wedges have not played a more central role in the theoretical development of quantum mechanics. Indeed, they first appear in papers whose primary purpose is classical electromagnetic wave diffraction (Buslaev, 1968; Keller and McLaughlin, 1975; Lee, 1978; Schulman, 1982, 1984; Knoll and Schaeffer, 1976; Crandall, 1983a, b; Low, 1984; Shiekh, 1985). They are based on functional integral techniques; some use them to solve a heat equation, others to solve a Schrödinger equation.

Path integral solutions of the wedge problem require use of the following tools:

(i) Computation of path integrals on multiply connected spaces. (Sect. 4.1)

(ii) Extension of the method of stationary phase to problems where the critical point of the action does not fall inside the domain of integration. (Sec. 4.2)

We present here two path integral calculations for the wedge problem in quantum mechanics: one (Shiekh, 1985; Sect. 4.1 consists mainly of unpublished results of Shiekh) uses linear combinations of explicit solutions of the Aharonov–Bohm effect (i.e., a path integral on multiply connected spaces) and the other (Schulman 1982, 1984) computes the propagator directly when a knife edge is present using a stationary phase method. Both solutions are exact.

(1) (i) In an Aharonov–Bohm configuration (Aharonov and Bohm, 1959), the probability amplitude K for a particle of mass M to go from a to b in time t is

$$K(b, t; a; \alpha) = \sum_{m=-\infty}^{\infty} \exp(2\pi i m\alpha)\, \kappa_m \tag{11}$$

where m is the number of times a path winds around the solenoid, and α is given by the flux in the solenoid, i.e., by the vector potential A integrated along a loop around the solenoid.

$$\alpha = \frac{e}{h} \oint A \cdot dl \tag{12}$$

The partial amplitude κ_m is the sum over all paths in the m-homotopy class (Schulman 1968, 1971, 1981; Laidlaw and DeWitt-Morette, 1971). It has been obtained independently by Edwards (1967), Berry (1980, see also 1972, 1981), Morandi and Menossi (1984), and Shiekh (1986). It is given by

$$\kappa_m = \frac{M}{2\pi i\hbar t} \exp\left(\frac{iM}{2\hbar t}(R'^2 + R^2)\right) \int_{-\infty}^{+\infty} d\lambda \exp\Big(i\lambda(\phi + 2\pi m)\Big)$$
$$\times\, I_{|\lambda|}\left(-\frac{i}{\hbar t} MR'R\right)$$

Using

$$\sum_{m=-\infty}^{\infty} \exp(im\theta) = 2\pi \sum_{n=-\infty}^{\infty} \delta(\theta + 2\pi n)$$

one obtains

$$K(\phi, R, t; \phi', R'; \alpha)$$
$$= \frac{M}{2\pi i\hbar t} \exp\left(\frac{iM}{2\hbar t}(R'^2 + R^2)\right)$$
$$\times \sum_{n=-\infty}^{\infty} \exp(-i(n+\alpha)(\phi - \phi'))(-i)^{|n+\alpha|} J_{|n+\alpha|}\left(\frac{1}{\hbar t} MR'R\right) \qquad (13)$$

(ii) This propagator can readily be used to obtain the propagator K_e for an absorbing (in a sense to be made precise shortly) knife edge. The Aharonov–Bohm propagator $K(b, t; a; \alpha)$ given in Eq. (13) is a function of the flux α, and hence is a function of the equivalence class $[A]$ of gauge-related potentials satisfying (12). We can describe the Aharonov–Bohm effect in terms of any one member of this class and here we choose A to be zero except along the half-plane barrier. The probability amplitude $K(b, t; a; \alpha)$ changes by a factor $\exp(2\pi i\alpha)$ when the detector located at b crosses the knife edge. Thus one can construct K_e (absorbing knife edge) by adding the contribution of $K(b, t; a; \frac{1}{2})$ and $K(b, t; a; 0)$; the latter kills the former on the far side of the half-plane barrier (away from the source). The barrier is absorbing, not in the sense that paths are destroyed at the half-plane barrier, but in the sense that their contribution interferes destructively on the far side.

$$K_{\text{edge}} \equiv K_e = \tfrac{1}{2}(K(\alpha = \tfrac{1}{2}) + K(\alpha = 0)) \qquad (14)$$

Using the identity (Gradshteyn and Ryzhik, 1980)

$$\tfrac{1}{2}J_0(z) + \sum_{n=1}^{\infty} (-i)^{n/2} J_{n/2}(z) \cos(n\theta)$$
$$= (i\pi)^{-1/2} \exp(-iz\cos(2\theta)) \int_{-\infty}^{(2z)^{1/2}\cos\theta} \exp(is^2)\, ds \qquad (15)$$

we obtain K_e as a Fresnel integral:

$$K_e(\phi, R, t; \phi', R')$$
$$= \frac{M}{2i\pi\hbar t} \exp\left(i\frac{M}{2\hbar t}(R^2 + R'^2)\right)(i\pi)^{-1/2}$$
$$\times \exp\left(-i\frac{R'RM}{\hbar t}\cos(\phi - \phi')\right) \int_{-\infty}^{(2R'RM/\hbar t)^{1/2}\cos((\phi-\phi')/2)} \exp(is^2)\, ds \qquad (16)$$

Since the Aharonov–Bohm propagator is a function only of the equivalence class $[A]$, any potential in this class gives the same result for the absorbing knife edge. It follows that the propagator is (to some extent) independent of the orientation and shape of the barrier.

(iii) We shall show that the propagator K_e constructed from two different Aharonov–Bohm propagators is in fact a free propagator defined on a two-sheeted Riemann surface. It follows that it can be used to construct the reflecting knife edge propagators: K_e^- for Dirichlet boundary conditions, K_e^+ for Neumann boundary conditions. Since K_e is bivalued when $\phi - \phi'$ is not restricted to the interval $[0, 2\pi)$, it is natural to consider it as a single valued propagator on a two-sheeted Riemann surface. The "top" sheet of this Riemann surface may be identified with the physical space in which the source and detector are located. Thus, the knife edge may be modeled by a free particle on this surface, changing sheets where the knife edge would have been. The linear combination (14) is such that

$$\lim_{t \to 0^+} K_e(\phi, R, t; \phi', R') = \delta(R - R') \sum_{n=-\infty}^{\infty} \delta(\phi - \phi' - 4\pi n)$$

i.e., such that the source is on the physical sheet.

A perfectly reflecting knife edge is modeled by introducing an image of the source on the lower sheet and adding the contributions of the free propagators K_e, one with source S_0, the other with source S_1. To achieve Dirichlet boundary conditions ($K_e^- = 0$ on the knife edge) the two contributions must be of equal amplitude and opposite phase. On the other hand, to achieve Neumann conditions ($\partial K_e^+/\partial n = 0$, n being the normal to the surface) the two contributions must be of equal amplitude and equal phase.

$$K_e^{\mp}(\phi, R, t; \phi', R') = K_e(\phi, R, t; \phi', R') \mp K_e(\phi, R, t; -\phi', R') \qquad (17)$$

where K_e is given by (16). This is precisely the result obtained by Schulman (1982) by an entirely different method (see Sect. 4.2). Schulman and C. M. Newman (private communication) have realized that (17) can be obtained from the Aharonov–Bohm propagator with the method of images. The derivation of (17) brings new insight to Wiegel and Boersma's derivation.

(iv) These results can be generalized to absorbing and reflecting wedges. It is clear mathematically what needs to be done. The fact that the free propagator on a Riemann surface can be obtained from a linear combination of different Aharonov–Bohm propagators [generalization of (14)] is not as intuitive as in the knife edge problem and is carried out in the Appendix.

First we determine the Riemann surface to be considered for computing propagators in the presence of a wedge of external angle $\theta = \mu\pi/\nu$. We need images obtained by reflecting the source in each boundary wall. However, the image which achieves proper boundary conditions on one wall disturbs the boundary conditions on the other one. So we add images of these images and repeat the process until the set of images closes on itself.

Images stand at $\psi = \pm\phi' \pm 2k\theta$ (k an integer). For instance, the image S_1 is at $-\phi$, $\bar{S}_1$ is at $-\phi + 2\theta$, S_2 is at $\phi + 2\theta$, $\bar{S}_2$ is at $\phi - 2\theta$, etc. These are conveniently grouped as those obtained after an even number of reflections and those after an odd number

Even	Odd	
$\psi = \phi' + 2\pi k \frac{\mu}{\nu}$,	$\psi = -\phi' + 2\pi k \frac{\mu}{\nu}$,	$k = 1, \dots, \nu$

This set has 2ν elements and exists upon a μ-sheeted Riemann surface.

