State Equations for Dynamic Circuits

Consider a dynamic circuit that does not contain any

- Loops consisting entirely of capacitors and voltage sources.
- Cutsets consisting entirely of inductors and current sources.

For example, consider

The input to this circuit is the source voltage, \( v_i \). The output is the current \( i_o \). The inductor current, \( i_1 \), and capacitor voltage, \( v_2 \), are called state variables. We can represent this circuit by differential equations of the form

\[
\frac{d i_1}{dt} = a_{11} i_1 + a_{12} v_2 + b_1 v_i
\]

\[
\frac{d v_2}{dt} = a_{21} i_1 + a_{22} v_2 + b_2 v_i
\]

Then the output can be expressed as

\[
i_o = c_1 i_1 + c_2 v_2 + d v_i
\]

The coefficients, \( a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2, c_1, c_2 \) and \( d \) are constants that depend on the values of \( L, C, R_3, R_4 \) and \( R_5 \). These equations are called state equations. We can write the state equations in matrix form:

\[
\begin{bmatrix}
\frac{d i_1}{dt} \\
\frac{d v_2}{dt}
\end{bmatrix} =
\begin{bmatrix}
a_{11} & a_{12} \\
a_{21} & a_{22}
\end{bmatrix}
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix} +
\begin{bmatrix}
b_1 \\
b_2
\end{bmatrix} v_i
\]

\[
i_o =
\begin{bmatrix}
c_1 & c_2
\end{bmatrix}
\begin{bmatrix}
i_1 \\
v_2
\end{bmatrix} + d v_i
\]
This looks pretty complicated. We might ask why we would want to do such a thing. We will see that

- Writing the state equations isn’t particularly difficult.
- MATLAB provides differential equation solvers that we can use to plot $i_1$, $v_2$ and $i_o$.

Let’s write the state equations. With the benefit of hind sight, we will replace the inductor by a current source and replace the capacitor by a voltage source to get

![Circuit Diagram]

(The orientations of the new sources are chosen to agree with the orientations of $i_1$ and $v_2$.) This circuit is a static circuit since it does not contain capacitors or inductors. We could analyze this circuit by writing node equations or mesh equation, but it seems easier to use superposition. The objective of the analysis is the express

$v_1$, the voltage across the current source that replaced the inductor,

$i_2$, the current is the voltage source that replaced the capacitor,

and

$i_o$, the output of the original circuit,

as functions of the input to this circuit, $v_i$, and the state variables, $i_1$, and, $v_2$.

Using superposition we find

$$v_1 = -\left(\frac{R_3}{R_3 + R_4}\right)i_1 + \left(\frac{R_4}{R_3 + R_4}\right)v_2 + \left(\frac{R_4}{R_3 + R_4}\right)v_i$$

$$i_2 = -\left(\frac{R_3}{R_3 + R_4}\right)i_1 - \left(\frac{1}{R_5 + \frac{R_3}{R_3 + R_4}}\right)v_2 + \left(\frac{1}{R_3 + R_4}\right)v_i$$

$$i_o = -\left(\frac{R_3}{R_3 + R_4}\right)i_1 - \left(\frac{1}{R_3 + R_4}\right)v_2 + \left(\frac{1}{R_3 + R_4}\right)v_i$$
These equations also represent the original circuit. In the original circuit, it is also true that

\[ v_1 = L \frac{d i_1}{dt} \quad \text{and} \quad i_2 = C \frac{d v_2}{dt} \]

Combining these equations gives

\[
\begin{align*}
L \frac{d i_1}{dt} &= -\left( \frac{R_3}{R_3 + R_4} \right) i_1 + \left( \frac{R_3}{R_3 + R_4} \right) v_2 + \left( \frac{R_4}{R_3 + R_4} \right) v_i \\
C \frac{d v_2}{dt} &= -\left( \frac{R_3}{R_3 + R_4} \right) i_1 - \left( \frac{1}{R_3 + R_4} \right) v_2 + \left( \frac{1}{R_3 + R_4} \right) v_i \\
i_o &= -\left( \frac{R_3}{R_3 + R_4} \right) i_1 - \left( \frac{1}{R_3 + R_4} \right) v_2 + \left( \frac{1}{R_3 + R_4} \right) v_i
\end{align*}
\]

Dividing both sides of the first equation by \( L \) and dividing both sides of the first equation by \( C \) we have the state equations:

\[
\begin{align*}
\frac{d i_1}{dt} &= -\left( \frac{R_3}{L (R_3 + R_4)} \right) i_1 + \left( \frac{R_3}{L (R_3 + R_4)} \right) v_2 + \left( \frac{R_4}{L (R_3 + R_4)} \right) v_i \\
\frac{d v_2}{dt} &= -\left( \frac{1}{C (R_3 + R_4)} \right) i_1 - \left( \frac{1}{C (R_3 + R_4)} \right) v_2 + \left( \frac{1}{C (R_3 + R_4)} \right) v_i \\
i_o &= -\left( \frac{R_3}{R_3 + R_4} \right) i_1 - \left( \frac{1}{R_3 + R_4} \right) v_2 + \left( \frac{1}{R_3 + R_4} \right) v_i
\end{align*}
\]
Here’s a MATLAB script based on this analysis:

```matlab
% solution time interval
tspan=[0,1];
% xo must be a column
xo=[1; 0];

% description of the circuit in example 1
Vi = 0;

% H
L = 4;
% F
C = 0.125;
% Ohms
R3 = 20;
R4 = 20;
R5 = 10;

a11 = -R3*R4/L/(R3+R4);
a12 = R3/L/(R3+R4);
a21 = -R3/C/(R3+R4);
a22 = -1/R5 - 1/C/(R3+R4);
b1 = R4/L/(R3+R4);
b2 = 1/C/(R3+R4);
c1 = -R3/(R3+R4);
c2 = -1/(R3+R4);
d = 1/(R3+R4);

A = [a11 a12; a21 a22];
B = [b1; b2];

[t,x]=ode45(@steqn, tspan, xo, [], A, B, Vi);

io = [c1 c2]*x' + d*Vi;
plot(t, io')
```

The above script uses this function:

```matlab
function xdot=steqn(t,x,A,B,Vi)

xdot = A*x + B*Vi;
```
Here's the plot produced by the MATLAB script: