Network Functions for Simple Circuits

Introduction

Each of the circuits in this problem set is represented by a network function. Network functions are defined, in the frequency-domain, to be quotient obtained by dividing the phasor corresponding to the circuit output by the phasor corresponding to the circuit input. We calculate the network function of a circuit by representing and analyzing the circuit in the frequency-domain.

Network functions are described in Section 13.3 of *Introduction to Electric Circuits* by R.C. Dorf and J.A Svoboda. Also, Table 10.7-1 summarizes the correspondence between the time-domain and the frequency domain.

Worked Examples

Example 1:

Consider the circuit shown in Figure 1. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the 8 Ω resistor, $v_o(t)$. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{0.66}{1 + j\dfrac{\omega}{30}} \tag{1}$$

Determine the value of the inductance, *L*.

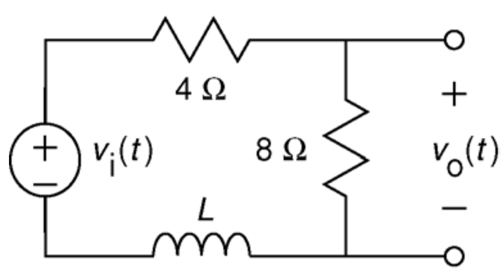

Figure 1 The circuit considered in Example 1.

Solution: The circuit has been represented twice, by a circuit diagram and also by a network function. The unknown inductance, *L*, appears in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown inductance. We will determine the value of the inductance by equating the two network functions.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 2 shows the frequency domain representation of the circuit from Figure 1.

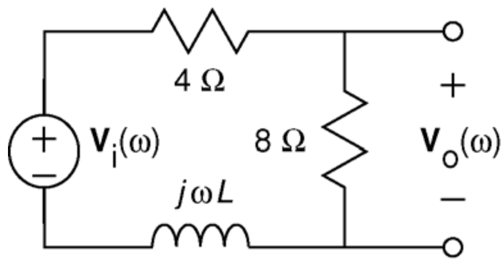

Figure 2 The circuit from Figure 1, represented in the frequency domain, using impedances and phasors.

The impedances of the inductor and the two resistors are connected in series in Figure 2. $\mathbf{V_i}(\omega)$ is the voltage across these three series impedances and $\mathbf{V_o}(\omega)$ is the voltage across one of the impedances. Apply the voltage division principle to get

$$\mathbf{V_o}(\omega) = \frac{8}{4+8+j\omega L}\mathbf{V_i}(\omega)$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{8}{12+j\omega L} \tag{2}$$

The network functions given in Equations 1 and 2 must be equal. That is

$$\frac{8}{12+j\omega L} = \frac{0.66}{1+j\dfrac{\omega}{30}}$$

$$8\left(1+j\frac{\omega}{30}\right) = 0.66\left(12+j\omega L\right)$$

$$8+j\frac{8\,\omega}{30} = 8+j\left(0.66\right)\omega L$$

$$\frac{8}{30} = \left(0.66\right)L$$

$$L = \frac{8}{30\left(0.66\right)}$$

$$L = 0.4 \text{ H}$$

We can simply the algebra required to find L by putting the network function in Equation 2 into the same form as the network function in Equation 1 before equating the two network functions. Notice that the real part of the denominator of the network function is 1 in Equation 1. Let's make the real part of the denominator be 1 in the network function given by Equation 2. Divide the numerator and denominator by 12 to get

$$\mathbf{H}(\omega)=\frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)}=\frac{\frac{8}{12}}{\frac{12+j\omega L}{12}}=\frac{0.66}{1+j\omega\frac{L}{12}} \tag{3}$$

Equating the network functions given by Equations 1 and 3 gives:

$$\frac{0.66}{1+j\omega\frac{L}{12}}=\frac{0.66}{1+j\frac{\omega}{30}} \quad\Rightarrow\quad \frac{L}{12}=\frac{1}{30} \quad\Rightarrow\quad L=0.4\ \text{ H}$$

The same result is obtained with less algebra.

Example 2:

Consider the circuit shown in Figure 3. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the 4 Ω resistor, $v_o(t)$. This circuit is an example of a "first order low-pass filter". The network function that represents a first order low-pass filter has the form

$$\mathbf{H}(\omega)=\frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)}=\frac{k}{1+j\frac{\omega}{p}} \tag{4}$$

This network function depends on two parameters, k and p. The parameter k is called the "dc gain" of the first order low-pass filter and p is the pole of the first order low-pass filter. Determine the values of k and of p for the first order low-pass filter in Figure 3.