On a μ-sheeted Riemann surface, paths must loop a multiple of μ times to go from the source to the detector which are both on the top (physical) sheet. We can repeat the calculation (Shiekh, 1986) leading from (11) to (13) when $\alpha = 0$ for paths which loop around a multiple of μ times.

$$K_\mu = \sum_{m=-\infty}^{\infty} \kappa_{\mu m}$$

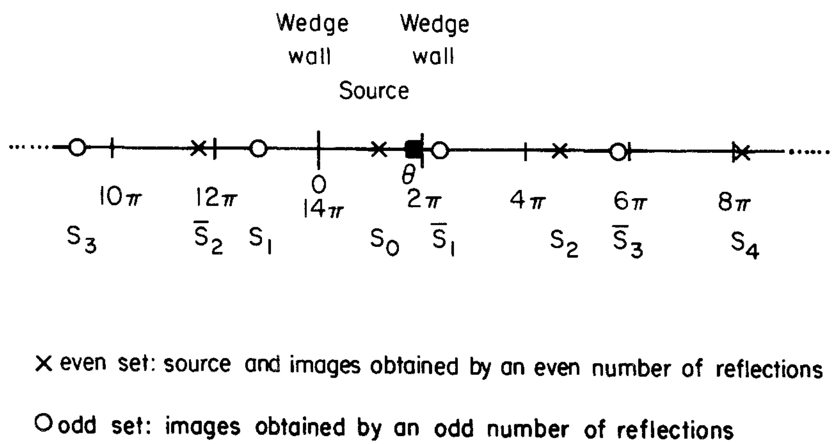

Fig. 2. Image positions on the μ-sheeted Riemannian surface for a wedge at external angle $\theta = 7\pi/4$ and source at $\phi' = 7\pi/6$. In this figure S_1 is the reflection of S_0 by the wall at $\psi = 0$, S_1 is the reflection of S_1 by the wall at $\psi = \theta$, S_2 is the reflection of S_1 by the wall at $\psi = \theta$, S_2 is the reflection of S_1 by the wall at $\psi = 0$, and so forth.

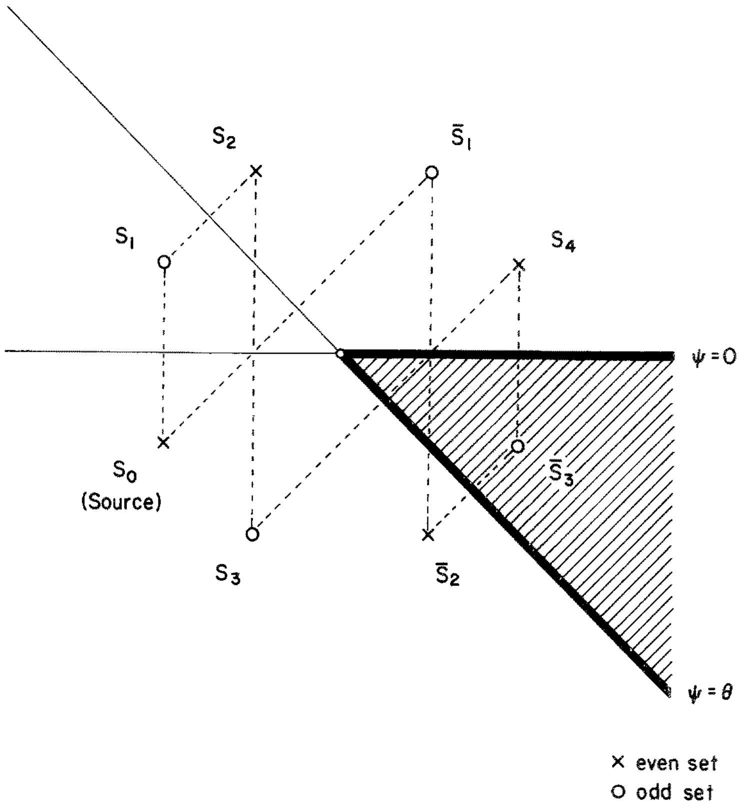

Fig. 3. The use of images to implement boundary conditions.

and obtain for the free propagator on a μ-sheeted Riemann surface

$$K_\mu(\phi, R, t; \phi', R') = \frac{1}{\mu} \frac{M}{2\pi i \hbar t} \exp\left[\frac{iM}{2\hbar t}(R^2 + R'^2)\right] \times \sum_{n=-\infty}^{\infty} \exp\left[-i\frac{n}{\mu}(\phi - \phi')\right](-i)^{|n/\mu|} J_{|n/\mu|}\left(\frac{MR'R}{\hbar t}\right) \tag{18}$$

With the source and all its images having the same strength, the Dirichlet boundary conditions are achieved by running the odd and even set defined above with opposite phase; the Neumann conditions are achieved by running them in phase. The propagator K^- which satisfies the Dirichlet boundary conditions and the propagator K^+ which satisfies the Neumann ones are given by

$$K_{\text{wedge}}^{\mp} \equiv K_w^{\mp} = \frac{M}{2\pi i\hbar t} \exp\left[\frac{iM}{2\hbar t}(R^2 + R'^2)\right] \frac{\pi}{\theta}$$
$$\times \left\{ J_0\left(\frac{MR'R}{\hbar t}\right) + 2 \sum_{n=1}^{\infty} \exp\left(\frac{-in\pi^2}{2\theta}\right) J_{n\pi/\theta}\left(\frac{MR'R}{\hbar t}\right) \right.$$
$$\left. \times \cos\left(\frac{n\pi}{\theta}(\phi - \phi')\right) \right\} \mp \{\phi' \to -\phi'\}$$
$$= \frac{M}{2\pi i\hbar t} \exp\left[\frac{iM}{2\hbar t}(R^2 + R'^2)\right] \frac{\pi}{\theta}$$
$$\times \left\{ J_0\left(\frac{MR'R}{\hbar t}\right)(1 \mp 1) + 2 \sum_{n=1}^{\infty} \exp\left(\frac{-in\pi^2}{2\theta}\right) J_{n\pi/\theta}\left(\frac{MR'R}{\hbar t}\right) \right.$$
$$\left. \times \cos\left(\frac{n\pi}{\theta}(\phi - \phi')\right) \mp \cos\left(\frac{n\pi}{\theta}(\phi + \phi')\right) \right\}, \qquad \text{where } \theta = \frac{\pi\mu}{\nu} \tag{19}$$

The perfectly absorbing wedge (in the sense given for the absorbing knife edge) can be obtained by superposing these two solutions. The absorbing edge and wedge propagators are not solutions of boundary value problems of partial differential equations because they are the sum of terms (14) which satisfy differential equations for different values of α. This may explain Sommerfeld's scanty remarks on the absorbing wedge.

As remarked after Eqs. (5) and (6) for the wedge problem in electromagnetic diffraction, the series in (19) converge slowly for large arguments of the Bessel functions. These series are equal to finite sums which are preferable for numerical calculations.

$$\frac{1}{2} J_0(z) + \sum_{n=1}^{\infty} \exp\left(-in\frac{\pi\nu}{2\mu}\right) J_{n\nu/\mu}(z) \cos\left(n\frac{\nu}{\mu}\phi\right)$$
$$= \frac{1}{2\nu} \sum_{k=1}^{\nu} \left[\exp(-iz\cos(\phi + 2\pi k\mu/\nu)) \left(-i \sum_{n=1}^{\mu-1} \cos\left[\left(\frac{\mu - n}{\mu}\right)\left(\phi + 2\pi k\frac{\mu}{\nu}\right)\right] \right.\right.$$
$$\times \int_0^z \left(\exp\left(+in\frac{\pi}{2\mu}\right) J_{-n/\mu}(u) - \exp\left(-in\frac{\pi}{2\mu}\right) J_{n/\mu}(u) \right)$$
$$\left.\left. \times \exp(+iu\cos(\phi + 2\pi k\mu/\nu))\, du + 1 \right) \right] \tag{20}$$

This equation is derived in the Appendix. It generalizes for rational wedges Eq. (15) used for edges where $\nu = 1$ and $\mu = 2$. The right-hand side of (20) is, however, cumbersome analytically and we retain the left-hand side for analytic manipulations.

We recall the key formulas which relate time-dependent Schrödinger kernels K and monochromatic solutions G of the wave equation with a point source so that we can relate the formulas in this section to formulas in Sect. 2.

