Power in AC Circuits

Introduction

Each problem in this problem set involves the steady state response of a circuit to a single sinusoidal input. The circuits in this problem set consist of resistors, capacitors and inductors. The input to each circuit is either the voltage of an independent voltage source or the current of an independent current source.

Each problem in this problem set asks a question regarding the complex power received or supplied by an element of the circuit. Circuit analysis in the frequency domain provides the answers to these questions.

Complex power is discussed in Sections 11.5 and 11.6 of Introduction to Electric Circuits by R.C. Dorf and J.A Svoboda. In particular, Table 11.5-1 summarizes the equations used to calculate complex power in the frequency domain. Circuit analysis in the frequency-domain is described in Sections 10.6 thru 10.11. Table 10.7-1 summarizes the correspondence between the time domain and the frequency domain. Also, Appendix B provides a review of complex arithmetic.

Worked Examples

Example 1:
Consider the circuit shown in Figure 1. The input to the circuit is the voltage of the voltage source, $v_s(t)$. The output is the voltage across the inductor, $v_o(t)$. Determine the following:

a. The average power supplied by the voltage source.
b. The average power received by the resistor.
c. The average power received by the inductor.
d. The power factor of the impedance of the series connection of the resistor and inductor.

\[ v_s(t) = 7.28 \cos (4t + 77^\circ) \text{ V} \]
\[ v_o(t) = 4.254 \cos (4t + 311^\circ) \text{ V} \]

Figure 1 The circuit considered in Example 1.
Solution: The input voltage is a sinusoid. The output voltage is also a sinusoid and has the same frequency as the input voltage. Apparently the circuit has reached steady state. Consequently, the circuit in Figure 1 can be represented in the frequency domain, using phasors and impedances. Figure 2 shows the frequency domain representation of the circuit from Figure 1. The voltages $V_s(\omega)$ and $V_o(\omega)$ in Figure 2 are the phasors corresponding to $v_s(t)$ and $v_o(t)$ from Figure 1. The inductor and the resistor are represented as impedances in Figure 2. The impedance of the inductor is $j\omega L = j(4)(0.54) = j2.16 \ \Omega$, as shown in Figure 2.

\[
\begin{align*}
V_s(\omega) &= 7.28 \angle 77^\circ \ V \\
V_o(\omega) &= 4.254 \angle 311^\circ \ V \\
I(\omega) &= \frac{V_o(\omega)}{j2.16} = \frac{4.254 \angle 311^\circ}{2.16 \angle 90^\circ} = \frac{4.254}{2.16} \angle (311^\circ - 90^\circ) = 1.969 \angle 221^\circ \ A
\end{align*}
\]

Figure 2 The circuit from Figure 1, represented in the frequency domain, using impedances and phasors.

The current $I(\omega)$ in Figure 2 is calculated from $V_o(\omega)$ and the impedance of the inductor using ohm's Law:

\[
I(\omega) = \frac{V_o(\omega)}{j2.16} = \frac{4.254 \angle 311^\circ}{2.16 \angle 90^\circ} = \frac{4.254}{2.16} \angle (311^\circ - 90^\circ) = 1.969 \angle 221^\circ \ A
\]

Once we know $I(\omega)$ we are ready to answer the questions asked in this problem.

a. The average power supplied by the source is calculated from $I(\omega)$ and $V_s(\omega)$. The average power of the source is given by

\[
\frac{|V_s(\omega)| |I(\omega)|}{2} \cos(\angle V_s(\omega) - \angle I(\omega)) = \frac{(7.28)(1.969)}{2} \cos(77^\circ - 221^\circ) = 7.167 \cos(-144^\circ) = -5.8 \ W
\]

Notice that $I(\omega)$ and $V_s(\omega)$ adhere to the passive convention. Consequently, Equation 1 gives the power received by the voltage source rather than the power supplied by the voltage source. The power supplied is the negative of the power received. Therefore, the power supplied by the voltage source is

\[
P_s = 5.8 \ W
\]
b. The resistor voltage, \( V_R(\omega) \), in Figure 2 is given by

\[
V_R(\omega) = R I(\omega) = 3(1.969 \angle 221^\circ) = 5.907 \angle 221^\circ \text{ V}
\]