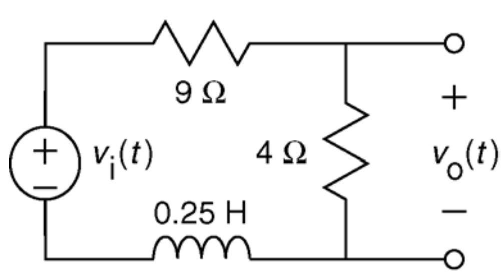

Figure 3 The circuit considered in Example 2.

Solution: We will analyze the circuit to determine its network function and then put the network function into the form given in Equation 4. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 4 shows the frequency domain representation of the circuit from Figure 3.

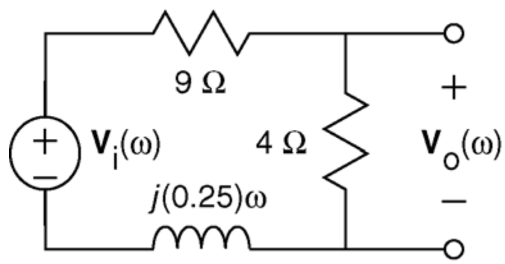

Figure 4 The circuit from Figure 3, represented in the frequency domain, using impedances and phasors.

The impedances of the inductor and the two resistors are connected in series in Figure 4. $V_i(\omega)$ is the voltage across these three series impedances and $V_o(\omega)$ is the voltage across one of the impedances. Apply the voltage division principle to get

$$\mathbf{V_o}(\omega) = \frac{4}{9+4+j(0.25)\omega}\mathbf{V_i}(\omega)$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{4}{13+j(0.25)\omega} \tag{5}$$

Next, we put the network function into the form specified by Equation 4. Notice that the real part of the denominator is 1 in Equation 4. Divide the numerator and denominator by 13 in Equation 5 to get

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{\frac{4}{13}}{\frac{13}{13}+\frac{j(0.25)\omega}{13}} = \frac{0.308}{1+\frac{j\omega}{52}} \tag{6}$$

Comparing the network functions given by Equations 4 and 6 gives

$k = 0.308$ V/V and $p = 52$ rad/s

Example 3:

Consider the circuit shown in Figure 5. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the 5 Ω resistor, $v_o(t)$. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = 0.208 \frac{j\omega}{1 + j\frac{\omega}{3}} \tag{7}$$

Determine the value of the capacitance, *C*.

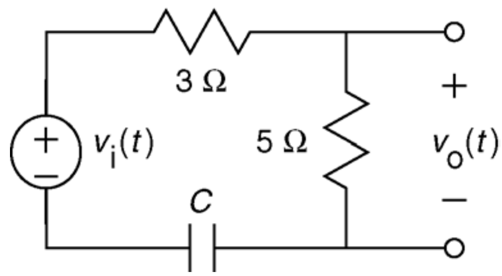

Figure 5 The circuit considered in Example 3.

Solution: The circuit has been represented twice, by a circuit diagram and also by a network function. The unknown capacitance, *C*, appears in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown capacitance. We will determine the value of the capacitance by equating the two network functions.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 6 shows the frequency domain representation of the circuit from Figure 5.

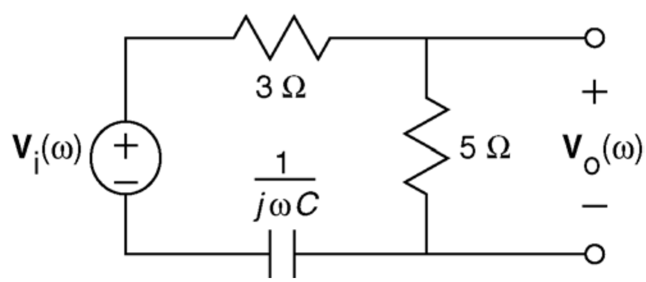

Figure 6 The circuit from Figure 5, represented in the frequency domain, using impedances and phasors.