The time-dependent Schrödinger equation for the kernel K

$$i\hbar \frac{\partial}{\partial t} K - \frac{(i\hbar)^2}{2M} \Delta K = i\hbar\, \delta(x - x_0)\, \delta(t)$$

can be converted to the Helmholtz equation with a point source

$$-k^2 G - \Delta G = \delta(x - x_0)$$

using the Fourier transform pair

$$G = \frac{i\hbar}{2M} \int_{-\infty}^{+\infty} K \exp(i\omega t)\, d\tau$$

and

$$K = \frac{1}{2\pi} \frac{2M}{i\hbar} \int_{-\infty}^{+\infty} G \exp(-i\omega t)\, d\omega, \qquad \omega = \hbar k^2/2M$$

The time-independent massless kernel G_w may then be obtained by Fourier transform of the time dependent nonrelativistic kernel K_w

$$G = \frac{i\hbar}{2M} \int_{-\infty}^{+\infty} K \exp(i\omega t)\, dt = \frac{i\hbar}{2M} \int_{0}^{\infty} K \exp(i\omega t)\, dt$$

since $K = 0$ for $t < 0$. This transforms our nonrelativistic scalar kernel K_w on a μ-sheeted Riemannian surface into

$$G_w = \frac{1}{2\pi} \frac{1}{\mu} \sum_{n=-\infty}^{\infty} \exp\left(-\frac{in}{\mu} (\phi - \phi') \right) (-i)^{|n/\mu|} J_{|n/\mu|}(kR)\, K_{|n/\mu|}(-ikR') \qquad (21)$$

when $R < R'$; while for $R > R'$ interchange R and R'. The following identity has been used in computing (21):

$$\int_0^\infty \frac{1}{\chi} \exp(-\alpha\chi - \beta/\chi)\, J_\nu(\gamma/\chi)\, d\chi$$
$$= 2J_\nu\{(2\alpha[(\beta^2 + \gamma^2)^{1/2} - \beta])^{1/2}\}\, K_\nu\{(2\alpha[(\beta^2 + \gamma^2)^{1/2} + \beta])^{1/2}\}$$

where Re $\alpha > 0$; Re $\beta > 0$; $\gamma > 0$.

The conditions on α and β make the integrals finite and eliminate solutions with waves incoming from infinity. Equation (21) agrees with Sommerfeld's solution.

Images may then be introduced as before to implement boundary conditions. For $R<R'$, the expression is

$$G_w^{\mp}=\frac{1}{2\theta}\left((1\mp 1)\,J_0(kR)\,K_0(-ikR')+2\sum_{n=1}^{\infty}\exp\left(\frac{-in\pi^2}{2\theta}\right)\right.$$
$$\left.\times\, J_{n\pi/\theta}(kR)\,K_{n\pi/\theta}(-ikR')\left\{\cos\left(\frac{n\pi}{\theta}(\phi-\phi')\right)\mp\cos\left(\frac{n\pi}{\theta}(\phi+\phi')\right)\right\}\right) \tag{22}$$

while for $R>R'$ interchange R and R'. This result agrees with Carslaw's (1919).

As remarked after Eqs. (5), (6), and (19), the series in (22) converges slowly for large arguments of the Bessel functions. One can reexpress (22) as finite sums by taking the Fourier transforms of the finite-term solutions for $K_w^{\mp}$ which are obtained by replacing the series in (19) by the finite sums given in (20).

In conclusion, the path integral calculation of the free propagator on a μ-sheeted Riemannian surface (18) together with the method of images makes it possible to compute explicitly propagators with boundary conditions appropriate to the wedge problem (see also Goodman (1981)).

2. The knife edge problem in quantum mechanics was first solved (Schulman, 1982) using path integration and stationary phase methods. The result turned out to be an exact solution of the Schrödinger equation with (reflecting) knife edge boundary conditions. For this problem the method of stationary phase introduced in functional integration in 1951 (Morette, 1951) had to be generalized because the path which extremizes the action (i.e., the critical point of the action) is not inside the domain of integration but on its boundary.

(i) Stationary phase in functional integration is a natural generalization of stationary phase in ordinary integrals[17] which in its simplest form gives an asymptotic approximation for large λ of integrals of the following form:

$$F(\lambda)=\int_X h(x)\exp(i\lambda f(x))\,d\mu(x) \tag{23}$$

where h is a real-valued smooth function of *compact* support on the n-dimensional riemannian manifold X with volume element $d\mu(x)$, and where the critical points of f (i.e., the solutions of $f'(y)=0$, $y\in X$) are assumed to

[17] See, for instance, Bleistein and Handelsman (1975) or Choquet-Bruhat *et al.* (1982), pp. 593–595.

be nondegenerate,[18] that is, the determinant of the Hessian $\partial^2 f/\partial y^i\,\partial y^j$ does not vanish. In this case,

$F(\lambda)=O(\lambda^{-N})$ for any N if f has no critical point on the support of h

$F(\lambda)=O(\lambda^{-n/2})$ if f has a finite number of nondegenerate critical points on the support of h.

These results are modified when h does not vanish on the boundary ∂X of X. If there is no critical point of f in X, then the leading term is given by boundary terms which are obtained by integration by parts. For example, for $n=1$ and $X=[a, b]$ the first integration by parts gives

$$F(k)=\frac{1}{i\lambda}\frac{h(x)}{f'(x)}\exp(ikf(x))\bigg|_a^b-\frac{1}{i\lambda}\int_a^b\left(\frac{h(x)}{f'(x)}\right)'\exp(i\lambda f(x))\,dx \tag{24}$$

After N integrations by parts, the boundary terms consist of a polynomial in $(i\lambda)^{-1}$ of order N and the remaining integral is of order $(\lambda)^{-N}$.

For X an n-dimensional Riemannian manifold,[19] use the identity

$$\frac{1}{i\lambda}\,\boldsymbol{\omega}(x)[\exp(i\lambda f(x))]\equiv\exp(i\lambda f(x)) \tag{25}$$

where $\boldsymbol{\omega}(x)$ is the derivation at x defined by

$$\boldsymbol{\omega}(x):=\frac{\mathbf{v}(x)}{|\mathbf{v}(x)|^2},\qquad \mathbf{v}(x)=g^{ij}(x)\frac{\partial f}{\partial x^i}\frac{\partial}{\partial x^j}=v^j\frac{\partial}{\partial x^j} \tag{26}$$

$\boldsymbol{\omega}$ is defined for all x which are not critical points.

It follows that, if there is no critical point $y\in X$,

$$\begin{aligned}F(\lambda)&=\frac{1}{i\lambda}\int_X h(x)\,\boldsymbol{\omega}(x)[\exp(i\lambda f(x))]\,d\mu(x)\\&=\frac{1}{i\lambda}\int_{\partial X} h(x)\exp(i\lambda f(x))\,\boldsymbol{\omega}(x)\cdot d\boldsymbol{\sigma}(x)\\&\quad-\frac{1}{i\lambda}\int_X \operatorname{div}(h(x)\,\boldsymbol{\omega}(x))\exp(i\lambda f(x))\,d\mu(x)\end{aligned} \tag{27}$$

where $d\boldsymbol{\sigma}$ is the $(n-1)$-dimensional surface element on ∂X.

[18] Generalizations when the critical points are degenerate (DeWitt-Morette *et al.*, 1983; DeWitt-Morette and Nelson, 1984; DeWitt-Morette, 1984) give, in particular, rainbows and glories (Ford and Wheeler, 1959).

[19] To determine the dependence of F on λ, it is sufficient to consider the case $X\subset\mathbb{R}^n$ and set $h(x)=1$. In that case one can use the identity

$$\frac{1}{i\lambda}\nabla\cdot(\exp(i\lambda f)\,\mathbf{u})=\exp(i\lambda f)+\frac{1}{i\lambda}\exp(i\lambda f)\,\nabla\cdot\mathbf{u}$$

with $\mathbf{u}\equiv\nabla f/|\nabla f|^2$. Then

$$F(\lambda)=\frac{1}{i\lambda}\int_{\partial X} d\boldsymbol{\sigma}\cdot\mathbf{u}\exp(i\lambda f)-\frac{1}{i\lambda}\int_X d^n x\exp(i\lambda f)\,\nabla\cdot\mathbf{u}$$

The dependence on λ is the same as the one-dimensional integral (24), and repeated integration by parts will produce the same dependence on λ as before. The leading term for large λ is the boundary contribution of (27).

$$F(\lambda) \sim (i\lambda)^{-1} \int_{\partial X} h(x) \exp(i\lambda f(x))\, \boldsymbol{\omega}(x) \cdot d\boldsymbol{\sigma}(x) \tag{28}$$

Under appropriate regularity and compactness conditions on X, $f|_{\partial X}$ has at least one extremal value (i.e., it has at least one critical point), and the $n-1$ dimensional integral in (28) can be approximated by the stationary phase method. It contributes a term of order $\lambda^{-(n-1)}$ and therefore

$$F(\lambda) \simeq \lambda^{-1} \lambda^{-(1/2)(n-1)} \simeq \lambda^{-(1/2)(n+1)}$$

$f|_{\partial X}$ attains its extrema at points $y \in \partial X$ when

$$(\partial f|_X)(y)/\partial y^i = 0$$

i.e., when the components of grad f in $T_y X$ vanish. This means that grad f attains its extrema when grad $f(y)$ is normal to ∂X.

