Note on the quantum recurrence theorem

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An essential step in the proof of the quantum recurrence theorem is shown to follow from the Poincaré recurrence theorem of classical mechanics.

The quantum recurrence theorem[1] is easy to state and prove. The only nontrivial step in the proof involves a theorem from the theory of almost-periodic functions. The purpose of this Note is to observe that this step can be replaced by a reference to the Poincaré recurrence theorem of *classical mechanics*.

Quantum recurrence theorem. Let $\psi(t)$ be a wave function evolving in time under the Hamiltonian H which has only discrete eigenvalues E_n, $n=1,2,\ldots$. Then for each ϵ, there is a $T>1$ such that

$$\| \psi(T)-\psi(0)\| < \epsilon . \tag{1}$$

Proof. $\psi(t)$ can be expressed as

$$\psi(t)=\sum_n r_n \exp(-iE_n t)u_n \tag{2}$$

with $u_n, n=1,\ldots$ eigenstates of H having phases chosen so that the coefficients r_n are real and positive. The numbers r_n satisfy

$$\sum_{n=1}^{\infty} r_n^2 = 1 . \tag{3}$$

Consider

$$\| \psi(t)-\psi(0)\|^2 = \sum_{n=1}^{\infty} r_n^2 2(1-\cos E_n t) . \tag{4}$$

By Eq. (3) there is an N such that

$$2\sum_{n=N+1}^{\infty} r_n^2(1-\cos E_n t) \leq 4 \sum_{n=N+1}^{\infty} r_n^2 < \frac{\epsilon}{3} . \tag{5}$$

It therefore remains to show that there is a $T>1$ such that

$$\sum_{n=1}^{N} |r_n e^{-iE_nT}-r_n|^2 = 2\sum_{n=1}^{N} r_n^2(1-\cos E_n T) < \frac{2\epsilon}{3} . \tag{6}$$

That such a T exists follows from the theory of almost-periodic functions, and by reference to that theory, Bocchieri and Loinger[1] complete their proof of the theorem.

It is interesting that the existence of such a T also follows from the classical Poincaré recurrence theorem. The idea is simple, although a few ϵ's are needed for technical reasons.

Consider a collection of N independent harmonic oscillators with frequencies E_n, $n=1,\ldots,N$. A point in the $2N$-dimensional phase space is $\gamma=(c_1,\varphi_1,c_2,\varphi_2,\ldots,c_N,\varphi_N)$, (c_i,φ_i) being the amplitude and phase of the ith oscillator, $c_i \geq 0$. Time evolution is given by $c_i=\text{const}$, $\varphi_i(t)=\varphi_i - E_i t$.

Let the set $\Gamma(\gamma,\alpha)$ be in the following neighborhood of γ:

$$\Gamma(\gamma,\alpha)=\left\{\gamma'=(c_1',\varphi_1',\ldots) \,\middle|\, \sum_{n=1}^{N} |c_i' e^{i\varphi_i'} - c_i e^{i\varphi_i}| < \alpha \right\} \tag{7}$$

and let $\rho=(r_1,0,r_2,0,\ldots,r_N,0)$.

Consider the set $\Gamma(\rho,\frac{1}{3}\epsilon)$. By the Poincaré recurrence theorem there is a point $\gamma=(c_1,\varphi_1,\ldots)$ in $\Gamma(\rho,\frac{1}{3}\epsilon)$ which returns to $\Gamma(\rho,\frac{1}{3}\epsilon)$ at some time T, $T>1$. (Regarding $T>1$, note that the proof in Ref. 2 uses discrete time steps.) For this point,

$$\sum_{n=1}^{N} |c_n e^{i\varphi_n} - r_n| < \frac{\epsilon}{3} \tag{8}$$

$$\sum_{n=1}^{N} |c_n e^{i\varphi_n - E_nT} - r_n| < \frac{\epsilon}{3} \tag{9}$$

It follows that

$$\sum_{n=1}^{N} |r_n e^{-iE_nT} - r_n| \leq \sum_{n=1}^{N} \left(|r_n e^{-iE_nT} - c_n e^{i\varphi_n - iE_nT}| + |c_n e^{i\varphi_n - iE_nT} - r_n|\right) < \frac{2\epsilon}{3} . \tag{10}$$

Since the sum on the left-hand side of Eq. (10) is less than one (for $\epsilon<\frac{3}{2}$) the individual terms must also be less than one. Hence

$$\sum_{n=1}^{N} |r_n e^{-iE_nT} - r_n|^2 \leq \sum_{n=1}^{N} |r_n e^{-iE_nT} - r_n| < \frac{2\epsilon}{3} , \tag{11}$$

which completes the proof.

We mention that ρ itself could not be used (in place of γ) since the Poincaré theorem only asserts recurrence for *almost all* points. In this sense the quantum analog seems more comprehensive. The proving of the inequality (6) also suggests the use of the Poincaré recurrence theorem—itself quite easily proved—in establishing other results concerning almost-periodic functions.

[1]P. Bocchieri and A. Loinger, Phys. Rev. 107, 337 (1957).

[2]M. Kac, *Probability and Related Topics in Physical Sciences*, (Interscience, New York, 1959).