The impedances of the capacitor and the two resistors are connected in series in Figure 6. $\mathbf{V_i}(\omega)$ is the voltage across these three series impedances and $\mathbf{V_o}(\omega)$ is the voltage across one of the impedances. Apply the voltage division principle to get

$$\mathbf{V_o}(\omega) = \frac{5}{5+3+\frac{1}{j\omega C}} \mathbf{V_i}(\omega)$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{5}{5+3+\frac{1}{j\omega C}} \tag{8}$$

We can simply the algebra required to find *C* by putting the network function in Equation 8 into the same form as the network function in Equation 7 before equating the two network functions. Let's multiply the numerator and denominator by $j\omega C$ to get

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{5}{8+\frac{1}{j\omega C}} \times \frac{j\omega C}{j\omega C} = 5\,C \frac{j\omega}{1+j\omega C(8)} \tag{9}$$

Equating the network functions given by Equations 7 and 9 gives:

$$5\,C \frac{j\omega}{1+j\omega C(8)} = 0.208 \frac{j\omega}{1+j\frac{\omega}{3}}$$

Comparing corresponding parts of this equation indicates that:

$$5\,C = 0.208 \quad \textbf{and} \quad 8\,C = \frac{1}{3}$$

The values of *C* obtained from these equations must agree. (If they do not, we've made an error.) Solving these equations gives

$$C = 41.60 \text{ mF} \quad \textbf{and} \quad C = 41.67 \text{ mF}$$

These values agree, but there is some uncertainty in the third significant figure. It's appropriate to report our result with two significant figures:

$$C = 42 \text{ mF}$$

Example 4:

Consider the circuit shown in Figure 7. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage across the 8 Ω resistor, $v_o(t)$. This circuit is an example of a "first order high-pass filter". The network function that represents a first order high-pass filter has the form

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = k\frac{j\omega}{1 + j\dfrac{\omega}{p}} \tag{10}$$

The network function depends on two parameters, *k* and *p*. The parameter *p* is called the pole of the first order high-pass filter. The parameter *k* is sometime referred to as a gain, but the high-frequency gain of the circuit is given by the product *kp*. Determine the values of *k* and of *p* for the first order high-pass filter in Figure 7.

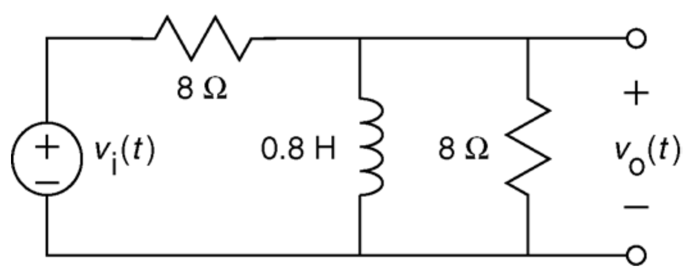

Figure 7 The circuit considered in Example 4.

Solution: We will analyze the circuit to determine its network function and then put the network function into the form given in Equation 10. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 8 shows the frequency domain representation of the circuit from Figure 7.

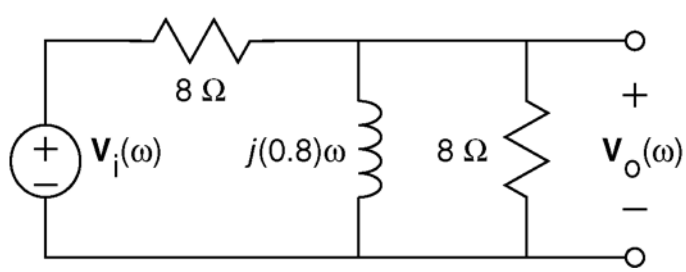

Figure 8 The circuit from Figure 7, represented in the frequency domain, using impedances and phasors.

The impedances of the inductor and one 8 Ω resistor are connected in parallel in Figure 8. The equivalent impedance is

$$\mathbf{Z_e}(\omega) = \frac{(8)\,j(0.8)\omega}{(8) + j(0.8)\omega}$$

The parallel impedance is connected in series with the other 8 Ω resistor. $\mathbf{V_i}(\omega)$ is the voltage across the series impedances and $\mathbf{V_o}(\omega)$ is the voltage across the equivalent impedance, $\mathbf{Z_e}(\omega)$. Apply the voltage division principle to get

$$\begin{aligned}
\mathbf{V_o}(\omega) &= \frac{\dfrac{(8)\,j(0.8)\omega}{(8) + j(0.8)\omega}}{8 + \dfrac{(8)\,j(0.8)\omega}{(8) + j(0.8)\omega}}\mathbf{V_i}(\omega) \\
&= \frac{(8)\,j(0.8)\omega}{8\big((8) + j(0.8)\omega\big) + (8)\,j(0.8)\omega}\mathbf{V_i}(\omega) \\
&= \frac{j(6.4)\omega}{64 + j(8)(2)(0.8)\omega}\mathbf{V_i}(\omega) \\
&= \frac{j(6.4)\omega}{64 + j(12.8)\omega}\mathbf{V_i}(\omega)
\end{aligned}$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{j(6.4)\omega}{64 + j(12.8)\omega} \tag{11}$$