In conclusion, we have

(A) (nondegenerate) critical points[20] h of compact support $F(\lambda) = O(\lambda^{-n/2})$

(B) no critical points $h \neq 0$ on the boundary $F(\lambda) = O(\lambda^{-n/2 - 1/2})$.

This means that for a regular compact boundary ∂X, the integral "loses" a power of k when there is no critical point of f in X.

(ii) We shall now investigate the stationary phase approximation of a path integral when there is no critical point of the action S in the domain of integration. Consider the knife edge problem, $\theta = 2\pi$, in which an otherwise free particle cannot pass through a thin barrier along the positive x-axis (see Fig. 4). Consider an N-dimensional approximation to the path integral representation of the propagator of a particle of mass 1 in $\mathbb{R}^2$.

$$K^N(b, t; a) = \int_{\Omega^N} d\mathbf{x}_1 \cdots d\mathbf{x}_N \left(\sqrt{\frac{1}{2\pi i\hbar\varepsilon}}\right)^{2N} \exp\left(\frac{i}{\hbar} S^N(\mathbf{x}_1, \ldots, \mathbf{x}_N)\right) \tag{29}$$

where $\varepsilon = t/N$, $\mathbf{x}_0 = a$, $\mathbf{x}_{N+1} = b$, and

$$S^N = \sum_{j=0}^{N} |\mathbf{x}_{j+1} - \mathbf{x}_j|^2 / 2\varepsilon \tag{30}$$

[20] If, but only if, the n-dimensional integral is separable, can one speak of "critical points in a lower-dimensional subspace of X." Otherwise a critical point is defined by $\partial f(y)/\partial y^j = 0$ for all j. The statement $\partial f(y)/\partial y^j = 0$ for some j is not coordinate independent.

The domain of integration Ω^N consists of all sequences of points $\{\mathbf{x}_1, ..., \mathbf{x}_N\}$ such that the broken line from a to b defined by these points does not cross the barrier. Convergence of K^N to the propagator is guaranteed by the Trotter product formula (Schulman 1981). A critical point of S^N is a solution of the discretized Euler–Lagrange equation

$$\mathbf{x}_{j+1} - 2\mathbf{x}_j + \mathbf{x}_{j-1} = 0, \qquad \forall j$$

i.e., a sequence of points in $\mathbb{R}^2$ which are aligned. If a and b are such that $|\phi' - \phi| > \pi$, then the critical point of S^N is not in Ω^N, and we expect the leading term[21] of K^N for small $\hbar$ to come from the boundary $\partial\Omega^N$ of Ω^N; but $\partial\Omega^N$, or its limit $\partial\Omega$ when $N = \infty$, is a more complex concept than the boundary of X. A possible approach for investigating this issue is indicated briefly after Eq. (38).

(iii) Although the role of the boundary $\partial\Omega^N$ is not yet well understood in the stationary-phase approximation of path integrals whose phase has no critical points, the propagator $K(b, t; a)$ for the knife edge has been computed directly by stationary phase methods (Schulman, 1984) and the result is fortuitously exact (Schulman, 1982). Here is an outline of the calculation.

$$K(b, t; a) = \int d^2c\, K(b, t - t_c; c)\, K(c, t_c; a) \tag{31}$$

where the arbitrary intermediate time t_c is chosen to be the time when a free particle going from $(a, t = 0)$ to the origin O to (b, t) reaches O. When c is visible both from a and b and located as in Fig. 4, we have direct contributions

$$K_{\text{Direct}}(b, t - t_c; c) = (2\pi i\hbar(t - t_c))^{-1} \exp(i\, |b - c|^2/2\hbar(t - t_c)) \tag{32}$$

$$K_{\text{Direct}}(c, t_c; a) = (2\pi i\hbar t_c)^{-1} \exp(i\, |c - a|^2/2\hbar t_c) \tag{33}$$

and one reflected contribution

$$K_{\text{Reflected}}(c, t_c; a) = (2\pi i\hbar t_c)^{-1} \exp(i\, |c' - a|^2/2\hbar t_c) \tag{34}$$

[21] Note in (29) that the $\hbar$ dependence of K^N comes both from the normalizaing factors and from the phase, in contrast to the λ dependence of F in (23) which comes only from the phase. Counting powers of h is facilitated by writing path integrals not formally as $\int \mathscr{D}x \exp(iS/\hbar)$ but as prodistributions (DeWitt-Morette 1972, 1974; DeWitt-Morette *et al.*, 1979).

$$\int_\Omega dw(\omega) \exp\left(-\frac{i}{\hbar}\int V(\omega(s))\, ds\right)$$

where w is the complex gaussian whose covariance equals $(i\hbar/m)$ times the Wiener (1929) covariance. Alternatively one can count the "relevant powers of $\hbar$" by comparing a propagator with its free counterpart.

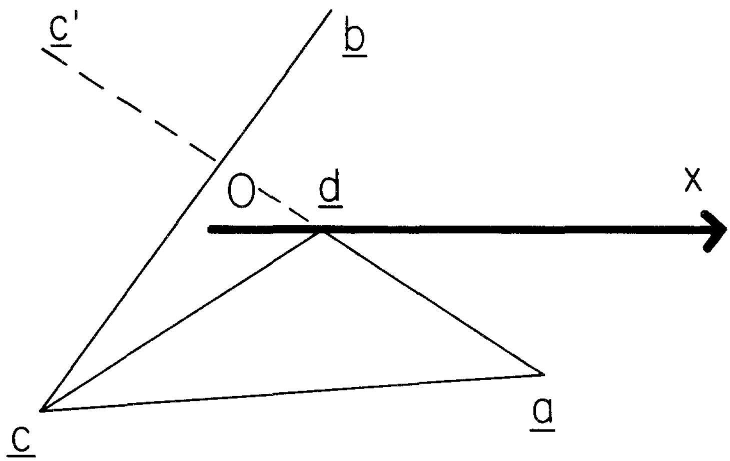

Fig. 4. For this geometry, the classical path contributions to the propagator are *acb* and *adcb*.

where c' is symmetric to c with respect to the barrier plane. It is understood that each term appears only in its classically allowed region. Inserting (32)–(34) into (31) gives

$$K^{\mp}(b, t; a) = K_{\rm DD}(b, t; a) \mp K_{\rm RD}(b, t; a)$$

where

$$K_{\rm DD}(b, t; a) = \int d^2c K_{\rm D}(b, t - t_c; c)\, K_{\rm D}(c, t_c; a)$$

$$K_{\rm RD}(b, t; a) = \int d^2c K_{\rm D}(b, t - t_c; c)\, K_{\rm R}(c, t_c; a)$$

$K^{-}(b, t; a)$ is the propagator for a reflecting knife edge with Dirichlet boundary condition, $K^{+}(b, t; a)$ for one with Neumann conditions. It is sufficient to compute $K_{\rm DD}$. After some algebra, we have

$$K_{\rm DD}(b, t; a) = K_0(2\pi i\hbar)^{-1}\gamma \int_0^\infty \rho\, d\rho \int_{\phi' - \pi}^{\phi + \pi} d\alpha \times \exp\left(\frac{i}{2\hbar}(\gamma\rho^2 - 2\rho v(\cos(\alpha - \phi') + \cos(\alpha - \phi)))\right) \quad (35)$$

where

$$\gamma = t_c^{-1} + (t - t_c)^{-1}$$

$$v = |\underline{a}0|/t_c = |0\underline{b}|/(t - t_c)$$

$$K_0 = (2\pi i\hbar t)^{-1} \exp\left(\frac{i}{2\hbar t}(|\underline{a}0|^2 + |0\underline{b}|^2)\right)$$

where $|\underline{a0}|$ and $|\underline{0b}|$ are the length of the line segments from a to O and b to O respectively in Fig. 4. The ρ and α coordinates of the critical point of the phase in the integrand of (35) satisfy

$$\gamma\rho - v(\cos(\alpha - \phi') + \cos(\alpha - \phi)) = 0 \tag{36}$$

$$\sin(\alpha - \phi') + \sin(\alpha - \phi) = 0 \tag{37}$$

The relevant solution of (37) is $\alpha = \frac{1}{2}(\phi + \phi')$ so that (36) yields

$$\rho = \frac{2v}{\gamma} \cos(\tfrac{1}{2}(\phi' - \phi))$$

ρ being positive, the coupled equations (36) and (37) have a solution if $\phi' - \phi < \pi$, i.e., if there is a straight line from a to b in the space Ω of allowed paths. If there is, the stationary phase approximation of (35) is the free propagator K_0 in $\mathbb{R}^2$ proportional to $\hbar^{-1}$. If not, the stationary phase approximation of (35) is dominated by the boundary terms, and, after some calculations (Schulman, 1984) one obtains

$$K^{\mp}(b, t; a) = K_0[\exp(-im^2)\, h(-m) \mp \exp(-in^2)\, h(-n)] \tag{38}$$

where

$$\begin{aligned} m &= (2/\hbar\gamma)^{1/2} \sin \omega_2, \qquad n = (2/\hbar\gamma)\, v \sin \omega_1 \\ \omega_2 &= \tfrac{1}{2}(\phi' - \phi) - \frac{\pi}{2}, \qquad \omega_1 = \tfrac{1}{2}(\phi' + \phi) \\ h(m) &= (\pi)^{-1/2} \exp(-i\pi/4) \int_{-\infty}^{m} \exp(it^2)\, dt \end{aligned} \tag{39}$$

Note that for $m > 0$, $h(-m)$ is of order unity, while for $m < 0$ the stationary point in the integral defining h is outside the range of integration. Thus for $m < 0$, $h = O(1/m) = O(\hbar^{1/2})$.

