Network Functions of Circuits Containing Dependents Sources

Introduction

Each of the circuits in this problem set is represented by a network function. Network functions are defined, in the frequency-domain, to be quotient obtained by dividing the phasor corresponding to the circuit output by the phasor corresponding to the circuit input. We calculate the network function of a circuit by representing and analyzing the circuit in the frequency-domain.

Network functions are described in Section 13.3 of Introduction to Electric Circuits by R.C. Dorf and J.A. Svoboda. Also, Table 10.7-1 summarizes the correspondence between the time-domain and the frequency domain.

Worked Examples

Example 1:
Consider the circuit shown in Figure 1. The input to the circuit is the voltage of the voltage source, \( v_i(t) \). The output is the voltage across the capacitor, \( v_o(t) \). The network function that represents this circuit is

\[
H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{3}{1 + j\frac{\omega}{2}} \left( 1 + j\frac{\omega}{5} \right)
\]  

(1)

Determine the value of the inductance, \( L \), and of the gain, \( A \), of the Voltage Controlled Voltage Source (VCVS).

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Figure 1 The circuit considered in Example 1.
Solution: The circuit has been represented twice, by a circuit diagram and also by the given
network function. The unknown parameters, \( L \) and \( A \), appear in the circuit diagram, but not in the
given network function. We can analyze the circuit to determine its network function. This
version of the network function will depend on the unknown parameters. We will determine the
value of these parameters by equating the two version of the network function.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in
the frequency domain. Consequently, our first step is to represent the circuit in the frequency
domain, using phasors and impedances. Figure 2 shows the frequency domain representation of
the circuit from Figure 1.

\[ j\omega L I_1(\omega) + 4I_1(\omega) - V_i(\omega) = 0 \]

Solve for \( I_1(\omega) \) to get

\[ I_1(\omega) = \frac{V_i(\omega)}{j\omega L + 4} = \frac{0.25}{1 + j\omega \frac{L}{4}} V_i(\omega) \]

Next use Ohm’s Law to obtain represent \( V_a(\omega) \) as

\[ V_a(\omega) = 4I_1(\omega) = \frac{1}{1 + j\omega \frac{L}{4}} V_i(\omega) \]  \hspace{1cm} (2)

Apply Kirchhoff’s Voltage Law (KVL) to the right-hand mesh to get

\[ 4I_2(\omega) + \frac{20}{j\omega} I_2(\omega) - A V_a(\omega) = 0 \]

Solve for \( I_2(\omega) \) to get
The output voltage is obtained by multiplying the mesh current \( I_2(\omega) \) by the impedance of the capacitor

\[
V_o(\omega) = \frac{20}{j \omega} I_2(\omega) = \frac{20}{j \omega} \frac{j \omega A}{1 + j \frac{\omega}{5}} V_a(\omega) = \frac{A}{1 + j \frac{\omega}{5}} V_a(\omega) \quad (3)
\]

Substituting the expression for \( V_a(\omega) \) from Equation 2 into Equation 3 gives

\[
V_a(\omega) = \frac{1}{1 + j \frac{\omega}{4}} A \frac{1}{1 + j \frac{\omega}{5}} V_i(\omega) = \frac{A}{\left(1 + j \frac{\omega}{4}\right) \left(1 + j \frac{\omega}{5}\right)} V_i(\omega)
\]

Divide both sides of this equation by \( V_i(\omega) \) to obtain the network function of the circuit

\[
H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{A}{\left(1 + j \frac{\omega}{4}\right) \left(1 + j \frac{\omega}{5}\right)} \quad (4)
\]

Comparing the network functions given by Equations 1 and 4 gives \( A = 3 \, \text{V/V} \) and \( L = 2 \, \text{H} \).
Example 2:
Consider the circuit shown in Figure 3. The input to the circuit is the voltage of the voltage source, \( v_i(t) \). The output is the voltage across the capacitor, \( v_o(t) \). This circuit is an example of a “second order low-pass filter”. The network function that represents a second order low-pass filter has the form

\[
H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{k}{\left(1 + \frac{j\omega}{p_1}\right)\left(1 + \frac{j\omega}{p_2}\right)}
\]  

(5)

This network function depends on three parameters, \( k, p_1 \) and \( p_2 \). The parameter \( k \) is called the “dc gain” of the second order low-pass filter. Both \( p_1 \) and \( p_2 \) are poles of the second order low-pass filter. Determine the values of \( k, p_1 \) and \( p_2 \) for the second order low-pass filter in Figure 3.