Next, we put the network function into the form specified by Equation 10. Notice that the real part of the denominator is 1 in Equation 10. Divide the numerator and denominator by 64 in Equation 11 to get

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{\dfrac{j(6.4)\omega}{64}}{\dfrac{64 + j(12.8)\omega}{64}} = 0.1\frac{j\omega}{1 + \dfrac{j(12.8)\omega}{64}} \tag{12}$$

Comparing the network functions given by Equations 10 and 12 gives

$$k = 0.1 \text{ V/V and } p = \frac{64}{12.8} = 5 \text{ rad/s}.$$

Example 5:

Consider the circuit shown in Figure 9. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage, $v_o(t)$, across the series connection of the capacitor and 16 kΩ resistor. The network function that represents a this circuit has the form

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}} \tag{13}$$

The network function depends on two parameters, z and p. The parameter z is called the zero of the circuit and the parameter p is called the pole of the circuit. Determine the values of z and of p for the circuit in Figure 9.

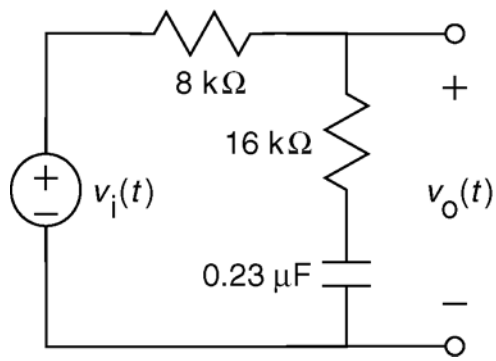

Figure 9 The circuit considered in Example 5.

Solution: We will analyze the circuit to determine its network function and then put the network function into the form given in Equation 13. A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 10 shows the frequency domain representation of the circuit from Figure 9.

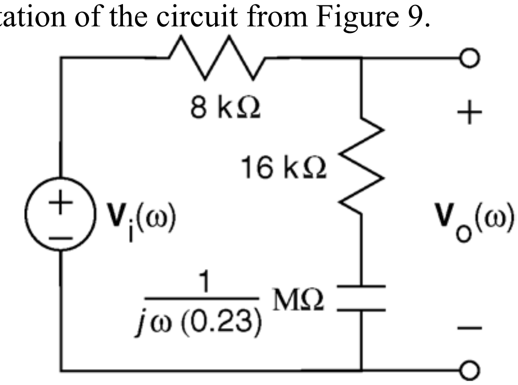

Figure 10 The circuit from Figure 9, represented in the frequency domain, using impedances and phasors.

The impedances of the capacitor and the 16 kΩ resistor are connected in series in Figure 10. The equivalent impedance is

$$\mathbf{Z_e}(\omega) = 16000 + \frac{10^6}{j(0.23)\omega}$$

The equivalent impedance is connected in series with the 8 kΩ resistor. $\mathbf{V_i}(\omega)$ is the voltage across the series impedances and $\mathbf{V_o}(\omega)$ is the voltage across the equivalent impedance, $\mathbf{Z_e}(\omega)$. Apply the voltage division principle to get

$$\begin{aligned}\mathbf{V_o}(\omega) &= \frac{16000 + \dfrac{10^6}{j(0.23)\omega}}{8000 + 16000 + \dfrac{10^6}{j(0.23)\omega}} \mathbf{V_i}(\omega) \\ &= \frac{10^6 + j(0.23)\omega(16000)}{10^6 + j(0.23)\omega(24000)} \mathbf{V_i}(\omega) \\ &= \frac{\dfrac{10^6 + j(3680)\omega}{10^6}}{\dfrac{10^6 + j(5520)\omega}{10^6}} \mathbf{V_i}(\omega) \\ &= \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega} \mathbf{V_i}(\omega)\end{aligned}$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega} \tag{14}$$