The path integral calculation of the knife edge problem reduces to a two-dimensional integral (31) because the path integral is a cylindrical one. The relationship between a cylindrical path integral and a finite-dimensional integral (DeWitt-Morette, 1976; DeWitt-Morette *et al.* 1979) suggests a method for investigating the boundary $\partial\Omega$ of the domain of integration and its contribution to the path integral. But the work remains to be done. The propagators $K^{\mp}$ are identical with the ones obtained by Carslaw in 1899. The wedge problem, a great problem of the nineteenth century, continues to serve as a source of inspiration today—and Feynman's path integrals developed when he was a graduate student with John A. Wheeler brings new insight to this classic problem. It shifts the focus from partial differential equations with boundary values to integrals over restricted spaces of paths. It brings stochastic calculus (averages over continuous paths) and differential calculus together. A path integral solves a partial differential equation with given boundary conditions. But it does more:

(i) It incorporates the boundary conditions in the definition of its domain of integration. In other words, boundary conditions are not, as in differential calculus, an *additional* requirement satisfied by a solution of a partial differential equation whose compatibility has to be checked.

(ii) It often can answer global questions more readily than a differential equation because the domain of integration consists of paths which take their values in the configuration space of the system and probes its global properties. A partial differential equation states only local relationships between a function and its derivatives.

5. STOCHASTIC DIFFUSIONS

Although solutions of the Schrödinger equation can be obtained by analytic continuation of solutions of the heat equation, their asymptotics are quite different. The reason that analytic continuation of the full solutions does not imply analytic continuation of their asymptotics for small diffusion coefficient D or small values of $\hbar$ is that these equations are singular in the limits $D=0$ and $\hbar=0$ respectively. This issue requires further study, but two simple remarks can readily be made.

(i) In a Schrödinger JWKB propagator the paths ξ_0 which minimize tha action $S(\xi_0)$ play no more an important role in asymptotic estimates than other extrema, whereas the minima and maxima of the "action" in a diffusion equation play different roles.

(ii) In a Schrödinger JWKB propagator the phase factor $\exp(iS(\xi_0)/\hbar)$ plays no role in asymptotic estimate probabilities; the relevant term is the Van Vleck determinant and the caustics, if any, dominate the answer. In the asymptotics of a diffusion equation the exponential term $\exp(-D^{-1}S(\xi_0))$ dominates the answer. A very instructive example has been analyzed by Michael Berry.[22] It arises from contemplating the question of how male moths use their sense of smell to find their way to females. Female moths emit pheromones which propagate by diffusion aided by convection. The concentration $C(\mathbf{r}, t)$ of particles with diffusion constant D in a wind of velocity $\mathbf{V}(\mathbf{r}, t)$ satisfies the equation

$$\frac{\partial C(\mathbf{r}, t)}{\partial t} = D\Delta C(\mathbf{r}, t) - \mathbf{V}(\mathbf{r}, t) \cdot C(\mathbf{r}, t) \tag{40}$$

[22] Berry (to appear). An example which deserves more than the brief summary given here. These paragraphs owe much to remarks made by Michael Berry during his visit to Austin.

In the limit $D=0$, the concentration $C(\mathbf{r}, t)$ vanishes everywhere except along the paths where the action $S(\xi_0)$ vanishes. The quantum mechanical counterpart of this equation is

$$i\hbar \frac{\partial \psi}{\partial t} = H(\mathbf{r}, -i\hbar\nabla)\, \psi \tag{41}$$

where the classical Hamiltonian $H = p^2 + \mathbf{V} \cdot \mathbf{p}$. The velocity field $\mathbf{V}$ is assumed to represent an incompressible flow, $\nabla \cdot \mathbf{V} = 0$, and there is no need to symmetrize the $\mathbf{V} \cdot \mathbf{V}$ term in the quantum Hamiltonian. Equation (41) is the equation for a particle in a magnetic field $\mathbf{B} = -\frac{1}{2}\nabla \times \mathbf{V}$ and an electric field $\mathbf{E} = \frac{1}{2}(\mathbf{V} \cdot \nabla\mathbf{V} + \mathbf{V} \times (\nabla + \mathbf{V}) + \partial\mathbf{V}/\partial t)$. Its short-wave limit is dominated by caustics of the trajectories generated by the classical hamiltonian.[23]

It should also be noted that for the Schrödinger equation semiclassical asymptotics are not necessarily identical to short-time asymptotics. For instance the JWKB propagator for a particle in a Riemannian manifold is not equal to the short-time propagator.[24] A more intuitive example is the propagator between two nearby points which can be connected by multiple geodesics. All geodesics contribute to the JWKB propagator, but only the shortest one contributes to the short-time propagator.

In spite of the differences between the Schrödinger and the heat equations, techniques and insight developed in stochastic calculus can sometimes be transferred[25] to Feynman path integrals, and we shall indicate some results in the theory of stochastic diffusion of interest for the wedge problem: namely Molchanov's results, the stochastic definitions of regular and irregular boundary points, and a Feynman–Kac formula for solutions of Dirichlet problems. Applications of these results to wedge problems are under consideration.

1. The wedge problem in diffusion was first studied by Molchanov (1975) who investigated, in particular, diffusion on a riemannian manifold M which is a nonconvex polygon (see Fig. 5).[26] He computed the new effects which occur when the geodesic γ from the source at a to the detector at b has points in common with the boundary ∂M. He worked out in detail

[23] Berry (1986).

[24] DeWitt-Morette *et al.* (1980), Eqs. (11) and (8) slightly modified to yield the position-to-position transition.

[25] DeWitt-Morette *et al.* (1979).

[26] We refer the reader to Ikeda (1982, 1985) for diffusion in the presence of smooth obstructions—as opposed to polygonal obstructions. One could argue that a change of scale makes a rounded wedge look like a sharp wedge. This is indeed correct when dealing with exact solutions, but not so when dealing with asymptotics: one cannot always interchange the semiclassical limit and the change of scale which makes a round wedge sharp.

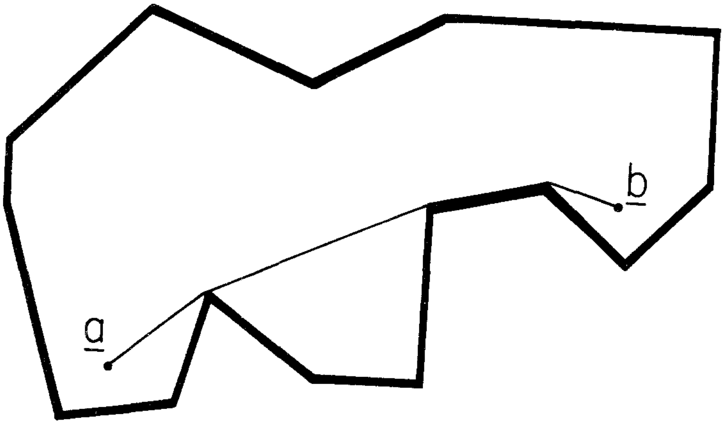

Fig. 5. A geodesic γ from a to b in a nonconvex polygonal domain M.

diffusion on two-dimensional spaces, but his method should carry over to higher-dimensional spaces.

It is possible to recast the wedge problem as a problem of diffusion in a nongeodesically convex Riemannian manifold M with boundary ∂M because the boundary has an effect[27] on the diffusion *only* at the points of contact of the geodesic γ with the boundary ∂M.

Kac called this situation the "principle of imperceptibility" of the boundary: Only if the geodesic *touches* the boundary do new effects appear; proximity is not enough. These new effects are the diffraction effects. That they are determined entirely by the points of contact justifies Keller's geometric theory of diffraction.

By computing upper and lower bounds of probabilities, Molchanov established the following results for M two dimensional.