The average power received by the resistor is calculated from \( I(\omega) \) and \( V_R(\omega) \):

\[
P_{R} = \frac{|V_R(\omega)||I(\omega)|}{2} \cos(\angle V_R(\omega) - \angle I(\omega))
= \frac{(5.907)(1.969)}{2} \cos(221^\circ - 221^\circ)
= 5.8 \cos(0^\circ) = 5.8 \text{ W}
\]  

(2)

Notice that \( I(\omega) \) and \( V_R(\omega) \) adhere to the passive convention. Consequently, \( P_R \) is the power received by the resistor, as required.

Alternately, the power received by a resistor can be calculated from the current \( I(\omega) \) and the resistance, \( R \). To see how, first notice that the voltage and current of a resistor are related by

\[
V_R(\omega) = R I(\omega) \Rightarrow |V_R(\omega)|\angle V_R(\omega) = R (|I(\omega)|\angle I(\omega)) \Rightarrow \begin{cases} |V_R(\omega)| = R |I(\omega)| \\ \angle V_R(\omega) = \angle I(\omega) \end{cases}
\]

Substituting these expressions for \( |V_R(\omega)| \) and \( \angle V_R(\omega) \) into Equation 2 gives

\[
P_R = \frac{R |I(\omega)||I(\omega)|}{2} \cos(\angle I(\omega) - \angle I(\omega))
= \frac{R |I(\omega)|^2}{2}
= \frac{(3)(1.969)^2}{2} = 5.8 \text{ W}
\]

c. The average power received by the inductor is calculated from \( I(\omega) \) and \( V_o(\omega) \):

\[
P_L = \frac{|V_o(\omega)||I(\omega)|}{2} \cos(\angle V_o(\omega) - \angle I(\omega))
= \frac{(4.254)(1.969)}{2} \cos(311^\circ - 221^\circ)
= 4.188 \cos(90^\circ) = 0 \text{ W}
\]  

(3)

The phase angle of the inductor voltage is always 90° greater than the phase angle of the inductor current. Consequently, the value of average power received by any inductor is zero.

d. The power factor of the impedance of the series connection of the resistor and inductor can be calculated from \( I(\omega) \) and the voltage across the impedance. That voltage is \( V_R(\omega)+V_o(\omega) \), which is calculated by applying Kirchhoff’s Voltage Law to the circuit in Figure 2:
\[ V_R(\omega) + V_a(\omega) + V_s(\omega) = 0 \]

\[ V_R(\omega) + V_s(\omega) = -V_s(\omega) = -7.28 \angle 77^\circ \]

\[ = (1 \angle 180^\circ)(7.28 \angle 77^\circ) \]

\[ = 7.28 \angle 257^\circ \]

Now the power factor is calculated as

\[ \text{pf} = \cos\left(\angle(V_R(\omega) + V_a(\omega)) - \angle I(\omega)\right) = \cos(257^\circ - 221^\circ) = 0.809^\circ \]

The power factor is said to be lagging because 257-221=36>0.

**Observation:** Average power is conserved. In this example, that means that the average power supplied by the voltage source must be equal to the sum of the average powers received by the resistor and the inductor. This fact provides a check on the accuracy of our calculations.

**Alternate Solution:** If the value of \( V_a(\omega) \) had not be given, then \( I(\omega) \) would be calculated by writing and solving a mesh equation. Referring to Figure 2, the mesh equation is

\[ 3 I(\omega) + j2.16 I(\omega) + 7.28 \angle 77^\circ = 0 \]

Solving for \( I(\omega) \) gives

\[ I(\omega) = \frac{-7.28 \angle 77^\circ}{3 + j2.16} = \frac{(1 \angle 180^\circ)(7.28 \angle 77^\circ)}{3.697 \angle 36^\circ} \]

\[ = \frac{(1)(7.28)}{3.697} \angle (180 + 77 - 36) = 1.969 \angle 221^\circ \text{ A} \]

as before.