Equating the network functions given by Equations 13 and 14 gives

$$\frac{1 + j(0.00368)\omega}{1 + j(0.00552)\omega} = \frac{1 + j\dfrac{\omega}{z}}{1 + j\dfrac{\omega}{p}}$$

Comparing these network functions gives

$$z = \frac{1}{0.00368} = 271.74 \text{ rad/s} \text{ and } p = \frac{1}{0.00552} = 181.16 \text{ rad/s}.$$

Example 6:
Consider the circuit shown in Figure 11. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage, $v_o(t)$, across series connection of the inductor the 2 Ω resistor. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = 0.2\frac{1 + j\dfrac{\omega}{5}}{1 + j\dfrac{\omega}{25}} \tag{15}$$

Determine the value of the inductance, *L*.

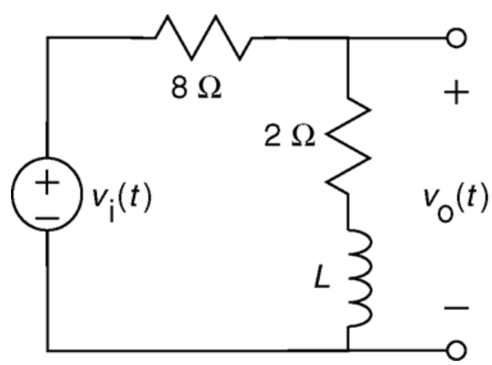

Figure 11 The circuit considered in Example 6.

Solution: The circuit has been represented twice, by a circuit diagram and also by a network function. The unknown inductance, *L*, appears in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown inductance. We will determine the value of the inductance by equating the two network functions.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 12 shows the frequency domain representation of the circuit from Figure 11.

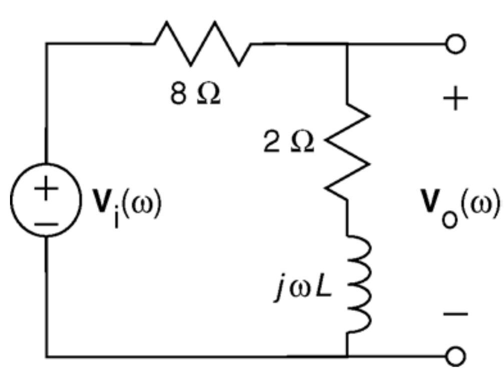

Figure 12 The circuit from Figure 11, represented in the frequency domain, using impedances and phasors.

The impedances of the inductor and the 2 Ω resistor are connected in series in Figure 12. The equivalent impedance is

$$\mathbf{Z_e}(\omega) = 2 + j\,\omega\,L$$

The equivalent impedance is connected in series with the 8 Ω resistor. $\mathbf{V_i}(\omega)$ is the voltage across the series impedances and $\mathbf{V_o}(\omega)$ is the voltage across the equivalent impedance, $\mathbf{Z_e}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V_o}(\omega) = \frac{2 + j\,\omega\,L}{8 + 2 + j\,\omega\,L}\mathbf{V_i}(\omega) = \frac{2 + j\,\omega\,L}{10 + j\,\omega\,L}\mathbf{V_i}(\omega)$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{2 + j\,\omega\,L}{10 + j\,\omega\,L}$$

Next, we put the network function into the form specified by Equation 15. Factoring 2 out of both terms in the numerator and also factoring 10 out of both terms in the denominator we get

$$\mathbf{H}(\omega) = \frac{2\left(1 + j\,\omega\,\frac{L}{2}\right)}{10\left(1 + j\,\omega\,\frac{L}{10}\right)} = 0.2\,\frac{1 + j\,\omega\,\frac{L}{2}}{1 + j\,\omega\,\frac{L}{10}} \tag{16}$$

Equating the network functions given by Equations 15 and 16 gives

$$0.2\,\frac{1 + j\,\omega\,\frac{L}{2}}{1 + j\,\omega\,\frac{L}{10}} = 0.2\,\frac{1 + j\,\frac{\omega}{5}}{1 + j\,\frac{\omega}{25}}$$

Comparing these network functions gives

$$\frac{L}{2} = \frac{1}{5} \text{ and } \frac{L}{10} = \frac{1}{25}$$

The values of L obtained from these equations must agree, and they do. (If they do not, we've made an error.) Solving each of these equations gives $L = 0.4$ H.