(i) If the polygonal geodesic γ has ρ sides and τ points in common with the boundary and if γ has a break at the end points of each arc and at each vertex, then the transition density $p(t, a, b)$ for diffusion from a to b in time t is

$$p(t, a, b) = O(t^{k-1} \exp(-L(\gamma)^2/2t)) \tag{42}$$

where $k = 2\rho + \frac{1}{2}\tau$ and $L(\gamma)$ is the length of $\gamma \in \mathscr{D}_{\rho,0,\tau}(a, b)$.

Note that k does not depend on the length of the sides ρ common to the geodesic and the boundary but only on their number. Note also that four vertices contribute the same amount to k, as one arc, however long. Molchanov calls the dependence of k on ρ and τ the "inertia" of Brownian

[27] See Molchanov (1975), Theorem 4.1, p. 24 for a precise statement.

motion: it is "harder" for a Brownian particle to go around $\tau > 4$ isolated vertices than to creep along one entire side of ∂M.

(ii) The angle of the boundary at a vertex does not affect the leading term of the transition probability expanded in powers of t, provided the part of ∂M lying in some ε-neighborhood of the vertex can be approximated (in the sense of inclusion) by two angles (see Fig. 6).

2. Lebesgue (1924) discovered that the classical Dirichlet problem is not well posed in $d > 2$ dimensions if the boundary does not satisfy some regularity condition. Indeed consider a $d \geqslant 3$ dimensional ball and push a sharp thorn (called noadays a "Lebesgue thorn") into its side, place a heater inside the ball at the tip O of the thorn, and maintain the walls at zero temperature away from O (see Fig. 7).

As time goes to infinite, the temperature u inside the ball should solve Dirichlet's problem

$$\Delta u(a) = 0 \qquad \text{for} \quad a \in M$$

$$\lim_{a=b} u(a) = f(b) \qquad \text{for} \quad b \in \partial M, \text{ here } f(0) = 1 \qquad \text{and} \qquad f(b \neq 0) = 0$$

But it can be shown that, if the thorn is sharp enough,

$$\lim_{a=0} u(a) < 1, \text{ hence } \lim_{a=0} u(a) \neq f(0)$$

A person inside the ball will be cold, no matter how close to the heater.

The definition of a regular boundary point in differential calculus is very complex,[28] but has a simple, equivalent, expression in stochastic calculus,[29] in terms of the first time a Brownian particle on the boundary

[28] Lebesgue (1924).

[29] See for instance Durett (1984), pp. 248–251, for an introduction to regular and irregular boundaries and Itô and McKean (1985), pp. 261–264, for the Dirichlet problem in stochastic calculus, and p. 259 for Wiener's necessary and sufficient condition for a boundary point to be regular. Other useful references are Barber and Ninham (1970); Gikhman and Skorokhod (1969); and Port and Stone (1978).

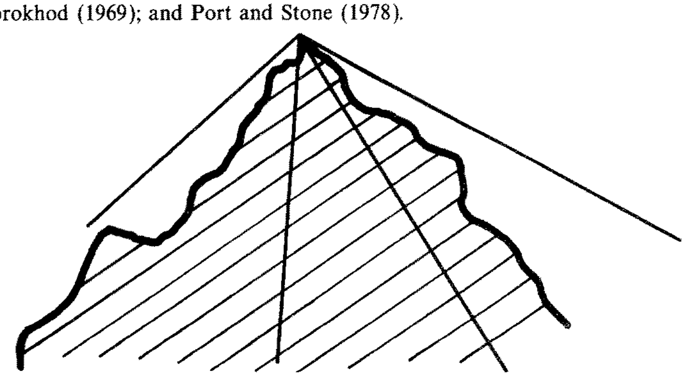

Fig. 6. An irregular wedge whose tip fits between two angles.

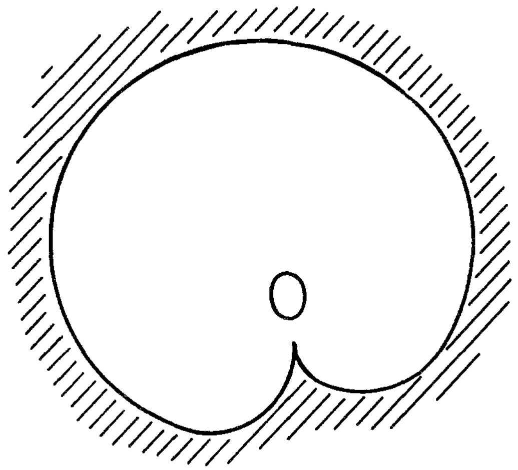

Fig. 7. A Lebesgue thorn.

∂M at time 0 leaves the domain M. The first exit time of a Brownian path $B(t, \omega)$, where ω is a point in a probability space Ω which labels the path, is naturally

$$\theta(\omega) = \inf\{t; B(t, \omega) \notin M\} \tag{43}$$

A point $b \in \partial M$ is said to be regular if the first exit time of a Brownian particle at b at time 0 vanishes with probability one, i.e., if

$$P_b(\theta = 0) := \int_{\chi_{\theta = 0}} dw(\omega) = 1 \tag{44}$$

where

$$\chi_{\theta = 0} = \{\omega; \theta(\omega) = 0\}$$

and w is the Wiener measure in the probability space Ω.

We shall construct a Lebesgue's thorn of variable "sharpness" so that we can state when it is "sharp enough" for its tip to be an irregular point. Consider a d ($d \geqslant 3$) dimensional block of side $[-1, 1]$ with a thorn removed (see Fig. 8).

$$\begin{aligned} M &:= [-1, 1]^d - \text{thorn} \\ \text{thorn} &:= \bigcup_{n=0}^{\infty} n\text{-block} \\ n\text{-block} &:= [-2^{-n}, -2^{-n-1}] \times [-a_n, a_n]^{d-1} \end{aligned}$$

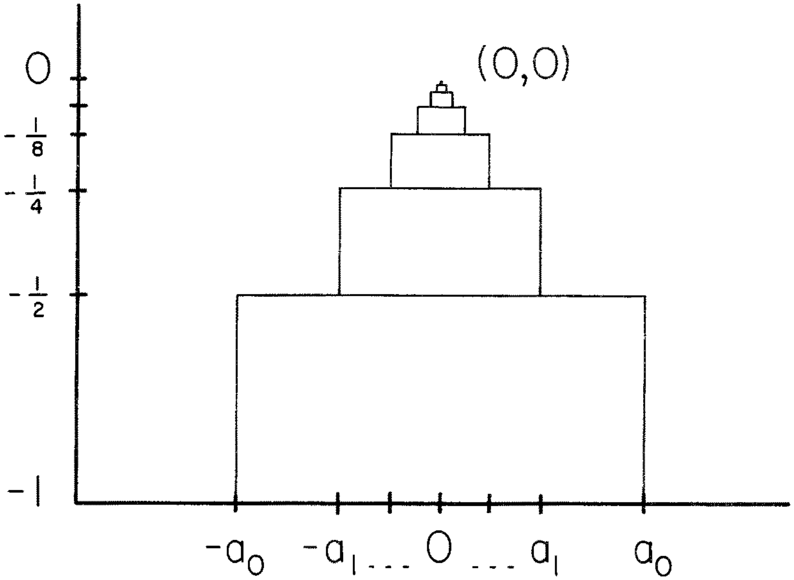

Fig. 8. Projection of a Lebesgue's thorn on two dimensions.

The origin is an irregular point $P_0(\theta > 0) > \frac{1}{2}$ if the sequence $\{a_n\}$ tends to zero sufficiently rapidly. Poincaré's (sufficient) "cone condition" says that 0 is regular if

$$\lim_{n=\infty} \inf 2^n a_n > 0$$

Wiener's necessary and sufficient condition says that 0 is irregular

$$\text{in } d=3, \qquad \text{if and only if } -\infty \leqslant \sum_{n=1} \log(2^n a_n) < \infty$$

$$\text{in } d \geqslant 4, \qquad \text{if and only if } \sum_{n=1}^{\infty} (2^n a_n)^{d-3} < \infty$$

3. Continuous solutions of the Dirichlet problem in a domain M with regular boundary ∂M

$$\begin{cases} \dfrac{1}{2} \Delta u + Vu = 0 \\ \qquad u|_{\partial M} = \phi \end{cases} \tag{45}$$

can be expressed, under fairly general assumptions for ϕ, as a path integral,[29] namely

$$u(x) = \int_\Omega dw(\omega) \exp\left(\int_0^{\theta(\omega)} V(B(t, \omega))\, dt\right) \phi(B(\theta, \omega))$$

$$\equiv \mathbb{E}_x \left(\exp\left(\int_0^\theta V(B(t))\, dt\right) \phi(B(\theta))\right) \tag{46}$$

where w is the Wiener measure on Ω, $B(\cdot, \omega)$ is a brownian path originating at x at time $t = 0$, θ is the first exit time (42), and $\mathbb{E}_x$ is defined by the path integral over Ω.