**Solution:** We will analyze the circuit to determine its network function and then put the network function into the form given in Equation 4. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 4 shows the frequency domain representation of the circuit from Figure 3.

**Figure 3** The circuit considered in Example 2.

**Figure 4** The circuit from Figure 3, represented in the frequency domain, using impedances and phasors.
The circuit in Figure 4 consists of two meshes. The mesh current of the left-hand mesh is labeled as $I_1(\omega)$ and the mesh current of the right-hand mesh is labeled as $I_2(\omega)$. Apply Kirchhoff’s Voltage Law (KVL) to the left-hand mesh to get

$$j \omega (0.66) I_1(\omega) + 4 I_1(\omega) - V_1(\omega) = 0$$

Solve for $I_1(\omega)$ to get

$$I_1(\omega) = \frac{V_1(\omega)}{j \omega (0.66) + 4} = \frac{0.25}{1 + j \omega (0.165)} V_1(\omega)$$

Next use Ohm’s Law to obtain represent $V_a(\omega)$ as

$$V_a(\omega) = 4 I_1(\omega) = \frac{1}{1 + j \omega (0.165)} V_1(\omega)$$

(6)

Apply Kirchhoff’s Voltage Law (KVL) to the right-hand mesh to get

$$6 I_2(\omega) + \frac{1000}{j \omega (12.82)} I_2(\omega) - 15 V_a(\omega) = 0$$

Solve for $I_2(\omega)$ to get

$$I_2(\omega) = \frac{15}{6 + \frac{1000}{j \omega (12.82)}} V_a(\omega) = \frac{j \omega (12.82) 15}{j \omega (12.82) 6 + 1000} V_a(\omega)$$

The output voltage is obtained by multiplying the mesh current $I_2(\omega)$ by the impedance of the capacitor

$$V_o(\omega) = \frac{1000}{j \omega (12.82)} I_2(\omega) = \frac{1000}{j \omega (12.82)} \times \frac{j \omega (12.82) 15}{j \omega (12.82) 6 + 1000} V_a(\omega)$$

$$= \frac{15}{1 + \frac{j \omega (12.82) 6}{1000}} V_a(\omega)$$

(7)

$$= \frac{15}{1 + \frac{j \omega}{13}} V_a(\omega)$$

Substituting the expression for $V_a(\omega)$ from Equation 6 into Equation 7 gives

$$V_o(\omega) = \frac{15}{1 + \frac{j \omega}{13}} \times \frac{1}{1 + j \omega (0.165)} V_1(\omega)$$
Divide both sides of this equation by $V_i(\omega)$ to obtain the network function of the circuit

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{15}{\left(1 + \frac{j\omega}{13}\right)(1 + j\omega(0.165))}$$

(8)

Comparing the network functions given by Equations 4 and 8 gives

$$k = 15 \text{ V/V}, \quad p_1 = 13 \text{ rad/s} \quad \text{and} \quad p_2 = 6.06 \text{ rad/s}$$

(Or perhaps $p_1 = 6.06 \text{ rad/s}$ and $p_2 = 13 \text{ rad/s}$. One pole is named $p_1$ and the other pole is named $p_2$.)

Example 3:
Consider the circuit shown in Figure 5. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the capacitor, $v_o(t)$. The network function that represents this circuit is

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{5}{j\omega\left(1 + \frac{j\omega}{10}\right)}$$

(9)

Determine the value of the inductance, $L$, and of the gain, $A$, of the Current Controlled Current Source (CCCS).