Example 7:

Consider the circuit shown in Figure 13. The input to the circuit is the voltage of the voltage source, $v_i(t)$. The output is the voltage, $v_o(t)$, across series connection of the capacitor the 4 kΩ resistor. The network function that represents this circuit is

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{1 + j\dfrac{\omega}{69.6}}{1 + j\dfrac{\omega}{55.7}} \tag{17}$$

Determine the value of the capacitance, C.

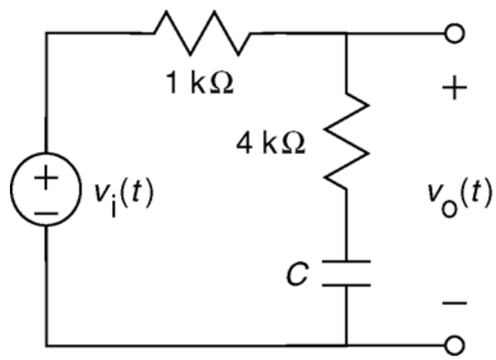

Figure 13 The circuit considered in Example 7.

Solution: The circuit has been represented twice, by a circuit diagram and also by a network function. The unknown capacitance, C, appears in the circuit diagram, but not in the given network function. We can analyze the circuit to determine its network function. This second network function will depend on the unknown capacitance. We will determine the value of the capacitance by equating the two network functions.

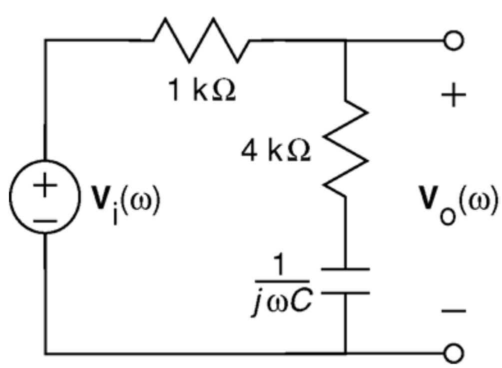

Figure 14 The circuit from Figure 13, represented in the frequency domain, using impedances and phasors.

A network function is the ratio of the output phasor to the input phasor. Phasors exist in the frequency domain. Consequently, our first step is to represent the circuit in the frequency domain, using phasors and impedances. Figure 14 shows the frequency domain representation of the circuit from Figure 13.

The impedances of the capacitor and the 4 kΩ resistor are connected in series in Figure 14. The equivalent impedance is

$$\mathbf{Z_e}(\omega) = 4000 + \frac{1}{j\omega C}$$

The equivalent impedance is connected in series with the 1 kΩ resistor. $\mathbf{V_i}(\omega)$ is the voltage across the series impedances and $\mathbf{V_o}(\omega)$ is the voltage across the equivalent impedance, $\mathbf{Z_e}(\omega)$. Apply the voltage division principle to get

$$\mathbf{V_o}(\omega) = \frac{4000 + \frac{1}{j\omega C}}{1000 + 4000 + \frac{1}{j\omega C}} \mathbf{V_i}(\omega) = \frac{1 + j\omega C(4000)}{1 + j\,\omega C(5000)} \mathbf{V_i}(\omega)$$

Divide both sides of this equation by $\mathbf{V_i}(\omega)$ to obtain the network function of the circuit

$$\mathbf{H}(\omega) = \frac{\mathbf{V_o}(\omega)}{\mathbf{V_i}(\omega)} = \frac{1 + j\omega C(4000)}{1 + j\,\omega C(5000)} \tag{18}$$

Equating the network functions given by Equations 17 and 18 gives

$$\frac{1 + j\omega C(4000)}{1 + j\,\omega C(5000)} = \frac{1 + j\frac{\omega}{69.6}}{1 + j\frac{\omega}{55.7}}$$

Comparing these network functions gives

$$4000\,C = \frac{1}{69.9} \text{ and } 5000\,C = \frac{1}{55.7}$$

The values of *C* obtained from these equations must agree, and they do. (If they do not, we've made an error.) Solving these equations gives

$$C = 3.577\ \mu\text{F} \quad \textbf{and} \quad C = 3.591\ \mu\text{F}$$

These values agree, but there is some uncertainty in the third significant figure. It's appropriate to report our result with two significant figures:

$$C = 3.6\ \mu\text{F}$$