The path integral solution of the Dirichlet problem bears some resemblance to the Feynman–Kac solution of the diffusion equation

$$\begin{cases} \dfrac{\partial K}{\partial t} = \dfrac{1}{2} \Delta K + VK \\ K(x, 0) = \phi(x) \end{cases}$$

which, written with the above notation, is

$$K(x, t) = \mathbb{E}_x \left(\exp\left(\int_0^t \left(V(B(s))\, ds\right) \phi(B(t))\right)\right) \tag{47}$$

The important difference between (46) and (47) is that the limit of integration θ is path dependent; computing explicitly (46) is thus much more difficult than computing (47). At present (46) has been used primarily to obtain bounds. Recalling that time-dependent solutions for monoenergetic waves are readily obtained from time-independent Helmholtz equations, Eq. (46) offers an interesting approach to investigating diffusion with boundary.

6. CONCLUSION

The propagators and the Green functions presented in this paper are the knife edge propagator for Dirichlet and Neumann boundary conditions, (4), (38), and (17), the absorbing knife edge propagator (16), the

[30] See, for instance, Friedman (1976) Vol I, p. 145, for analytical details. Note also the stochastic solution of Poisson's equation:

$$u(x) = \mathbb{E}_x \left(\phi(B\theta) + \int_0^\theta c(B(s))\, ds\right) \text{ satisfies } 1/2\Delta u + c = 0,\ u|_{\partial M} = \phi.$$

wedge propagators (19). The propagator for a free particle on a μ-sheeted Riemann surface (18), and the Dirichlet and the Neumann wedge Green function (5), (6), and (22).

Each section presents the wedge problem in a particular context. In each case, identifying the relevant mathematical problem is the key to the solution; we supplement Sect. 1 with mentions of the corresponding mathematical problems.

2. Diffraction by reflecting wedge. Boundary value problem of differential equation. Solutions defined on multiply connected spaces.

3. Geometrical theory of diffraction. Identification of function spaces.

4. Path integral solutions. Path integration on multiply connected spaces. Asymptotics on the boundaries of function spaces.

5. Probing the shape of the wedge and the roughness of its surface. Stochastic calculus.

Problems 2 and 3 are by now well understood; progress is expected in problems 4 and 5. We would be delighted to write *Wedges II* for the next Wheeler *Festschrift*.

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APPENDIX TO SECTION 4.1

1. Free propagator on a μ-sheeted Riemann surface constructed as a linear combination of Aharonov–Bohm propagators for different fluxes (generalization of 4.4).

As discussed previously, our use of images leads to the consideration of a μ-sheeted Riemannian surface. Only solenoids whose strength (α) are a multiple of $1/\mu$ exist upon such a surface. Combine these such that the source lies only within the physical sheet by requiring that

$$\lim_{t \to 0^+} K = \delta(R - R') \sum_{n=-\infty}^{\infty} \delta((\phi - \phi') - 2\pi\mu n) \tag{A.1}$$

Starting from the arbitrary linear combination

$$K=\frac{M}{2\pi i\hbar t}\exp\left[\frac{iM}{2\hbar t}(R^2+R'^2)\right]\sum_{n=-\infty}^{\infty}a_n\exp\left[-\frac{in}{\mu}(\phi-\phi')\right]$$
$$\times(-i)^{|n/\mu|}J_{|n/\mu|}\left(\frac{MR'R}{\hbar t}\right)$$
$$=\frac{M}{2\pi i\hbar t}\exp\left[\frac{iM}{2\hbar t}(R^2+R'^2)\right]\sum_{n=-\infty}^{\infty}a_n\exp\left[-\frac{in}{\mu}(\phi-\phi')\right]$$
$$\times I_{|n/\mu|}\left(\frac{-iMR'R}{\hbar t}\right) \tag{A.2}$$

and noting that

$$I_\nu(z)_{|z|\to\infty}\to\frac{\exp(z)}{\sqrt{2\pi z}}\,|\arg z|<\frac{\pi}{2}$$

(regularization being implicitly used throughout), this may be cast in the form

$$\lim_{t\to 0^+}K=\lim_{t\to 0^+}\frac{M}{2\pi i\hbar t}\exp\left[\frac{iM}{2\hbar t}(R^2+R'^2)\right]$$
$$\times\sqrt{\frac{i\hbar t}{2\pi R'RM}}\exp\left[-\frac{iR'RM}{\hbar t}\right]\sum_{n=-\infty}^{\infty}a_n\exp\left[-\frac{in}{\mu}(\phi-\phi')\right]$$

The angular factor is required to be a series of delta functions. The a_n's are in fact nothing but the Fourier coefficients. Fourier analysis yields the Poisson sum

$$\sum_{m=-\infty}^{\infty}\exp(im\theta)=2\pi\sum_{n=-\infty}^{\infty}\delta(\theta+2\pi n)$$

from which one can deduce that the a_n's are equal and so can be denoted simply as a, leading to

$$\lim_{t\to 0^+}K=\lim_{t\to 0^+}a\mu\sqrt{\frac{M}{2\pi i\hbar R'Rt}}\exp\left[\frac{iM}{2\hbar t}(R-R')^2\right]\sum_{n=-\infty}^{\infty}(\delta(\phi-\phi')-2\pi\mu n)$$

An inspection of

$$\lim_{t\to 0^+}\int_0^\infty\sqrt{\frac{M}{2\pi i\hbar R'Rt}}\exp\left[\frac{iM}{2\hbar t}(R-R')^2\right]R\,dR \tag{A.3}$$

reveals that the integrand contributes only in the neighborhood of $R = R'$. Therefore, (A.3) reduces to

$$\sqrt{\frac{M}{2\pi i\hbar t}} \int_{-\infty}^{\infty} \exp\left[\frac{iM}{2\hbar t}(R-R')^2\right] dR = 1$$

and so behaves as $\delta(R-R')$. Hence

$$\lim_{t\to 0^+} K = a\mu\,\delta(R-R') \sum_{n=-\infty}^{\infty} \delta((\phi-\phi')-2\pi\mu n)$$

and so $a = 1/\mu$ by using a comparison with A.1. This yields the complete kernel

$$K = \frac{1}{\mu}\frac{M}{2\pi i\hbar t}\exp\left[\frac{iM}{2\hbar t}(R^2+R'^2)\right]$$
$$\times \sum_{n=-\infty}^{\infty} \exp\left[-i\frac{n}{\mu}(\phi-\phi')\right](-i)^{|n/\mu|} J_{|n/\mu|}\left(\frac{MR'R}{\hbar t}\right)$$

from which we proceed as previously.

2. Proof of Eq. (20) [generalization of (15)]
To evaluate a finite sum form for

$$S(z,\phi) = \tfrac{1}{2}J_0(z) + \sum_{n=1}^{\infty} \exp\left[-i\frac{n}{2}\frac{\pi}{\mu}\right] J_{n/\mu}(z)\cos\left(\frac{n}{\mu}\phi\right)$$

where μ is a positive integer.

The aproach taken is to develop a partial differential equation (w.r.t. z) satisfied by S, which is then integrated to yield an integral form for S.

Look at

$$S_\phi := \tfrac{1}{2}J_0 + \sum_{n=1}^{\infty} \exp\left[i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[-i\frac{n}{\mu}\phi\right], \qquad S = \tfrac{1}{2}(S_\phi + S_{-\phi})$$

$$\frac{\partial S_\phi}{\partial z} = -\tfrac{1}{2}J_1 + \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] \tfrac{1}{2}(J_{(n/\mu)-1} - J_{(n/\mu)+1}) \exp\left[i\frac{n}{\mu}\phi\right]$$

having used

$$J'_\nu = \tfrac{1}{2}(J_{\nu-1} - J_{\nu+1}), \qquad \nu \neq 0, \qquad J'_0 = -J_1$$

So,

$$\frac{\partial S_\phi}{\partial z} = -\tfrac{1}{2}J_1 + \tfrac{1}{2}\left\{\sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{(n/\mu)-1} \exp\left[i\frac{n}{\mu}\phi\right]\right.$$
$$\left. - \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{(n/\mu)+1} \exp\left[i\frac{n}{\mu}\phi\right]\right\}$$

Let $n \to n+\mu$ in the first summation and $n \to n-\mu$ in the second:

$$\frac{\partial S_\phi}{\partial z} = -\tfrac{1}{2}J_1 + \tfrac{1}{2}\left\{ -i\exp[i\phi] \sum_{n=1-\mu}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right] \right.$$
$$\left. - i\exp[-i\phi] \sum_{n=1+\mu}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right] \right\}$$