![Figure 5 The circuit considered in Example 3.](image)

Solution: The circuit has been represented twice, by a circuit diagram and also by the given network function. The unknown parameters, $L$ and $A$, appear in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This
version of the network function will depend on the unknown parameters. We will determine the value of these parameters by equating the two version of the network function.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 6 shows the frequency domain representation of the circuit from Figure 5.

The circuit in Figure 6 consists of two meshes. The mesh current of the left-hand mesh is the same current the controlling current of the CCCS, \( I_a(\omega) \). Apply Kirchhoff’s Voltage Law (KVL) to the left-hand mesh to get

\[
20I_a(\omega) + j\omega L I_a(\omega) - V_i(\omega) = 0
\]

Solve for \( I_a(\omega) \) to get

\[
I_a(\omega) = \frac{V_i(\omega)}{20 + j\omega L} = \frac{0.05}{1 + j\omega \frac{L}{20}} V_i(\omega)
\]  

(10)

The mesh current of the right-hand mesh is the same current the controlled current of the CCCS, \( A I_a(\omega) \). The output voltage is obtained by multiplying this mesh current by the impedance of the capacitor

\[
V_o(\omega) = \frac{1}{j\omega} A I_a(\omega)
\]

(11)

Substituting the expression for \( I_a(\omega) \) from Equation 10 into Equation 11 gives

\[
V_o(\omega) = \frac{1}{j\omega} A \times \frac{0.05}{1 + j\omega \frac{L}{20}} V_i(\omega) = \frac{0.05 A}{j\omega \left(1 + j\omega \frac{L}{20}\right)} V_i(\omega)
\]
Divide both sides of this equation by $V_i(\omega)$ to obtain the network function of the circuit

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{0.05 A}{j \omega \left(1 + j \omega \frac{L}{20}\right)} \quad (12)$$

Comparing the network functions given by Equations 9 and 12 gives $A = 100$ V/V and $L = 2$ H.

**Example 4:**
Consider the circuit shown in Figure 7. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the capacitor, $v_o(t)$. The network function that represents this circuit is

$$H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{9}{j \omega \left(1 + j \omega \frac{\omega}{18}\right)} \quad (13)$$

Determine the value of the capacitance, $C$, and of the gain, $A$, of the Current Controlled Voltage Source (CCVS).

**Solution:** The circuit has been represented twice, by a circuit diagram and also by the given network function. The unknown parameters, $C$ and $A$, appear in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This version of the network function will depend on the unknown parameters. We will determine the value of these parameters by equating the two version of the network function.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 8 shows the frequency domain representation of the circuit from Figure 7.

**Figure 7** The circuit considered in Example 4.
The circuit in Figure 6 consists of two meshes. The mesh current of the left-hand mesh is the same current the controlling current of the CCVS, \( I_a(\omega) \). Apply Kirchhoff’s Voltage Law (KVL) to the left-hand mesh to get

\[
j \omega I_a(\omega) - V_i(\omega) = 0
\]

Solve for \( I_a(\omega) \) to get

\[
I_a(\omega) = \frac{1}{j \omega} V_i(\omega)
\]

(14)

Next consider the right-hand mesh. Use the voltage division principle to get

\[
V_o(\omega) = \frac{1}{4 + \frac{1}{j \omega C}} A I_a(\omega) = \frac{A}{1 + j \omega C} I_a(\omega)
\]

(15)

Substituting the expression for \( I_a(\omega) \) from Equation 14 into Equation 15 gives

\[
V_o(\omega) = \frac{1}{j \omega} \times \frac{A}{1 + j \omega C} V_i(\omega) = \frac{A}{j \omega (1 + j \omega C)} V_i(\omega)
\]

Divide both sides of this equation by \( V_i(\omega) \) to obtain the network function of the circuit

\[
H(\omega) = \frac{V_o(\omega)}{V_i(\omega)} = \frac{A}{j \omega (1 + j \omega C)}
\]

(16)

Comparing the network functions given by Equations 13 and 16 gives \( A = 9 \) V/V and

\[
C = \frac{1}{18 (4)} = 13.89 \text{ mF}.
\]