Complete the summations from 1 to ∞:

$$\frac{\partial S_\phi}{\partial z} = -\tfrac{1}{2}J_1 - \frac{i}{2}\exp[i\phi] \sum_{n=1-\mu}^{0} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right]$$
$$+ \frac{i}{2}\exp[-i\phi] \sum_{n=1}^{\mu} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right]$$
$$- i\cos(\phi) \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right]$$

$$\frac{\partial S_\phi}{\partial z} = -\tfrac{1}{2}J_1 - \frac{i}{2}\exp[i\phi] \left\{ J_0 + \sum_{n=1-\mu}^{-1} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right] \right\}$$
$$+ \frac{i}{2}\exp[-i\phi] \left\{ -iJ_1\exp[i\phi] + \sum_{n=1}^{\mu-1} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right] \right\}$$
$$- i\cos(\phi) \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right]$$

Let $n \to -n$ in the first summation:

$$\frac{\partial S_\phi}{\partial z} = -\frac{i}{2}\exp[i\phi]\, J_0 - \frac{i}{2} \sum_{n=1}^{\mu-1} \exp\left[+i\frac{n\pi}{2\mu}\right] J_{-n/\mu} \exp\left[i\frac{\mu-n}{\mu}\phi\right]$$
$$+ \frac{i}{2} \sum_{n=1}^{\mu-1} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[-i\frac{\mu-n}{\mu}\phi\right]$$
$$- i\cos(\phi) \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \exp\left[i\frac{n}{\mu}\phi\right]$$

Recall that

$$\frac{\partial S}{\partial z} = \frac{1}{2}\frac{\partial}{\partial z}(S_\phi + S_{-\phi})$$
$$= -\frac{i}{2}\cos(\phi)J_0 - \frac{i}{2} \sum_{n=1}^{\mu-1} \left\{ \exp\left[+i\frac{n\pi}{2\mu}\right] J_{-n/\mu} - \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \right\} \cos\left(\frac{\mu-n}{\mu}\phi\right)$$
$$- i\cos(\phi) \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \cos\left(\frac{n}{\mu}\phi\right)$$

This leads to the finite-term partial differential equation for S

$$\frac{\partial S}{\partial z}(z, \phi) = -\frac{i}{2}\sum_{n=1}^{\mu-1}\left\{\exp\left[+i\frac{n\pi}{2\mu}\right]J_{-n/\mu}(z) - \exp\left[-i\frac{n\pi}{2\mu}\right]J_{n/\mu}(z)\right\}$$
$$\times \cos\left(\frac{\mu-n}{\mu}\phi\right) - i\cos(\phi)\, S(z, \phi)$$

which is of the form

$$\frac{\partial S}{\partial z}(z, \phi) = p(z, \phi)\, S(z, \phi) + q(z, \phi)$$

and has the solution

$$S(z, \phi) = \exp\left[\int p(z, \phi)\, dz\right]\left\{\int^{z} q(u, \phi)\exp\left[-\int^{u} p(v, \phi)\, dv\right]du + f(\phi)\right\}$$

where f is an arbitrary function of ϕ. Hence,

$$S(z, \phi) = \exp[-iz\cos(\phi)]\left\{-\frac{i}{2}\sum_{n=1}^{\mu-1}\cos\left(\frac{\mu-n}{\mu}\phi\right)\right.$$
$$\times \int_0^z\left(\exp\left[+i\frac{n\pi}{2\mu}\right]J_{-n/\mu}(u) - \exp\left[-i\frac{n\pi}{2\mu}\right]J_{n/\mu}(u)\right)$$
$$\left.\times \exp[+iu\cos(\phi)]\, du + f(\phi)\right\}$$

So,

$$\tfrac{1}{2}J_0(z) + \sum_{n=1}^{\infty}\exp\left[-i\frac{n\pi}{2\mu}\right]J_{n/\mu}(z)\cos\left(\frac{n}{\mu}\phi\right)$$
$$= \exp[-iz\cos(\phi)]\left\{-\frac{i}{2}\sum_{n=1}^{\mu-1}\cos\left(\frac{\mu-n}{\mu}\phi\right)\int_0^z\left(\exp\left[+i\frac{n\pi}{2\mu}\right]J_{-n/\mu}(u)\right.\right.$$
$$\left.\left.-\exp\left[-i\frac{n\pi}{2\mu}\right]J_{n/\mu}(\mu)\right)\exp[+iu\cos(\phi)]\, du + f(\phi)\right\} iME$$

Put $z = 0$ to find $f(\phi)$. This gives $f(\phi) = 1/2$; having used the fact that, we have

$$J_\nu(0) = 0, \qquad \nu > 0, \qquad J_0(0) = 1$$

Finally, the identity is obtained as

$$\tfrac{1}{2}J_0(z) + \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu}(z) \cos\left(\frac{n}{\mu}\phi\right)$$

$$= \exp[-iz\cos(\phi)] \left\{ -\frac{i}{2} \sum_{n=1}^{\mu-1} \cos\left(\frac{\mu-n}{\mu}\phi\right) \int_0^z \left(\exp\left[+i\frac{n\pi}{2\mu}\right] J_{-n/\mu}(u) \right.\right.$$

$$\left.\left. -\exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu}(u) \right) \exp[+iu\cos(\phi)]\, du + \frac{1}{2} \right\}$$

This reduces to the known result for $\mu = 2$ [Gradshteyn and Ryzhik (1980), p. 973, 8.511(5)], namely,

$$\tfrac{1}{2}J_0(z) + \sum_{n=1}^{\infty} \exp\left[-in\frac{\pi}{4}\right] J_{n/2}(z) \cos\left(n\frac{\phi}{2}\right) = (i/\pi)^{-1/2} \exp[iz\cos(\phi)]$$

$$\times \int_{-\infty}^{\sqrt{(2z)}\cos(\phi/2)} \exp(+it^2)\, dt$$

on using

$$J_{-1/2}(z) = \sqrt{(2/\pi z)}\cos(z), \qquad J_{1/2}(z) = \sqrt{(2/\pi z)}\sin(z)$$

the equality

$$\frac{1}{2} = (i/\pi)^{-1/2} \int_{-\infty}^{0} \exp(-it^2)\, dt$$

and the variable change

$$t = \sqrt{(2u)}\cos(\phi/2)$$

One may further generalize this identity by adding together this expression for angles given by

$$\phi + 2\pi k\frac{\mu}{\nu}, \qquad k = 1,\ldots, \nu \text{ where } \nu \text{ is a positive integer}$$

The left-hand side of the identity developed becomes

$$\frac{1}{2}\sum_{k=1}^{\nu} \left\{ J_0 + \sum_{n=1}^{\infty} \exp\left[-i\frac{n\pi}{2\mu}\right] J_{n/\mu} \left(\exp\left[-i\frac{n}{\mu}\left(\phi + 2\pi k\frac{\mu}{\nu}\right)\right] \right.\right.$$

$$\left.\left. + \exp\left[+i\frac{n}{\mu}\left(\phi + 2\pi k\frac{\mu}{\nu}\right)\right] \right) \right\}$$

But

$$\sum_{k=1}^{\nu} \exp\left[-i2\pi k\frac{n}{\nu}\right] \quad \begin{cases} = \nu & \text{if } n \text{ is a multiple of } \nu \\ = 0 & \text{if not} \end{cases}$$

Which is most easily seen by noticing that this is the sum of evenly spaced unit vectors in the complex plane.

So the left-hand side becomes

$$\nu\left\{\tfrac{1}{2}J_0+\sum_{n=1}^{\infty}\exp\left[-in\frac{\pi\nu}{2\mu}\right]J_{n\nu/\mu}\cos\left(n\frac{\nu}{\mu}\phi\right)\right\}$$

and one obtains the generalization of the first identity developed:

$$\tfrac{1}{2}J_0(z)+\sum_{n=1}^{\infty}\exp\left[-in\frac{\pi\nu}{2\mu}\right]J_{n\nu/\mu}(z)\cos\left(n\frac{\nu}{\mu}\phi\right)$$
$$=\frac{1}{2\nu}\sum_{k=1}^{\nu}\left\{\exp\left[-iz\cos\left(\phi+2\pi k\frac{\mu}{\nu}\right)\right]\left\{-i\sum_{n=1}^{\mu-1}\cos\left[\left(\frac{\mu-n}{\mu}\right)\left(\phi+2\pi k\frac{\mu}{\nu}\right)\right]\right.\right.$$
$$\times\int_0^z\left(\exp\left[+i\frac{n\pi}{2\mu}\right]J_{-n/\mu}(u)-\exp\left[-i\frac{n\pi}{2\mu}\right]J_{n/\mu}(u)\right)$$
$$\left.\left.\times\exp\left[+iu\cos\left(\phi+2\pi k\frac{\mu}{\nu}\right)\right]du+1\right\}\right\}$$

These results are useful in that they convert an infinite series which suffers from slow convergence (for large Bessel function argument) into an expression with a finite number of terms (for μ, ν finite).

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